INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

(m, q)-isometries on metric spaces

Teresa Bermúdez
Antonio Martinón
Vladimír Müller

Preprint No. 9-2013

PRAHA 2013

(m,q)-ISOMETRIES ON METRIC SPACES

TERESA BERMÚDEZ, ANTONIO MARTINÓN, AND VLADIMIR MÜLLER

ABSTRACT. We show that there exist a linear m-isometry on a Hilbert space which is not continuous, and a continuous m-isometry on a Hilbert space which is not affine. Further we define (m,q)-isometries on metric spaces and prove their basic properties.

1. Introduction

The notion of an m-isometry in the setting of Hilbert spaces was introduced by J. Agler [2]: a bounded linear operator $T : H \longrightarrow H$, on a Hilbert space H is an m-isometry ($m \geq 1$ integer) if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k} = 0\ , \tag{1.1}$$

where T^* denotes the adjoint operator of T. These operators were further studied extensively by Agler and Stankus [4, 5, 6].

The m-isometric operators were studied by many authors. For example: in [8, 16, 17, 20, 21, 19, 23, 24, 29, 33] various results about m-isometries were given; in [9, 11, 22] the dynamics of m-isometries was studied; in [7, 12, 18, 25, 30, 32] certain types of operators (composition, multiplication, shift) were considered and some conditions under which these operators are m-isometries were given; in [15, 26, 35] some special spaces were considered.

If H is finite-dimensional, then the situation is very simple: a linear operator $T : H \longrightarrow H$ is an m-isometry, but not an $(m-1)$-isometry, if and only if m is odd and $T = A+Q$, where A and Q are commuting operators on H, A is an isometry and Q a linear nilpotent operator of order $\frac{m+1}{2}$, [3, 14].

It is clear that (1.1) is equivalent to

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^{2} = 0 \qquad (x \in H)\ . \tag{1.2}$$

Notice that it is possible to use (1.2) as the definition of m-isometries for operators on Banach spaces, as Bayart [9] and Sid Ahmed [34] have observed. Moreover, Bayart [9] has noted that there is no reason

Date: January 15, 2013.

1991 *Mathematics Subject Classification.* 54E40, 47B99.

Key words and phrases. m-isometry, metric space.

why the exponent 2 in (1.2) should play a particular role. Then the following definition was introduced: a bounded linear operator $T : X \longrightarrow X$, on a Banach space X is an (m, q)-isometry ($m \geq 1$ integer, $q \geq 1$ real) if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^q = 0 \qquad (x \in X) . \tag{1.3}$$

Hoffmann and Mackey [27] considered the above definition with $q > 0$ real and studied the role of the second parameter q. They proved that, for $q > 0$, there is a $(2, q)$-isometry which is not a $(2, q')$-isometry for any $q' \neq q$.

In [12] it was introduced a notion of an m-isometric map on certain hyperspaces of a Banach space. In this paper we study the notion of (m, q)-isometry for maps on a metric space: a map $T : E \longrightarrow E$, on a metric space E with distance d, is called an (m, q)-isometry ($m \geq 1$ integer, $q > 0$ real) if

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} d(T^k x, T^k y)^q = 0 \qquad (x, y \in E) . \tag{1.4}$$

Of course, if E is a Banach space and T is linear, then (1.4) is equivalent to (1.3).

The paper is organized as follows. In sections 2 and 3 we include a general theory of (m, q)-isometries on metric spaces. Many results known in the Banach space setting are now established for metric spaces. For example, for an (m, q)-isometry $T : E \longrightarrow E$ we show that it is an $(m + 1, q)$-isometry and any power T^r is an (m, q)-isometry; moreover, if T is power bounded, then is an isometry.

