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The Planar Dirichlet Problem for the Stokes Equations

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The Dirichlet problem for the Stokes equations is studied in a planar domain. We construct a solution of this problem in form of appropriate potentials and determine the unknown source densities via integral equation systems on the boundary of the domain. The solution is given explicitly in the form of a series. As a consequence we determine a solution of the Dirichlet problem for the compressible Stokes equations and a solution of one boundary problem on a domain with cracks.

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1 Introduction

One of the most important problems of mathematical physics is the Dirichlet problem of the Stokes equations

$$-\Delta u + \nabla p = 0 \quad \text{in} \quad G, \qquad \nabla \cdot u = 0 \quad \text{in} \quad G, \qquad u = b \quad \text{on} \quad \partial G \qquad (1)$$

(see [7], [6], [1], [12]). One traditional way how to study this problem is the integral equation method (see for example [13], [17], [3], [15], [2], [4], [9]). In the present paper we construct a solution of the Dirichlet problem for the Stokes equations on planar domains with compact boundary of class $C^{1+\alpha}$ (not necessarily connected) using methods of hydrodynamical potential theory. It is usual to look for a solution of the Dirichlet problem in the form of a double layer potential. But it does not work for domains with holes. There are many ways how mathematicians overcome this difficulty. The authors studied in [11] the Dirichlet problem of the Stokes equations in domains in R^m with $m > 2$. They looked for a solution in the form of a sum of a double layer potential and a single layer potential with the same density. They reduced the original problem to the equivalent integral equation. They derived necessary and sufficient conditions for the solvability and constructed the solution in the form of the Neumann series. But this method does not work for planar domains because a double layer potential is not bounded in the plane. We look for a solution in a modified form. We were inspired by [17], where the Dirichlet problem for the Stokes equations was studied on an exterior planar domain with connected boundary, and by [10], where the Dirichlet problem for the Laplace equation was studied by integral equation method on planar domains and a solution of the corresponding integral equation was given in the form of a Neumann series.

In the present paper we construct a solution of (1). The construction we use is based on an explicit representation of the fundamental tensor for Stoke's equations. It includes a detailed study of the corresponding boundary layer potentials and ends up with a boundary integral equations method reducing (1) to a system of second kind Fredholm boundary integral equations. If G is unbounded then this system of integral equations is uniquely solvable and we express its solution in the form of a Neumann series. If G is a bounded domain then this system of integral equations is solvable if and only if

$$\int_{\partial G} b \cdot N_G \,\mathrm{do} = 0.$$

(This condition is a necessary and sufficient condition for the solvability of the problem (1).) From the numerical reasons we need to have a uniquely solvable system of equations. (The round errors may make that the necessary and sufficient condition of the solvability is not satisfied but we would like to obtain an approximate solution which is very close to the solution. - The similar problem has been recently studied for boundary value problems in elastostatics in [8].) We modified the integral equation. Thus we got a uniquely solvable integral equation. Moreover, if the Dirichlet problem for the Stokes system is solvable then the solution of the modified integral equation is also a solution of the original problem. At the end we obtained a solution of the corresponding integral equations using the successive approximation method.

As an easy consequence we apply the result for the construction of a solution for the boundary value problem of the Stokes equations in case of a cracked domain

$$-\Delta u + \nabla p = 0 \quad \text{in} \quad \Omega, \qquad \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \qquad u = b \quad \text{on} \quad \partial\Omega \setminus S, \tag{2}$$

$$u^+ - u^- = f, \quad (T_p^v N_{V^+})^+ - (T_p^v N_{V^+})^- = h \qquad \text{on } S \cap G \tag{3}$$

Here $\Omega = G \setminus S$, where $G \subset \mathbb{R}^2$ is a bounded domain with boundary ∂G of class $C^{1,\alpha}$, $\alpha > 0$. The crack S is a closed subset (empty or nonempty) of a surface of class $C^{2+\alpha}$, and might reach the boundary. The Dirichlet condition $u = b$ is prescribed on the boundary of G. The jump of the velocity and the jump of the stress tensor in the normal direction are prescribed on the crack S. The same problem in the higher-dimensional case was studied in [11].

As a second application we construct a solution of the problem

$$-\Delta v + \nabla q \;=\; f \quad \text{in } G, \qquad \nabla \cdot v \;=\; c \quad \text{in} \quad G,$$

$$v \;=\; g \quad \text{on} \quad \partial G$$

which was studied in [5].

As in classical potential theory we need the Green formulas to start with. To formulate these identities we define the formally adjoint differential operators $S,\ S'$ by

$$S \;:\; \binom{u}{p} \;\longrightarrow\; S\genfrac{}{}{0pt}{}{u}{p} \;\;=\;\; \begin{pmatrix} -\Delta\, u + \nabla p \\ \nabla \cdot u \end{pmatrix}, \tag{4}$$

$$S' \;:\; \binom{u}{p} \;\longrightarrow\; S'\genfrac{}{}{0pt}{}{u}{p} \;\;=\;\; \begin{pmatrix} -\Delta\, u - \nabla p \\ -\nabla \cdot u \end{pmatrix}. \tag{5}$$

The corresponding formally adjoint stress tensors are denoted by

$$\begin{aligned} T \;\;:\; \binom{u}{p} \;\;&\longrightarrow\;\; T\genfrac{}{}{0pt}{}{u}{p} \;:=\; -2Du + pI, \\ T' \;:\; \binom{u}{p} \;\;&\longrightarrow\;\; T'\genfrac{}{}{0pt}{}{u}{p} \;:=\; -2Du - pI, \end{aligned} \tag{6}$$

where the deformation tensor is given by

$$Du \;:=\; \tfrac{1}{2}(\nabla u + (\nabla u)^T) \tag{7}$$

with $(\nabla u)^T$ as the matrix transposed to $\nabla u \;:=\; (\partial_i u_k)_{k,i=1,2}$, and where I denotes the identity matrix. For $a, b \in \mathbb{R}^2$ and matrices $C, D \in \mathbb{R}^{2\times 2}$ with $C = (C_{ij})$, $D = (D_{ij})$ we use

$$a \cdot b := \sum_{i=1}^{2} a_i b_i, \qquad C : D := \sum_{i,j=1}^{2} C_{ij} D_{ij}.$$

With these notations, for solenoidal vector functions $u \in C^2(\operatorname{cl} G, \mathbb{R}^2)$, $v \in C^1(\operatorname{cl} G, \mathbb{R}^2)$ and scalar functions $p \in C^1(\operatorname{cl} G, \mathbb{R}^1)$, $q \in C^0(\operatorname{cl} G, \mathbb{R}^1)$ in a bounded open set $G \subset \mathbb{R}^2$ with boundary ∂G of class C^1 we have Green's first and second formula

$$\int\limits_G \left(S\genfrac{}{}{0pt}{}{u}{p}\right) \cdot \binom{v}{q}\, dy = \int\limits_{\partial G} (T\genfrac{}{}{0pt}{}{u}{p} N_G) \cdot v \,\mathrm{do} + 2 \int\limits_G Du : Dv \, dy, \tag{8}$$

$$\int\limits_G \left\{ \left(S\genfrac{}{}{0pt}{}{u}{p}\right) \cdot \binom{v}{q} \;-\; \binom{u}{p} \cdot \left(S'\genfrac{}{}{0pt}{}{v}{q}\right) \right\} dy = \int\limits_{\partial G} \left\{ (T\genfrac{}{}{0pt}{}{u}{p} N_G) \cdot v \;-\; u \cdot (T'\genfrac{}{}{0pt}{}{v}{q} N_G) \right\} \mathrm{do}. \tag{9}$$

Here and in the following, $N_G = N_G(y)$ denotes the exterior (with respect to the open set G) unit surface normal vector in $y \in \partial G$ and $\mathrm{cl}\, G$ denotes the closure of G. If u, p is a solution of the Stokes system in G, then (8) holds under the weaker assumption $u \in C^1(\mathrm{cl}\, G, \mathbb{R}^2)$, $p \in C^0(\mathrm{cl}\, G, \mathbb{R}^1)$. If G is an unbounded open set with bounded boundary of class C^1, and u, p is a solution of the Stokes system satisfying

$$|(T^u_p N_G(x)) \cdot v(x)| = o(|x|^{-1})$$

as $|x| \to \infty$, then (8) also holds.