In section 4 we study properties related to the continuity. First, (m, q)-isometries on metric spaces are not necessarily continuous maps. Moreover, we prove that there is a Hilbert space Y and a linear unbounded $(2, 2)$-isometry $T : Y \longrightarrow Y$, so for (m, q)-isometries linearity is not sufficient to guarantee the continuity. Finally, we show that there exists a continuous non-affine $(2, 2)$-isometry $T : \ell_2 \longrightarrow \ell_2$ on the Hilbert space ℓ_2. Recall that the Mazur-Ulam theorem affirms that if $T : X \longrightarrow Y$ is a surjective isometry between two real normed spaces X and Y, then T is affine. A natural question related with this topics is the following: if T is a continuous and surjective (m, q)-isometry on a real normed space X, is then T affine? That is, does a version of the Mazur-Ulam theorem for (m, q)-isometries hold?

In the last section we consider some distances associated with $T : E \longrightarrow E$ and prove that T is an (m, q)-isometry if and only if T is an isometry for certain semi-distance ρ_T.

2. Definition and first properties

Throughout this paper, E denotes a metric space and d its distance, $T : E \longrightarrow E$ a map, $m \geq 1$ an integer and $q > 0$ a real number, unless said otherwise.

We give the main definition of this paper:

Definition 2.1. *Let E be a metric space. A map $T : E \longrightarrow E$ is called an* (m, q)-isometry *($m \geq 1$ integer, $q > 0$ real) if, for all $x, y \in E$,*

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}d(T^k x, T^k y)^q = 0 \ . \tag{2.5}$$

For $m \geq 2$, T is a strict (m, q)-isometry *if it is an (m, q)-isometry, but is not an $(m-1, q)$-isometry.*

For any q, $(1, q)$-isometries coincide with isometries; that is, maps T satisfying $d(Tx, Ty) = d(x, y)$, for all $x, y \in E$. Every isometry is an (m, q)-isometry, for all $m \geq 1$ and $q > 0$.

If X is a normed space with norm $\|\cdot\|$ and $T : X \longrightarrow X$ is a linear operator, then T is an (m, q)-isometry if and only if (1.3) holds. Clearly m-isometries on Hilbert spaces agree with $(m, 2)$-isometries. It is well known that there exist strict $(m, 2)$-isometries on ℓ_2, for $m = 2, 3, 4...$ [8, 12].

The following simple example shows that there exist strict $(3, 2)$-isometries.

Example 2.2. There exists a bijective $(3, 2)$-isometry which is not an isometry. Indeed, let H be the Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$. Let $T : H \longrightarrow H$ be the weighted bilateral shift defined by

$$Te_n := \sqrt{\frac{(n+1)^2+1}{n^2+1}}e_n \quad (n \in \mathbb{Z}) \ .$$

It is easy to verify that T is a strict $(3, 2)$-isometry.

The following remark contains a simple but very useful result.

Remark 2.3. Every (m, q)-isometry T is injective. Indeed, if $Tx = Ty$, then $T^k x = T^k y$ for $k = 2, ..., m$ and from (2.5) we obtain $x = y$. Hence T is an injective map.

We have the following result about bijective (m, q)-isometries.

Proposition 2.4. *If T is a bijective (m, q)-isometry, then T^{-1} is also an (m, q)-isometry.*

Proof. Let $x, y \in E$. Let $u, v \in E$ satisfy $T^m u = x$ and $T^m v = y$; that is, $u = T^{-m}x$ and $v = T^{-m}y$. Since T is an (m, q)-isometry, (2.5) implies

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}d(T^{-k}x, T^{-k}y)^q = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{m-k}d(T^{m-k}u, T^{m-k}v)^q = 0 \ .$$

Hence T^{-1} is an (m,q)-isometry. ☐

We introduce the following notation: given $T: E \longrightarrow E$, $h \geq 0$ integer, $q > 0$ real and $x, y \in E$, we put

$$f_T(h,q;x,y) := \sum_{k=0}^{h} (-1)^{h-k} \binom{h}{k} d(T^k x, T^k y)^q \ .$$

Notice that $f_T(0,q;x,y) = d(x,y)^q$.

Clearly T is an (m,q)-isometry if and only if $f_T(m,q;x,y) = 0$, for all $x, y \in E$.

The following result relate certain values of f_T:

Proposition 2.5. *For any integer $h \geq 1$, real number $q > 0$ and $x, y \in E$,*

$$f_T(h,q;x,y) = f_T(h-1,q;Tx,Ty) - f_T(h-1,q;x,y) \ .$$

Proof. Fix $x, y \in E$. Then

$$f_T(h,q;x,y) = \sum_{k=0}^{h} (-1)^{h-k} \binom{h}{k} d(T^k x, T^k y)^q =$$

$$= (-1)^h d(x,y) + \sum_{k=1}^{h-1} (-1)^{h-k} \binom{h-1}{k} d(T^k x, T^k y)^q$$

$$+ \sum_{k=1}^{h-1} (-1)^{h-k} \binom{h-1}{k-1} d(T^k x, T^k y)^q + d(T^h x, T^h y)$$

$$= -f_T(h-1,q;x,y) + f_T(h-1,q;Tx,Ty) \ .$$

Hence $f_T(h-1,q;x,y) + f_T(h,q;x,y) = f_T(h-1,q;Tx,Ty)$. ☐

Corollary 2.6. *If T is an (m,q)-isometry, then T is an $(m+1,q)$-isometry.*

3. Basic properties

The next results give some expressions for $d(T^n x, T^n y)$, where T is an (m,q)-isometry. Recall that for integers $n, k \geq 0$,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \ ,$$

so $\binom{n}{k} = 0$ if $n < k$.

We will use the following lemmas.

Lemma 3.1. *Let $(e_k)_{k\geq 0}$ and $(d_j)_{j\geq 0}$ be sequences of real numbers and let $(c_{k,j})_{k,j\geq 0}$ be a double sequence of real numbers. Then*

$$\sum_{k=0}^{n} e_k \sum_{j=0}^{k} c_{k,j} d_j = \sum_{j=0}^{n} d_j \sum_{k=j}^{n} c_{k,j} e_k \ .$$

for any $n = 0, 1, 2...$

Proof. Note that both expressions are equal to

$$\sum_{0\leq j\leq k\leq n} d_j e_k c_{k,j} \ ,$$

so the proof is completed. □

Lemma 3.2. *Let s, r be integers with $0 \leq r < s$. Then*

$$\sum_{h=0}^{r} (-1)^h \binom{s}{h} = (-1)^r \binom{s-1}{r} \ . \tag{3.6}$$

Proof. By induction:

(1) Case $s = r + 1$. We have

$$\sum_{h=0}^{r} (-1)^h \binom{r+1}{h} = -(-1)^{r+1} \binom{r+1}{r+1} = (-1)^r \ ,$$

so (3.6) holds.

(2) Assume that (3.6) is true for certain $s \geq r + 1$ and we prove that is also true for $s + 1$. Indeed,

$$\sum_{h=0}^{r} (-1)^h \binom{s+1}{h} = \binom{s+1}{0} - \left(\binom{s}{0} + \binom{s}{1}\right) + \cdots + (-1)^r \left(\binom{s}{r-1} + \binom{s}{r}\right) = (-1)^r \binom{s}{r} \ .$$

This finishes the proof. □

Lemma 3.3. *Let n, m, k be integers such that $0 \leq k \leq m - 1 < n$. Then*

$$\sum_{j=k}^{m-1} (-1)^{j-k} \binom{n}{j} \binom{j}{k} = (-1)^{m-k-1} \frac{n(n-1)\cdots \overbrace{(n-k)}\cdots (n-m+1)}{k!(m-k-1)!} \ , \tag{3.7}$$

where $\overbrace{(n-k)}$ denotes that the factor $(n-k)$ is omitted.