With help of Green's second formula (9) a representation of the solution u, p of the Stokes equations can be obtained, if the corresponding fundamental tensor $E = \big(E_{jk}\big)_{j,k=1,\ldots,3}$ is known. This tensor can be determined as the solution of

$$SE \;=\; \delta I \tag{10}$$

in the sense of distributions, where $SE = (SE_1, \ldots, SE_3)$ means the application of S to the column vectors $E_k := (E_{jk})_{j=1,\ldots,3}$ for each $k = 1, \ldots, 3$, and where δ is Dirac's distribution in $\mathbb{R}^2$. In the present case $E = \big(E_{jk}(x)\big)$ has the following form:

$(j, k = 1, \ldots, 2)$:

$$\begin{aligned}
E_{jk}(x) &= \frac{1}{4\pi}\Big\{\delta_{jk}\, \ln\frac{1}{|x|} + \frac{x_j x_k}{|x|^2}\Big\}, \\
E_{3,k}(x) &= E_{k,3}(x) \;=\; \frac{x_k}{2\pi|x|^2}\;, \\
E_{3,3}(x) &= \delta(x)
\end{aligned} \tag{11}$$

Here $|x| = \sqrt{x_1^2 + x_2^2}$.

With help of the fundamental tensor E now we obtain a representation of a solution u, p of the Stokes equations

$$S^u_p = \tbinom{0}{0} \quad \text{in} \quad G,$$

where here $G \subset \mathbb{R}^2$ is some bounded (not necessarily connected) open set with boundary ∂G of class $C^{1,\alpha}$, $\alpha > 0$. This representation has the form (compare [16])

$$\int_{\partial G} E^{(c)}(x-y)\, T_p^u N_G(y)\, \mathrm{do}_y \quad - \quad \int_{\partial G} D_G(x,y)\, u(y)\, \mathrm{do}_y$$

$$= \begin{cases} -\begin{pmatrix} u \\ p \end{pmatrix}(x) & , \quad x \in G, \\ \\ 0 & , \quad x \notin \mathrm{cl}\, G. \end{cases} \tag{12}$$

Here the 3×2 matrix $E^{(c)}(x-y)$ is obtained from $E(x-y)$ by eliminating the last column, and the 3×2 double layer tensor $D_G(x,y)$ is defined by

$$D_G(x,y) \;:=\; \Big((-T_x E_k(x-y))_{ij}\; (N_G)_j(y) \Big)_{ki}$$

using the column vectors $E_k := (E_{jk})_{j=1,\ldots,3}$ for $k = 1,\ldots,3$. The tensor $D_G = \big(D_{ki}(x,y)\big)_{k=1,\ldots,3;\; i=1,\ldots,2}$ has the following form (for abbreviation we set $z := x-y,\; N_G := N_G(y)$):

$(k, i = 1,\ldots,2)$:

$$\begin{aligned} D_{ki}(x,y) &= -\frac{1}{\pi}\frac{z_k z_i z \cdot N_G}{|z|^4}, \\ D_{3,i}(x,y) &= -\frac{1}{\pi}\left\{ \frac{2 z_i (z \cdot N_G)}{|z|^4} - \frac{(N_G)_i}{|z|^2} \right\}. \end{aligned} \tag{13}$$

If G is an unbounded open set with bounded boundary ∂G of class $C^{1,\alpha}$, $0 < \alpha < 1$, we again define the tensor D_G by the prescription (13).

2 The Surface Potentials

Starting from now, throughout the paper G denotes an open set (bounded or unbounded) with compact boundary ∂G of class $C^{1,\alpha}$, $0 < \alpha < 1$, and $G^* := \mathbb{R}^2 \setminus \mathrm{cl}\, G$ denotes its complement with $\partial G^* = \partial G$. With help of the tensors E and D_G calculated above now the surface potentials with vector-valued source densities $\Psi \in C^0(\partial G, \mathbb{R}^2)$ are constructed. We need the single layer potential

$$(E_G \Psi)(x) \;=\; \int_{\partial G} E^{(c)}(x-y)\, \Psi(y)\, \mathrm{do}_y, \qquad x \notin \partial G, \tag{14}$$

and the double layer potential

$$(D_G \Psi)(x) \;=\; \int_{\partial G} D_G(x,y)\, \Psi(y)\, \mathrm{do}_y, \qquad x \notin \partial G. \tag{15}$$

The 2–componential velocity parts of theses potentials are supported with a dot, to obtain

$$(E^\bullet_G \Psi)(x) \;=\; \int\limits_{\partial G} E^{(r,c)}(x-y)\,\Psi(y)\,\mathrm{do}_y, \qquad x \notin \partial G, \tag{16}$$

$$(D^\bullet_G \Psi)(x) \;=\; \int\limits_{\partial G} D^{(r)}_G(x,y)\,\Psi(y)\,\mathrm{do}_y, \qquad x \notin \partial G. \tag{17}$$

Here the 2×2 matrix $E^{(r,c)}(x-y)$ is obtained from $E(x-y)$ by eliminating the last row and the last column. Moreover, we need the normal stresses of the single layer potential $E_G\Psi$, defined in a neighborhood $U \subset \mathbb{R}^2$ of the surface ∂G by

$$\begin{aligned}(H^\bullet_G \Psi)(x) \;&=\; \int\limits_{\partial G} T_x\Big(E^{(r)}(x-y)\,\Psi(y)\Big)\,N_G(\tilde{x})\,\mathrm{do}_y \\ &=: \int\limits_{\partial G} H_G(x,y)\,\Psi(y)\,\mathrm{do}_y, \qquad x \notin \partial G.\end{aligned} \tag{18}$$

Here $\tilde{x} \in \partial G$ is the projection of $x \in U$ onto ∂G, and for the 2×2 kernel matrix $H_G(x,y)$ in $x, y \in \partial G$ the following identity holds:

$$H_G(x,y) \;=\; \Big(D^{(r)}_G(y,x)\Big)^T \;=\; \Big(D^{(r)}_G(y,x)\Big).$$

Here $D^{(r)}_G$ is the $n\times n$ kernel matrix resulting from D_G by canceling the last row.

We need further statements regarding the continuity behavior of some surface potentials with special densities. In particular, we consider the velocity part of the double layer potential with constant density and the single layer potential having the unit normal field as density.

Lemma 2.1. *Let G be a bounded open set.*

1. *For the double layer potential $D^\bullet_G b$ (see (17)) with some constant density $b \in \mathbb{R}^2$ we have*

$$(D^\bullet_G b)(x) = \begin{cases} b, & x \in G, \\ \frac{1}{2} b, & x \in \partial G, \\ 0, & x \in G^*. \end{cases} \tag{19}$$

2. *For the single layer potential $E_G N_G$ (see (14)) with the exterior (with respect to G) unit normal field N_G as density we have*

$$(E_G N_G)(x) \;=\; \int\limits_{\partial G} E^{(c)}(x-y)\,N_G(y)\,\mathrm{do}_y = \begin{cases} -\binom{0}{1}, & x \in G, \\ -\frac{1}{2}\binom{0}{1}, & x \in \partial G, \\ \binom{0}{0}, & x \in G^*. \end{cases} \tag{20}$$

Consequently, $E^{\bullet}_G N_G = 0$ in $\mathbb{R}^2$.

Lemma 2.1 does not hold if G is unbounded. For the calculation of these potentials in this case we can use the fact that $E_G\Psi = E_{G^*}\Psi$, $D_{G^*}\Psi = -D_G\Psi$.

The continuity and jump relations of the Stokes surface potentials on the boundary ∂G are described in the next proposition. Here we need to define the following limiting values:

$$\begin{aligned} w^+(z) &= \lim_{G \ni x \to z \in \partial G} w(z), \\ w^-(z) &= \lim_{G^* \ni x \to z \in \partial G} w(z). \end{aligned}$$

Proposition 2.2. *Let $\Psi \in C^0(\partial G, \mathbb{R}^2)$ and let $E^{\bullet}_G\Psi,\ D^{\bullet}_G\Psi,\ H^{\bullet}_G\Psi$ denote the surface potentials defined in (16), (17), (18), respectively. Then on the boundary ∂G the following continuity and jump relations are satisfied:*

$$\begin{aligned} (E^{\bullet}_G\Psi)^+ &= E^{\bullet}_G\Psi &&= (E^{\bullet}_G\Psi)^-, \\ (D^{\bullet}_G\Psi)^+ - D^{\bullet}_G\Psi &= +\tfrac{1}{2}\Psi &&= D^{\bullet}_G\Psi - (D^{\bullet}_G\Psi)^-, \\ (H^{\bullet}_G\Psi)^+ - H^{\bullet}_G\Psi &= -\tfrac{1}{2}\Psi &&= H^{\bullet}_G\Psi - (H^{\bullet}_G\Psi)^-, \end{aligned} \tag{21}$$

hence

$$(D^{\bullet}_G\Psi)^+ - (D^{\bullet}_G\Psi)^- \;=\; \Psi \;=\; (H^{\bullet}_G\Psi)^- - (H^{\bullet}_G\Psi)^+. \tag{22}$$

Finally, we need some statements concerning the decay properties of the surface potentials at infinity.

Lemma 2.3. *Let*

$$\int\limits_{\partial G} \Psi(y)\,\mathrm{do}_y = 0. \tag{23}$$

For the single layer potential $E^{\bullet}_G\Psi$ and the double layer potential $D^{\bullet}_G\Psi$ (see (16) and (17)) we have the following decay behavior as $|x| \to \infty$:

$$(E^{\bullet}_G\Psi)(x), (D^{\bullet}_G\Psi)(x) \;=\; \mathcal{O}(|x|^{-1}), \tag{24}$$

$$[E_G\Psi]_3(x),\ |(\nabla E^{\bullet}_G\Psi)(x)|,\ |(\nabla D^{\bullet}_G\Psi)(x)|,\ [D_G\Psi]_3(x) \;=\; \mathcal{O}(|x|^{-2}). \tag{25}$$

Here $E^\bullet_G\Psi$, $D^\bullet_G\Psi$ denote the velocity parts of the potentials, and $[E_G\Psi]_3(x)$, $[D_G\Psi]_3(x)$ the pressure parts.

We have seen that for $\Psi \in C^0(\partial G, \mathbb{R}^2)$ the velocity part of the single layer potential is continuous in the whole space, i.e. it holds $E^\bullet_G\Psi \in C^0(\mathbb{R}^2, \mathbb{R}^2)$. Since $\partial G \in C^{1,\alpha}$ we even have $E^\bullet_G\Psi \in C^\alpha(\partial G, \mathbb{R}^2)$ and $H^\bullet_G\Psi \in C^\alpha(\partial G, \mathbb{R}^2)$. If, in addition, $\Psi \in C^\gamma(\partial G, \mathbb{R}^2)$ with $\gamma > 0$, then $E_G\Psi$ can be extended to functions in $C^0(\mathrm{cl}\, G, \mathbb{R}^3)$ and in $C^0(\mathrm{cl}\, G^*, \mathbb{R}^3)$; $\nabla E^\bullet_G\Psi$ can be extended to functions in $C^0(\mathrm{cl}\, G, \mathbb{R}^4)$ and in $C^0(\mathrm{cl}\, G^*, \mathbb{R}^4)$. (See [13] and [9], p. 80.)

3 The Method of Integral Equations

First we study the problem for a non cracked domain. Let $G \subset \mathbb{R}^2$ be a domain with a nonempty bounded boundary $\partial G \in C^{1,\alpha}$, $\alpha > 0$.

We call u, p a solution for the Dirichlet problem of the Stokes system with boundary value $b \in C^0(\partial G, \mathbb{R}^2)$, if $u \in C^2(G, \mathbb{R}^2) \cap C^0(\mathrm{cl}\, G, \mathbb{R}^2)$, $p \in C^1(G, \mathbb{R}^1)$ satisfy

$$-\Delta u + \nabla p = 0 \quad \text{in } G, \qquad \nabla \cdot u = 0 \quad \text{in} \quad G, \tag{26}$$

$$u = b \quad \text{on} \quad \partial G. \tag{27}$$

If G is unbounded we suppose that $u(x) = \mathcal{O}(1)$, $p(x) = \mathcal{O}(|x|^{-1})$, $|\nabla u(x)| = \mathcal{O}(|x|^{-1})$, $|\nabla p(x)| = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$.

Now fix $\eta > 0$. For $\Psi \in C^0(\partial G, \mathbb{R}^2)$ denote

$$\Psi_M = \frac{1}{|\partial G|} \int_{\partial G} \Psi(y)\, \mathrm{do}_y, \qquad |\partial G| = \int_{\partial G} 1\, \mathrm{do}_y, \tag{28}$$

$$M\Psi = \Psi - \Psi_M. \tag{29}$$

We look for a classical solution of the Dirichlet problem in the form $(u,p)^T = D_G M\Psi + \eta E_G M\Psi + (\Psi_M, 0)$ with an unknown density $\Psi \in C^0(\partial G, \mathbb{R}^2)$. Using the continuity properties of potentials we obtain the integral equation $L_{0,\eta}\Psi = b$,

$$L_{0,\eta}\Psi := \frac{1}{2} M\Psi + D^\bullet_G M\Psi + \eta E^\bullet_G M\Psi + \Psi_M = \frac{1}{2}\Psi + D^\bullet_G M\Psi + \eta E^\bullet_G M\Psi + \frac{1}{2}\Psi_M.$$

Remark that $L_{0,\eta}\Psi = \frac{1}{2}\Psi + D^\bullet_G\Psi + \eta E^\bullet_G M\Psi$ for G bounded and $L_{0,\eta}\Psi = \frac{1}{2}\Psi + D^\bullet_G\Psi + \eta E^\bullet_G M\Psi + \Psi_M$ for G unbounded.

If G is bounded then a necessary condition for the solvability of the Dirichlet problem (26), (27) is

$$\int b \cdot N_G\, \mathrm{do} = 0,$$

and therefore the operator $L_{0,\eta}$ is not invertible. To overcome this difficulty, instead of the original integral equation $\frac{1}{2}\Psi + D_G^\bullet M\Psi + \eta E_G^\bullet M\Psi + \frac{1}{2}\Psi_M = b$, we shall study the modified integral equation

$$\frac{1}{2}\Psi + D_G^\bullet M\Psi + \eta E_G^\bullet M\Psi + \frac{1}{2}\Psi_M + aN\Psi = b,$$

where a is a fixed positive constant and

$$N\Psi := \left(\frac{1}{|\partial G|}\int_{\partial G} N_G \cdot \Psi \,\mathrm{do}\right) N_G$$

(compare [17]).

Proposition 3.1. *Let G be connected. Let (u, p) be a solution of the Dirichlet problem for the Stokes system with zero boundary condition. Then $u \equiv 0$, p is constant.*

Proof. If G is bounded then $u \equiv 0$ by [9], Theorem 5.3.

Let now G be unbounded. Since u is bounded we obtain

$$\int_{\{x\in G; |x|<r\}} |u(x)|^2(1+|x|)^{-4}\,dx = o(\ln r),$$

$$u(x) = o(\ln|x|)$$

as $|x| \to \infty$. Hence [9], Theorem 6.5 gives $u \equiv 0$.

Since $\nabla p = 0$ in G by the Stokes equations, the function p is constant in G.

4 Spectrum of the integral operator

Notation 4.1. *Let W be a bounded linear operator in a complex Banach space X. Denote by $\sigma(W)$ the spectrum of W and by $r(W) = \sup\{|\lambda|; \lambda \in \sigma(W)\}$ the spectral radius of W. Denote by I the identity operator in X.*

Lemma 4.2. *Let $\gamma > 0$, $\Psi \in C^\gamma(\partial G, \mathbb{C}^2)$ satisfying (23). Then*

$$\int_{\partial G} (TE_G\Psi)N_G \,\mathrm{do}_x = 0, \tag{30}$$

$$\int_{\partial G} \Psi \cdot E_G^\bullet \overline{\Psi}\,\mathrm{do}_x = \int_{\partial G} \Psi \cdot ME_G^\bullet \overline{\Psi}\,\mathrm{do}_x = 2\int_{\mathbb{R}^2\setminus\partial G} |DE_G^\bullet\Psi|^2 dx \geq 0.$$

If

$$\int_{\partial G} \Psi \cdot E_G^\bullet \overline{\Psi}\,\mathrm{do}_x = 0$$

then $E_G\varphi$ is constant on each component of $\mathbb{R}^2 \setminus \partial G$ and $E^\bullet_G\Psi = 0$ in $\mathbb{R}^2$.