Proof. Notice that

$$A := \sum_{j=k}^{m-1} (-1)^{j-k} \binom{n}{j} \binom{j}{k} = \sum_{j=k}^{m-1} (-1)^{j-k} \frac{n(n-1)\cdots(n-k+1)(n-k)!}{k!(n-j)!(j-k)!} =$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \sum_{j=0}^{m-k-1} (-1)^j \binom{n-k}{j} \ .$$

Applying (3.6) we obtain

$$A = \frac{n(n-1)\cdots(n-k+1)}{k!}(-1)^{m-k-1}\binom{n-k-1}{m-k-1}\,.$$

On the other hand,

$$B := (-1)^{m-k-1}\frac{n(n-1)\cdots\overbrace{(n-k)}\cdots(n-m+1)}{k!(m-k-1)!} =$$

$$= (-1)^{m-k-1}\frac{n(n-1)\cdots(n-k+1)(n-k-1)!}{k!(m-k-1)!(n-m)!} =$$

$$= (-1)^{m-k-1}\frac{n(n-1)\cdots(n-k+1)}{k!}\binom{n-k-1}{m-k-1}\,.$$

Hence $A = B$ and the proof is finished. □

Lemma 3.4. *Let $T : E \longrightarrow E$ be a map. Then, for every integer $n \geq 1$, real number $q > 0$ and $x, y \in E$, we have*

$$d(T^n x, T^n y)^q = \sum_{k=0}^{n}\binom{n}{k} f_T(k, q; x, y)\,. \tag{3.8}$$

Proof. Clearly (3.8) is true for $n = 0$ and $n = 1$. Assume that (3.8) is true for any $0, 1, ..., n$. We shall prove that is also true for $n + 1$. Fix $x, y \in E$ and put $f(h) = f_T(h, q; x, y)$. We have

$$d(T^{n+1}x, T^{n+1}y)^q = f(n+1) - \sum_{k=0}^{n}(-1)^{n+1-k}\binom{n+1}{k} d(T^k x, T^k y)^q$$

$$= f(n+1) - \sum_{k=0}^{n}(-1)^{n+1-k}\binom{n+1}{k}\sum_{j=0}^{k}\binom{k}{j} f(j)\,.$$

We apply Lemma 3.1 and obtain

$$d(T^{n+1}x, T^{n+1}y)^q = f(n+1) - \sum_{j=0}^{n} f(j)\sum_{k=j}^{n}(-1)^{n+1-k}\binom{n+1}{k}\binom{k}{j} =$$

$$= f(n+1) - \sum_{j=0}^{n}\binom{n+1}{j} f(j)\sum_{k=j}^{n}(-1)^{n+1-k}\frac{(n+1-j)!}{(k-j)!(n+1-k)!} =$$

$$= f(n+1) - \sum_{j=0}^{n}\binom{n+1}{j} f(j)\sum_{k=j}^{n}(-1)^{n+1-k}\binom{n+1-j}{k-j} = \sum_{j=0}^{n+1}\binom{n+1}{j} f(j)$$

which finishes the proof. □

Theorem 3.5. *A map T is an (m, q)-isometry if and only if, for every integer $n \geq 1$ and all $x, y \in E$, we have*

$$d(T^n x, T^n y)^q = \sum_{k=0}^{m-1}\binom{n}{k} f_T(k, q; x, y)\,. \tag{3.9}$$

Proof. If T is an (m,q)-isometry, then $f_T(k,q;x,y)=0$ for all $k\geq m$ and $x,y\in E$. Hence we derive (3.9) from (3.8).

On the other hand, if (3.9) holds for all $n\geq 1$ and $x,y\in E$, then $f_T(k,q;x,y)=0$ for $k\geq m$, by (3.8), so T is an (m,q)-isometry. □

The following result is similar to [12, Theorem 2.1]. We give its proof based on Lemma 3.7.

Theorem 3.6. *A map T is an (m,q)-isometry if and only if, for all $x,y\in E$ and every integer $n\geq 0$, we have*

$$d(T^nx,T^ny)^q=\sum_{k=0}^{m-1}(-1)^{m-k-1}\frac{n\cdots\overbrace{(n-k)}\cdots(n-m+1)}{k!(m-k-1)!}d(T^kx,T^ky)^q\ . \tag{3.10}$$

Therefore for each $k=0,1,...,m-1$, the coefficient at $d(T^kx,T^ky)^q$ is a polynomial in n of degree $\leq m-1$.

Proof. Firstly, assume that T is an (m,q)-isometry. The equality (3.10) is clear if $n<m$. Assume $n\geq m$.