Proof. We get (30) from (8) for $(u,p) = E_G\Psi$, $q = 0$ and a constant v using Lemma 2.3.

Let $\Psi = \varphi + i\phi$ where $\varphi, \phi \in C^\gamma(\partial G, \mathbb{R}^2)$. We get from the symmetry of $E^{(r,c)}$ and from Fubini's theorem

$$\int_{\partial G} \Psi \cdot (E^\bullet_G \overline{\Psi})\, \mathrm{do}_x = \int_{\partial G} [\varphi \cdot E^\bullet_G\varphi + \phi \cdot E^\bullet_G\phi]\, \mathrm{do}_x$$

$$= -\int_{\partial G} \left[\left(-\frac{1}{2}\varphi + H^\bullet_G\varphi \right) \cdot E^\bullet_G\varphi + \left(-\frac{1}{2}\varphi + H^\bullet_{G^*}\varphi \right) \cdot E^\bullet_G\varphi \right.$$

$$\left. + \left(-\frac{1}{2}\phi + H^\bullet_G\phi \right) \cdot E^\bullet_G\phi + \left(-\frac{1}{2}\phi + H^\bullet_{G^*}\phi \right) \cdot E^\bullet_G\phi \right] \mathrm{do}_x.$$

Using (8) for G and for G^* we obtain

$$\int_{\partial G} \Psi \cdot (E^\bullet_G \overline{\Psi})\, \mathrm{do}_x = 2 \int_{\mathbb{R}^2 \setminus \partial G} [|DE^\bullet_G\varphi|^2 + |DE^\bullet_G\phi|^2] dx = 2 \int_{\mathbb{R}^2 \setminus \partial G} |DE^\bullet_G\Psi|^2 dx.$$

Suppose now that

$$\int_{\partial G} \Psi \cdot (E^\bullet_G \overline{\Psi})\, \mathrm{do}_x = 0.$$

Then $DE^\bullet_G\Psi = 0$ in $\mathbb{R}^2 \setminus \partial G$. This gives that $E^\bullet_G\Psi$ is linear in each component of $\mathbb{R}^2 \setminus \partial G$. Since $(E^\bullet_G\Psi)(x) \to 0$ as $|x| \to \infty$ we get that $E^\bullet_G\Psi = 0$ in the unbounded component of $\mathbb{R}^2 \setminus \partial G$. Since $E^\bullet_G\Psi$ is continuous in $\mathbb{R}^2$, linear in each component of $\mathbb{R}^2 \setminus \partial G$ and $E^\bullet_G\Psi = 0$ in the unbounded component of $\mathbb{R}^2 \setminus \partial G$ we deduce that $E^\bullet_G\Psi = 0$ in $\mathbb{R}^2$. Since $SE_G\Psi = 0$ in $\mathbb{R}^2 \setminus \partial G$ we obtain that $E_G\Psi$ is constant in each component of $R^2 \setminus \partial G$.

Lemma 4.3. *The operator $E^\bullet_G : \Psi \mapsto E^\bullet_G\Psi$ is a compact linear operator in $C^0(\partial G, \mathbb{C}^2)$. Denote by $\mathrm{diam}(\partial G)$ the diameter of ∂G,*

$$V_{j,k}(z) = E_{jk}(z) + \delta_{jk} \frac{\ln(\mathrm{diam}(\partial G))}{4\pi} \tag{31}$$

$$A_0 = \max_{j=1,2} \sup_{x \in \partial G} \int_{\partial G} \sum_{k=1}^{2} |V_{jk}(x-y)|\, \mathrm{do}_y. \tag{32}$$

Then $A_0 < \infty$. Let $\gamma > 0$, $\Psi \in C^\gamma(\partial G, \mathbb{C}^2)$ satisfying (23). Then

$$\int_{\partial G} |ME^\bullet_G\Psi|^2\, \mathrm{do}_x \le A_0 \int_{\partial G} \Psi \cdot (ME^\bullet_G \overline{\Psi})\, \mathrm{do}_x. \tag{33}$$

If B is a such positive constant that

$$\int_{\{y\in\partial G;|x-y|<r\}} 1 \, \mathrm{do}_y \le Br \tag{34}$$

for each $x \in \partial G$ and $0 < r < \operatorname{diam} \partial G$ then

$$A_0 \le \frac{1}{4\pi}\left[B \operatorname{diam} \partial G + \int_{\partial G} 2 \, \mathrm{do}_x\right].$$

Proof. $E_G^\bullet$ is a compact linear operator in $C^0(\partial G, \mathbb{C}^2)$ because it is an integral operator with weakly singular kernel.

Denote by $\mathcal{F}$ the space of all $\Psi \in C^\gamma(\partial G, \mathbb{C}^2)$ satisfying (23) and by $\mathcal{K}$ the set of all $\varphi \in \mathcal{F}$ such that

$$\int_{\partial G} |\varphi|^2 \, \mathrm{do}_x \le 1. \tag{35}$$

Denote for $\varphi, \psi \in \mathcal{F}$

$$\langle \varphi, \psi\rangle = \int_{\partial G} \varphi \cdot (ME_G^\bullet \overline{\psi}) \, \mathrm{do}_x.$$

Then $\langle \cdot, \cdot\rangle$ is a scalar product on $\mathcal{F}$ by Lemma 4.2. If $\Psi \in \mathcal{F}$ then we get using the Schwartz inequality

$$\int_{\partial G} |ME_G^\bullet \Psi|^2 \, \mathrm{do}_x = \sup_{\varphi\in\mathcal{K}} |\langle \varphi, \Psi\rangle|^2 \le \langle \Psi, \Psi\rangle \sup_{\varphi\in\mathcal{K}} \langle \varphi, \varphi\rangle.$$

For the proof of (33) it is enough to prove

$$\sup_{\varphi\in\mathcal{K}} \langle \varphi, \varphi\rangle \le A_0. \tag{36}$$

Let now $\varphi \in \mathcal{K}$ be fixed. Since φ satisfy (23) we have

$$\langle \varphi, \varphi\rangle = \int_{\partial G} \sum_{j=1}^{2} \varphi_j(y) \int_{\partial G} \sum_{k=1}^{2} V_{jk}(x-y)\overline{\varphi}_k(y) \, \mathrm{do}_y \, \mathrm{do}_x.$$

(35), Hölder's inequality and Fubini's theorem give

$$\langle \varphi, \varphi\rangle \le \left[\int_{\partial G} \sum_{j=1}^{2} \left(\int_{\partial G} \sum_{k=1}^{2} |V_{jk}(x-y)||\varphi_k(y)| \, \mathrm{do}_y\right)^2 \mathrm{do}_x\right]^{1/2}$$

$$\leq \Bigg[\int\limits_{\partial G}\sum_{j=1}^{2}\Bigg(\int\limits_{\partial G}\sum_{k=1}^{2}|V_{jk}(x-y)|\,\mathrm{do}_y\Bigg)\Bigg(\int\limits_{\partial G}\sum_{k=1}^{2}|V_{jk}(x-y)||\varphi_k(y)|^2\,\mathrm{do}_y\Bigg)\,\mathrm{do}_x\Bigg]^{1/2}$$

$$\leq \sqrt{A_0}\Bigg[\int\limits_{\partial G}\int\limits_{\partial G}\sum_{j=1}^{2}\sum_{k=1}^{2}|V_{jk}(x-y)||\varphi_k(y)|^2\,\mathrm{do}_y\,\mathrm{do}_x\Bigg]^{1/2}$$

$$= \sqrt{A_0}\Bigg[\int\limits_{\partial G}\sum_{k=1}^{2}|\varphi_k(y)|^2\int\limits_{\partial G}\sum_{j=1}^{2}|V_{jk}(x-y)|\,\mathrm{do}_x\,\mathrm{do}_y\Bigg]^{1/2}\leq A_0.$$

This gives the estimate (36).

Suppose now that the inequality (34) is true. Using [18], Lemma 1.5.1 we get

$$A_0 \leq \frac{1}{4\pi}\sup_{x\in\partial G}\int\limits_{\partial G}[\ln\frac{\operatorname{diam}\partial G}{|x-y|}+2]\,\mathrm{do}_y$$

$$\leq \frac{1}{4\pi}\sup_{x\in\partial G}\int_0^\infty \int\limits_{\{y\in\partial G;\ln[(\operatorname{diam}\partial G)/|x-y|]>t\}} 1\,\mathrm{do}_y dt+\frac{1}{2\pi}\int\limits_{\partial G}1\,\mathrm{do}_x$$

$$\leq \frac{1}{4\pi}\int_0^\infty B(\operatorname{diam}\partial G)e^{-t}\,dt+\frac{1}{2\pi}\int\limits_{\partial G}1\,\mathrm{do}_x=\frac{1}{4\pi}\Bigg[B\operatorname{diam}\partial G+\int\limits_{\partial G}2\,\mathrm{do}_x\Bigg].$$

Since ∂G is of class $C^{1,\alpha}$ there is a constant B such that the inequality (34) holds for each $x\in\partial G$ and $0<r<\operatorname{diam}\partial G$. This gives $A_0<\infty$.

Theorem 4.4. *Suppose that G is connected. Fix $a\geq 0$, $\eta\geq 0$. For $\Psi\in C^0(\partial G,\mathbb{C}^2)$ define*

$$L_{a,\eta}\Psi=\frac{1}{2}\Psi+D_G^\bullet M\Psi+\eta E_G^\bullet M\Psi+\frac{1}{2}\Psi_M+aN\Psi, \tag{37}$$

$$L^*_{a,\eta}\Psi=\frac{1}{2}\Psi+MH_G^\bullet\Psi+\eta ME_G^\bullet\Psi+\frac{1}{2}\Psi_M+aN\Psi, \tag{38}$$

where

$$N\Psi=\Bigg(\frac{1}{|\partial G|}\int\limits_{\partial G}N_G\cdot\Psi\,\mathrm{do}\Bigg)N_G.$$

Then $L^*_{a,\eta}$ is the adjoint operator of $L_{a,\eta}$. Let λ be an eigenvalue of the operator $L^*_{a,\eta}$ with an eigenfunction Ψ. Then $0\leq\lambda\leq 1+\eta A_0+a$ where A_0 is given by (32). If (23) does not hold then $\lambda=1$.

Proof. Fubini's theorem yields that $L^*_{a,\eta}$ is the adjoint operator of $L_{a,\eta}$.

Suppose first that (23) does not hold. Then

$$\lambda\int\limits_{\partial G}\Psi\,\mathrm{do}_x=\int\limits_{\partial G}L^*_{a,\eta}\Psi\,\mathrm{do}_x=\int\limits_{\partial G}\Psi L_{a,\eta}1\,\mathrm{do}_x=\int\limits_{\partial G}\Psi\,\mathrm{do}_x$$

and $\lambda = 1$.

Let now (23) holds. Then $L^*_{a,\eta}\Psi = \frac{1}{2}\Psi + H^\bullet_G\Psi + \eta M E^\bullet_G\Psi + aN\Psi$ by Lemma 4.2 (see (30)). We can suppose that $\lambda \neq \frac{1}{2}$. Since ∂G is of class $C^{1,\alpha}$ we have $N_G \in C^\alpha(\partial G, \mathbb{C}^2)$, $E^\bullet_G\Psi \in C^\alpha(\partial G, \mathbb{C}^2)$ and $H^\bullet_G\Psi \in C^\alpha(\partial G, \mathbb{C}^2)$. Since $\lambda \neq \frac{1}{2}$ we obtain that $\Psi \in C^\alpha(\partial G, \mathbb{C}^2)$.

Suppose first that

$$\int_{\partial G} \Psi \cdot (E^\bullet_G\overline{\Psi})\, \mathrm{do}_x \neq 0.$$

Since $\nabla \cdot E^\bullet_G\Psi = 0$ we get using the divergence theorem, (8) and Lemma 2.3

$$\int_{\partial G} \lambda\Psi \cdot E^\bullet_G\overline{\Psi}\, \mathrm{do}_x = \int_{\partial G} \left(\frac{1}{2}\Psi - H^\bullet_{G^*}\Psi + \eta M E^\bullet_G\Psi + aN_n\Psi\right) \cdot (E^\bullet_G\overline{\Psi})\, \mathrm{do}_x$$

$$= 2\int_{G^*} |DE^\bullet_G\Psi|^2 dx + \eta \int_{\partial G} |ME^\bullet_G\Psi|^2\, \mathrm{do}_x.$$

Using Lemma 4.2 we get

$$\lambda = \frac{\int\limits_{G^*} |DE^\bullet_G\Psi|^2 dx}{\int\limits_{\mathbb{R}^2\setminus\partial G} |DE^\bullet_G\Psi|^2 dx} + \eta \frac{\int\limits_{\partial G} |ME^\bullet_G\Psi|^2\, \mathrm{do}_x}{\int\limits_{\partial G} \Psi \cdot (ME^\bullet_G\overline{\Psi})\, \mathrm{do}_x} \geq 0.$$

Lemma 4.3 gives that $\lambda \leq 1 + \eta A_0$.

Suppose now that

$$\int_{\partial G} \Psi \cdot (E^\bullet_G\overline{\Psi})\, \mathrm{do} = 0.$$

Then $E^\bullet_G\Psi = 0$ in R^2 and $E_G\Psi$ is constant in each component of $R^2 \setminus \partial G$ by Lemma 4.2. Since $(E_G\Psi)(y) \to 0$ as $|y| \to \infty$ we deduce that $(E_G\Psi)(y) = 0$ on the unbounded component of $R^2 \setminus \partial G$. If V is a component of G^* then

$$\Psi = -\left[\left(-\frac{1}{2}\Psi + H_G\Psi\right) + \left(-\frac{1}{2}\Psi + H_{G^*}\Psi\right)\right] = c_V N_G$$

on ∂V where c_V is a complex constant. Put

$$b = \sum_{V \text{ component of } G^*} ac_V \frac{|\partial V|}{|\partial G|}. \tag{39}$$

If V is the unbounded component of G^* then

$$\lambda c_V N_G = L^*_{a,\eta}\Psi = bN_G \tag{40}$$

on ∂V. If V is a bounded component of G^* then

$$\lambda c_V N_G = L^*_{a,\eta}\Psi = (c_V + b)N_G \tag{41}$$

on ∂V. (See Lemma 2.1.) Suppose first that $b = 0$. Since Ψ is an eigenfunction there is a component V of G^* such that $c_V \neq 0$. If V is unbounded then $\lambda = 0$ by (40). If V is bounded then $\lambda = 1$ by (41). Suppose now that $b \neq 0$. If $c_V = 0$ for V unbounded then $bN_G = 0$ by (40), what is a contradiction. If $c_V = 0$ for V bounded then again $bN_G = 0$ by (41), what is a contradiction. Therefore $c_V \neq 0$ for each component V of G^*. Thus

$$\lambda = b/c_V \tag{42}$$

for the unbounded component of G^* and

$$\lambda = 1 + (b/c_V) \tag{43}$$

for a bounded component of G^*. If G^* is connected than the relations (43), (42) and (39) yield that $\lambda = a$ or $\lambda = 1 + a$. Suppose now that G^* is not connected. The relation (43) gives that there are complex nonzero numbers c_b, c_u such that $c_V = c_b$ for a bounded component V of G^* and $c_V = c_u$ for the unbounded component V of G^*. If G^* is bounded we get from (39) and (43) that $\lambda = 1+a$. Suppose now that G^* is unbounded. Denote $c = c_b/c_u$. Denote $t = |\partial V|/|\partial G|$ where V is the unbounded component of G^*. Then $0 < t < 1$. We rewrite (43) and (42) using (39) as

$$\lambda = a[t + (1-t)c], \tag{44}$$

$$\lambda = 1 + a[tc^{-1} + (1-t)]. \tag{45}$$

Since the imaginary parts of the expressions (44), (45) have opposite signs we deduce that λ is real and therefore c is real. If $c \le 1$ we get using (44) that $\lambda \le a$. If $c > 1$ we get using (45) that $\lambda \le 1 + a$. Suppose now that $\lambda < 0$. Then $t + (1-t)c < 0$ by (44). But then $c < 0$ and $[t + (1-t)c]c^{-1} > 0$. The relation (45) gives $\lambda = 1 + a[t + (1-t)c]c^{-1} \ge 1$ what is a contradiction.

Theorem 4.5. *Suppose that G is connected. Fix $a \ge 0$, $\eta \ge 0$. Suppose that $L_{a,\eta}$ given by (37) is invertible in $C^0(\partial G, \mathbb{C}^2)$. Fix $\gamma > (1 + \eta A_0 + a)/2$. Then there are constants $d \in \langle 1, \infty)$, $t \in (0,1)$ such that for each nonnegative natural number k*

$$\|(I - \gamma^{-1}L_{a,\eta})^k\|_{C^0(\partial G,\mathbb{C}^2)} \le dt^k, \tag{46}$$

and

$$L^{-1}_{a,\eta} = \gamma^{-1}\sum_{k=0}^{\infty}(I - \gamma^{-1}L_{a,\eta})^k. \tag{47}$$

Proof. The operator $L_{a,\eta} - \frac{1}{2}I$ is a compact operator in $C^0(\partial G, \mathbb{C}^2)$. Since the operator $L^*_{a,\eta}$ given by (38) is the adjoint operator of $L_{a,\eta}$ in $C^0(\partial G, \mathbb{C}^2)$

we have $\sigma(L_{a,\eta}) = \sigma(L^*_{a,\eta}) \subset (0, 1 + \eta A_0 + a\rangle \subset (0, 2\gamma)$ in $C^0(\partial G, \mathbb{C}^2)$ by Theorem 4.4 and [14], Chapter 1, § 3.7. The spectral mapping theorem gives $\sigma(I - \gamma^{-1} L_{a,\eta}) \subset (-1, 1)$. Since $r(I - \gamma^{-1} L_{a,\eta}) < 1$ there are constants $d \in \langle 1, \infty)$, $t \in (0, 1)$ such that (46) holds. Since $L_{a,\eta} = \gamma[(I - (I - \gamma^{-1} L_{a,\eta})]$ easy calculation yields (47).

5 Solutions of the Dirichlet problem

Lemma 5.1. *Fix $\eta \geq 0$, $a \geq 0$. If G is not a bounded domain with connected boundary suppose that $\eta > 0$. Let $L^*_{a,\eta}$ be given by (38). If $\Psi \in C^0(\partial G, \mathbb{R}^2)$, $L^*_{a,\eta} \Psi = 0$ then (23) holds, $E^\bullet_G \Psi = 0$ in $\mathbb{R}^2$ and $E_G \Psi$ is constant in each component of $\mathbb{R}^2 \setminus \partial G$.*

Proof. Theorem 4.4 gives that (23) holds. Hence $\Psi = -2(H^\bullet_G \Psi + \eta M E^\bullet_G \Psi + a N_n \Psi) \in C^\alpha(\partial G, \mathbb{R}^2)$ by Lemma 4.2 (see (30)). Using (8) and the divergence theorem we get

$$0 = \int_{\partial G} \left(\frac{1}{2} \Psi - H^\bullet_{G^*} \Psi + \eta M E^\bullet_G \Psi + a N_n \Psi \right) \cdot E^\bullet_G \Psi \, \mathrm{do}_x$$

$$= 2 \int_{G^*} |D E^\bullet_G \Psi|^2 dx + \eta \int_{\partial G} |M E^\bullet_G \Psi|^2 \, \mathrm{do}_x.$$

This gives $D E^\bullet_G \Psi = 0$ in G^* and $\eta M E^\bullet_G \Psi = 0$ on ∂G.

If $\eta > 0$ then $M E^\bullet_G \Psi = 0$ on $\partial \Omega$ which forces that $E^\bullet_G \Psi$ is constant on ∂G. Suppose now that $\eta = 0$. Then G is a bounded domain with connected boundary. Since $D E^\bullet_G \Psi = 0$ in G^* the mapping $E^\bullet_G \Psi(x)$ is affine in G^*. Since $E^\bullet_G \Psi(x) \to 0$ as $|x| \to \infty$ we deduce that $E^\bullet_G \Psi = 0$ in G^*. The continuity of $E^\bullet_G \Psi$ gives that $E^\bullet_G \Psi = 0$ on ∂G.

Since $E^\bullet_G \Psi$ is constant on ∂G and Ψ satisfies (23), we have

$$\int_{\partial G} \Psi(E^\bullet_G \Psi) \, \mathrm{do}_x = 0.$$

Lemma 4.2 gives that $E_G \Psi$ is constant on each component of $\mathbb{R}^2 \setminus \partial G$ and $E^\bullet_G \Psi = 0$ in $\mathbb{R}^2$.

Theorem 5.2. *Let G be unbounded and connected, $b \in C^0(\partial G, \mathbb{R}^2)$. Then there is a solution (u, p) of the Dirichlet problem with the boundary condition b. The vector function u is unique, the function p is unique up to an additive constant. Fix $\eta > 0$. Then $L_{0,\eta}$ given by (37) is invertible in $C^0(\partial G, \mathbb{R}^2)$ and*

$$(u, p) = D_G M \Phi + \eta E_G M \Phi + (\Phi_M, c) \tag{48}$$

where $\Phi = L_{0,\eta}^{-1} b$ and c is an arbitrary constant. If we fix $\gamma > (1+\eta A_0)/2$ then $L_{0,\eta}^{-1}$ is given by (47).

Proof. $L_{0,\eta}^*$ given by (38) is the adjoint operator of $L_{0,\eta}$ in $C^0(\partial G, \mathbb{R}^2)$. Let $\Psi \in C^0(\partial G, \mathbb{R}^2)$ be such that $L_{0,\eta}^* \Psi = 0$. Since (23) holds by Lemma 5.1, we have $L_{0,\eta}^* \Psi = \frac{1}{2}\Psi + H_G^\bullet \Psi + \eta M E_G^\bullet \Psi$ by Lemma 4.2 (see (30)). Lemma 5.1 gives that $E_G \Psi$ is constant on each component of $\mathbb{R}^2 \setminus \partial G$ and $E_G^\bullet \Psi = 0$ in $\mathbb{R}^2$. Since G is an unbounded domain and $E_G \Psi(x) \to 0$ as $|x| \to \infty$ we have $E_G \Psi = 0$ in G. Using Proposition 2.2 we obtain $-\frac{1}{2}\Psi + H_G^\bullet \Psi = 0$. Thus $0 = L_{0,\eta}^* \Psi = \Psi + (-\frac{1}{2}\Psi + H_G^\bullet \Psi) = \Psi$.

We see that the operator $L_{0,\eta}^*$ is injective. Since $E_G^\bullet M$, $H_G^\bullet M$ are compact operators, the Fredholm theory gives that $L_{0,\eta}^*$ is invertible. Since the operator $L_{0,\eta}^*$ is invertible and $L_{0,\eta}$ is the adjoint operator of $L_{0,\eta}^*$, the operator $L_{0,\eta}$ is invertible (see [14], § 3.7). If we put $\Phi = L_{0,\eta}^{-1} b$ then (u,p) given by (48) is a solution of the problem. Theorem 4.5 gives that $L_{0,\eta}^{-1}$ is given by (47). The rest is a consequence of Proposition 3.1.

Theorem 5.3. *Let G be bounded and connected, $b \in C^0(\partial G, \mathbb{R}^2)$. Then there is a solution (u,p) of the Dirichlet problem with the boundary condition b if and only if*

$$\int_{\partial G} b \cdot N_G \, \mathrm{do} = 0. \tag{49}$$

The vector function u is unique, the function p is unique up to an additive constant. Fix $\eta > 0$, $a > 0$. If $L_{a,\eta}$ is given by (37) then $L_{a,\eta}$ is invertible and (u,p) is given by (48), where $\Phi = L_{a,\eta}^{-1} b$ and c is an arbitrary constant. If we fix $\gamma > (1+\eta A_0 + a)/2$ then $L_{a,\eta}^{-1}$ is given by (47).

Proof. If there is a solution of the Dirichlet problem with the boundary condition b then we get (49) from the divergence theorem.

Suppose now that (49) holds. We will prove the existence of a solution of the Dirichlet problem with the boundary condition b. The operator $L_{a,\eta}^*$ given by (38) is the adjoint operator of $L_{a,\eta}$ in $C^0(\partial G, \mathbb{R}^2)$. Let $\Psi \in C^0(\partial G, \mathbb{R}^2)$ be such that $L_{a,\eta}^* \Psi = 0$. Since (23) holds by Lemma 5.1, we have $L_{a,\eta}^* \Psi = \frac{1}{2}\Psi + H_G^\bullet \Psi + \eta M E_G^\bullet \Psi + aN\Psi$ by Lemma 4.2 (see (30)). Lemma 5.1 gives that $E_G \Psi$ is constant on each component of $\mathbb{R}^2 \setminus \partial G$ and $E_G^\bullet \Psi = 0$ in $\mathbb{R}^2$. The jump relation (22) gives that for each component V of G^* there is a constant c_V such that $\Psi = c_V N_G$ on ∂V. Denote by $\tilde{V}$ the unbounded component of G^*. According to Lemma 2.1 we have $0 = L_{a,\eta}^* \Psi = a N_n \Psi$ on $\partial \tilde{V}$. Since $a > 0$ we get $N_n \Psi = 0$. Let now V be a bounded component of G^*. Since $N_n \Psi = 0$ Lemma 2.1 gives $0 = L_{a,\eta}^* \Psi = (c_V - c_{\tilde{V}}) N^G$ on ∂V. Therefore there is a constant c such that $\Psi = c N_G$. Since $N_n \Psi = 0$ we obtain $c = 0$ and thus $\Psi \equiv 0$.

We see that the operator $L_{a,\eta}^*$ is injective. Since $M E_G^\bullet$, $M H_G^\bullet$ and N are compact operators, the Fredholm theory gives that $L_{a,\eta}^*$ is invertible. Since the operator $L_{a,\eta}^*$ is invertible and the operator of $L_{a,\eta}$ is the adjoint operator of

$L^*_{a,\eta}$, we deduce that the operator $L_{a,\eta}$ is invertible (see [14], §3.7). If we put $\Phi = L^{-1}_{a,\eta} b$ then (u,p) given by (48) is a solution of the problem. Theorem 4.5 gives that $L^{-1}_{a,\eta}$ is given by (47). The rest is a consequence of Proposition 3.1.

If we want to calculate $L^{-1}_{a,\eta}$ using (47) we need an estimation of A_0. The estimation of A_0 might be unpleasant. We need not it in a special case of a bounded domain with connected boundary.

Theorem 5.4. *Let G be bounded and ∂G be connected. Fix $a > 0$. Put $L_{a,0}\Psi = \frac{1}{2}\Psi + D^\bullet_G \Psi + aN\Psi$. Then $L_{a,0}$ is invertible in $C^0(\partial G, \mathbb{R}^2)$. If $b \in C^0(\partial G, \mathbb{R}^2)$ fulfills (49) put $\Phi = L^{-1}_{a,0} b$. If c is constant then $(u,p) = D_G \Phi + (0,\dots,0,c)$ is a solution of the Dirichlet problem with the boundary condition b. If we fix $\gamma > (1+a)/2$ then $L^{-1}_{a,0}$ is given by (47).*

Proof. Let $\Psi \in C^0(\partial G, \mathbb{R}^2)$ be such that $L^*_{a,0}\Psi = 0$. Lemma 5.1 gives that $E_G \Psi$ is constant on each component of $\mathbb{R}^2 \setminus \partial G$ and $E^\bullet_G \Psi = 0$ in $\mathbb{R}^2$. Since G^* is an unbounded domain and $E_G\Psi(x) \to 0$ as $|x| \to \infty$ we have $E_G \Psi = 0$ in G^*. The jump relation (22) gives that $\Psi = cN_G$. Thus $0 = L^*_{a,0}\Psi = acN^G$. Since $a > 0$ we get $c = 0$ and thus $\Psi = 0$.

We see that the operator $L^*_{a,0}$ is injective. Since N, $H^\bullet_G$ are compact operators, the Fredholm theory gives that $L^*_{a,0}$ is invertible. Since the operator $L^*_{a,0}$ is invertible and $L_{a,0}$ is the adjoint operator of $L^*_{a,0}$, we infer that the operator $L_{a,0}$ is invertible too (see [14], §3.7). If we put $\Phi = L^{-1}_{a,0} b$ then $(u,p) = D_G \varphi + (0,\dots,0,c)$ is a solution of the problem. Theorem 4.5 gives that $L^{-1}_{a,0}$ is given by (47).

Remark 5.5. Let $b \in C^0(\partial G, \mathbb{R}^2)$ be such that the Dirichlet problem for the Stokes system with the boundary condition b is solvable. Using Theorem 5.2, Theorem 5.3 or Theorem 5.4 we can reduce the original problem to solving the equation $L_{a,\eta}\Phi = b$. Let γ be a constant satisfying the conditions of the corresponding theorem. We can use the successive approximation method for solving the equation $L_{a,\eta}\Phi = b$. First we must rewrite the equation $L_{a,\eta}\Phi = b$ as $\Phi = (I - \gamma^{-1} L_{a,\eta})\Phi + \gamma^{-1} b$. Fix arbitrary $\Phi_0 \in C^0(\partial G, \mathbb{R}^2)$ and put

$$\Phi_{k+1} = (I - \gamma^{-1} L_{a,\eta})\Phi_k + \gamma^{-1} b$$

for a nonnegative integer k. According to the Theorem 4.5 there are constants $d \in \langle 1, \infty)$, $t \in (0,1)$ such that the estimate (46) holds for each nonnegative natural number k. Since

$$\Phi_{k+1} - \Phi_k = (I - \gamma^{-1} L_{a,\eta})(\Phi_k - \Phi_{k-1}) = \dots = (I - \gamma^{-1} L_{a,\eta})^k (\Phi_1 - \Phi_0)$$

we have

$$\|\Phi_{k+1} - \Phi_k\| \le dt^k \|\Phi_1 - \Phi_0\|.$$

If $j < k$ then

$$\|\Phi_k - \Phi_j\| \le d(t^j + \dots t^{k-1})\|\Phi_1 - \Phi_0\| = \frac{dt^j}{1-t}\|\Phi_1 - \Phi_0\|.$$

Thus $\{\Phi_k\}$ is a Cauchy sequence and converges to the solution Φ of the equation $L_{a,\eta}\Phi = b$. Moreover,

$$\|\Phi - \Phi_j\| \le \frac{dt^j}{1-t}\|\Phi_1 - \Phi_0\|.$$

6 More general problems

In the rest of the paper we shall suppose that G is bounded. We shall study two more general problems.

First we shall study a problem on a domain with cracks. Suppose that V^+ is a nonempty bounded open set with boundary of class $C^{2+\alpha}$, $\alpha > 0$. Let $S \subset \partial V^+ \cap \mathrm{cl}\, G$ be a nonempty compact set such that $\mathrm{cl}(G \cap S) = S$. We shall suppose that $\partial G \cap \partial V^+ \setminus S = \emptyset$. Denote $V^- = \mathbb{R}^2 \setminus \mathrm{cl}\, V^+$, $\Omega = G \setminus S$.

We shall solve the following problem: Find $(v, q) \in C^\infty(\Omega, \mathbb{R}^3)$ such that $v \in C^0(\mathrm{cl}\,\Omega \setminus S, \mathbb{R}^2)$, v is continuously extendible onto $\mathrm{cl}(V^+ \cap \Omega)$ and $\mathrm{cl}(V^- \cap \Omega)$, ∇v, q are continuously extendible onto $\mathrm{cl}\, V^+ \cap G$ and $\mathrm{cl}\, V^- \cap G$ and

$$-\Delta v + \nabla q = 0 \qquad \text{in } \Omega \tag{50}$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega \tag{51}$$

$$v = f \qquad \text{on } \partial\Omega \setminus S \tag{52}$$

$$v^+ - v^- = g \qquad \text{on } S \tag{53}$$

$$(T_q^v N_{V^+})^+ - (T_q^v N_{V^+})^- = h \qquad \text{on } S \cap G. \tag{54}$$

By superscripts "+" and "-" we denote limiting values of functions on S with respect to V^+ and V^-, respectively.

If (v, p) is a solution of the problem (50)–(54) then the Divergence theorem gives

$$0 = \int\limits_{\partial(G\cap V^+)} N_{G\cap V^+} \cdot v^+ \,\mathrm{do} + \int\limits_{\partial(G\cap V^-)} N_{G\cap V^-} \cdot v^- \,\mathrm{do}.$$

The boundary conditions (52), (53) give that

$$\int\limits_{\partial\Omega\setminus S} f \cdot N_\Omega \,\mathrm{do} + \int\limits_S N_{V^+} \cdot g \,\mathrm{do} = 0 \tag{55}$$

is a necessary condition for the solvability of the problem (50)–(54).

Theorem 6.1. *Let (v,q) be a solution of the problem (50) – (54) with $g \equiv 0$, $h \equiv 0$. Then (v,q) can be extended onto G such that (v,q) is a solution of the Stokes system in G. If moreover $f \equiv 0$ then $v \equiv 0$ and q is constant.*

Proof. Fix $x \in S \cap G$. Choose $r > 0$ small enough such that $\operatorname{dist}(x, \partial G) > r$ and ∂V^+ slits $B_r(x) = \{y \in \mathbb{R}^2; |y - x| < r\}$ into 2 components B^+, B^- so that $\partial B^+ \cap \partial B^- = \partial V^+ \cap \operatorname{cl} B_r(x)$. Fix arbitrary $z \in B^+$ and choose $\rho > 0$ such that $B_{2\rho}(z) \subset B^+$. Using Green's formula we get (compare (12))

$$(v,q)(z) = -\int_{\partial B_\rho(z)} E^{(c)}(z-y) T^v_q N_{B_\rho(z)}(y)\, \mathrm{do}_y + \int_{\partial B_\rho(z)} D_{B_\rho(z)}(z,y) v(y)\, \mathrm{do}_y, \tag{56}$$

$$0 = -\int_{\partial(B^+ \setminus B_\rho(z))} E^{(c)}(z-y) T^v_q N_{B^+ \setminus B_\rho(z)}(y) \mathrm{do}_y + \int_{\partial(B^+ \setminus B_\rho(z))} D_{(B^+ \setminus B_\rho(z))}(z,y) v(y) \mathrm{do}_y, \tag{57}$$

$$0 = -\int_{\partial B^-} E^{(c)}(z-y) T^v_q N_{B^-}(y)\, \mathrm{do}_y + \int_{\partial B^-} D_{B^-}(z,y) v(y)\, \mathrm{do}_y. \tag{58}$$

Adding (56), (57) and (58) we get

$$(v,q)(z) = -\int_{\partial B_r(x)} E^{(c)}(z-y) T^v_q N_{B_r(x)}(y)\, \mathrm{do}_y + \int_{\partial B_r(x)} D_{B_r(x)}(z,y) v(y)\, \mathrm{do}_y. \tag{59}$$

By a similar way we get (59) for $z \in B^-$. We can define (v,q) by the limit on S. Then (59) holds for each $z \in B_r(x)$. According to (59) the vector field (v,q) is a solution of the Stokes system in $B_r(x)$.

Suppose now $f \equiv 0$. Since (v,q) is a classical solution of the Dirichlet problem for the Stokes system in G with zero boundary condition we get the proposition of Theorem 6.1 from Proposition 3.1.

Suppose that the functions g, h are such that $g \in C^{1+\gamma}(\partial V^+, \mathbb{R}^2)$, $h \in C^\gamma(\partial V^+, \mathbb{R}^2)$, where $0 < \gamma < \alpha$, and $g = h = 0$ on $G \cap \partial V^+ \setminus S$. Suppose moreover that $f \in C^0(\partial G \setminus S, \mathbb{R}^2)$, f is continuously extendible onto $\operatorname{cl} V^+ \cap \partial G$ and $\operatorname{cl} V^- \cap \partial G$ and $f^+ - f^- = g$ on $S \cap \partial G$. We can suppose, moreover, that the supports of g and h are compact subsets of ∂V^+.

Denote on Ω

$$(w,r) = -E_{V^+} h + D_{V^+} g. \tag{60}$$

It is well known that $\nabla D_{V^+} g$ is continuously extendible onto $\operatorname{cl} V^+ \cap \operatorname{cl} G$ and $[(T D_{V^+} g) N_{V^+}]^+ = [(T D_{V^+} g) N_{V^+}]^-$. Thus (w,r) is a solution of the Stokes system outside the crack S, w is continuously extendible onto $\operatorname{cl} V^+ \cap \operatorname{cl} G$ and $\operatorname{cl} V^- \cap \operatorname{cl} G$, ∇w, r are continuously extendible onto $\operatorname{cl} V^+ \cap G$ and $\operatorname{cl} V^- \cap G$. Using properties of potentials we see that (w,r) satisfies the jump conditions (53), (54).

Put $b = f - w$ on $\partial G \setminus S$. Since $g = f^+ - f^-$, the function b can be continuously extended onto ∂G. We look for a solution of the problem (50)–(54) in the form $(v, q) = (u, p) + (w, r)$. Theorem 6.1 gives that (v, q) is a solution of the problem (50)–(54) if and only if (u, p) is a classical solution of the problem

$$-\Delta u + \nabla p = 0, \qquad \nabla \cdot u = 0 \qquad \text{in } G,$$

$$u = b \qquad \text{on } \partial G.$$

The condition (55) is necessary for the solvability of the problem (50)–(54). Suppose that this condition is fulfilled. Put $\tilde{f} = w$ on $\partial\Omega \setminus S$. Since (w, r) is a solution of the problem

$$-\Delta w + \nabla r = 0, \qquad \nabla \cdot w = 0 \qquad \text{in } \Omega$$

$$w = \tilde{f} \qquad \text{on } \partial\Omega \setminus S$$

$$w^+ - w^- = g \qquad \text{on } S$$

$$(T_r^w N_{V^+})^+ - (T_r^w N_{V^+})^- = h \qquad \text{on } S \cap G$$

we have

$$\int\limits_{\partial\Omega\setminus S} \tilde{f} \cdot N_\Omega \, \mathrm{do} + \int\limits_S N_{V^+} \cdot g \, \mathrm{do} = 0.$$

Subtracting this and (55) we get (49). Theorem 5.3 gives that there is a classical solution (u, p) of the Dirichlet problem (26), (27). We constructed (u, p) in the paragraph 5.

As a second application we shall study the following problem

$$-\Delta v + \nabla q \ = \ f \quad \text{in } G, \qquad \nabla \cdot v \ = \ c \quad \text{in} \quad G, \tag{61}$$

$$v \ = \ g \quad \text{on} \quad \partial G \tag{62}$$

which was studied in [5]. Here c is constant, $g \in C^\beta(\partial G)$, $f \in C^\beta(\mathbb{R}^2)$ has compact support. We reduce this problem to the Dirichlet problem for the Stokes system. Put

$$\begin{pmatrix} w \\ r \end{pmatrix} = \int\limits_{R^2} E(x - y) f(y) dy.$$

Then $(w, r) \in C^{2+\beta}(\mathbb{R}^2)$ and satisfies $-\Delta w + \nabla r = f$, $\nabla \cdot w = 0$. If we look for a solution of (61), (62) in the form $(v, q) = (u, p) + (w, r) + \frac{c}{2}(x_1, x_2, 0)$ then the original problem is equivalent to the problem (26), (27) with $b(x) = g(x) - w(x) - \frac{c}{2}x$ on ∂G. According to Theorem 5.3 this problem is solvable if and only if (49) holds. Since the divergence theorem gives

$$\int\limits_{\partial G} \left[w(x) + \frac{c}{2} x \right] \cdot N_G \, \mathrm{do} = \int\limits_G \nabla \cdot \left[w(x) + \frac{c}{2} x \right] dx = c \int\limits_G 1 dx,$$

we deduce that the problem (61), (62) is solvable if and only if

$$\int_{\partial G} g \cdot N_G \, \mathrm{do} = c \int_G 1 dx.$$

We constructed (u, p) in the paragraph 5.

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