Fix $x,y\in E$. Put $a_n:=d(T^nx,T^ny)^q$ and $f(j):=f_T(j,q;x,y)$. Theorem 3.5 and Lemma 3.1 imply

$$a_n=\sum_{j=0}^{m-1}\binom{n}{j}f(j)=\sum_{j=0}^{m-1}\binom{n}{j}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}a_k=\sum_{k=0}^{m-1}a_k\sum_{j=k}^{m-1}(-1)^{j-k}\binom{n}{j}\binom{j}{k}\ .$$

¿From Lemma 3.3 we obtain (3.10).

Suppose that (3.10) holds for $n\geq 0$ and all $x,y\in E$. In particular, for $n=m$ we have that equality (3.10) agrees with (2.5) and the proof is completed. □

Theorem 3.7. *A map T is an (m,q)-isometry if and only if, for all $x,y\in E$, there exist real numbers $\gamma_0(x,y),...,\gamma_{m-1}(x,y)$ such that, for every integer number $n\geq 0$, we have*

$$d(T^nx,T^ny)^q=\sum_{k=0}^{m-1}\gamma_k(x,y)n^k\ . \tag{3.11}$$

Proof. Suppose that T is an (m,q)-isometry. Fix $x,y\in E$ and set $a_n:=d(T^nx,T^ny)^q$, for $n=0,1,2,3....$ As T is an (m,q)-isometry we obtain the recursive relation

$$\sum_{h=0}^{m}(-1)^h\binom{m}{h}a_{n+m-h}= \tag{3.12}$$

$$a_{n+m}-\binom{m}{1}a_{n+m-1}+\binom{m}{2}a_{n+m-2}-\cdots+(-1)^{m-1}\binom{m}{m-1}a_{n-1}+(-1)^na_n=0\ .$$

It is well known that the solutions $(a_n)_{n\geq 0}$ of (3.12) verify

$$a_n=\gamma_{m-1}(x,y)n^{m-1}+\cdots+\gamma_1(x,y)n+\gamma_0(x,y)\ , \tag{3.13}$$

for some real numbers $\gamma_0(x,y),...,\gamma_{m-1}(x,y)$ (see, for example, [1] and [28]), so (3.11) holds. Conversely, if the sequence $(a_n)_{n\geq 0} := (d(T^n x, T^n y)^q_{n\geq 0}$ verifies (3.13), then it also verifies (3.12), so (3.11) holds. □

Proposition 3.8. *Let T be an (m,q)-isometry. Then, for all $x,y \in E$, the sequence $(d(T^n x, T^n y))_{n\geq 0}$ is eventually increasing; that is, there is a positive integer n_0 such that*

$$d(T^n x, T^n y) \leq d(T^{n+1}x, T^{n+1}y)$$

for all $n \geq n_0$. Moreover, if T is not an isometry, then

$$\lim_{n\to\infty} d(T^n x, T^n y) = \infty \ ,$$

for all $x,y \in E$ with $x \neq y$.

Proof. If T is an isometry then the result is clear. Assume that T is not an isometry, so it is a strict (m,q)-isometry for some $m \geq 2$. The sequence $(d(T^n x, T^n y)^q)_{n\geq 0}$ verifies (3.11) with positive leading coefficient $\gamma_{m-1}(x,y)$, where $m-1 \geq 1$, so $d(T^n x, T^n y) \to \infty$ as $n \to \infty$ and the sequence is eventually increasing. □

It is possible that the sequence $(d(T^n x, T^n y))_{n\geq 0}$ is not increasing, as we show in the next example.

Example 3.9. Consider the norm $\|\cdot\|_2$ on $\mathbb{C}^2$. The map $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by $T(x,y) := (x+y,y)$ is a $(3,2)$-isometry and we have

$$\|T(-1,1)\|_2 = 1 < 2 = \|(-1,1)\|_2 \ .$$

Corollary 3.10. *Let E be a bounded metric space. If T is an (m,q)-isometry, then T is an isometry.*

Proposition 3.11. *If T is a bijective (m,q)-isometry and m is even, then T is an $(m-1,q)$-isometry*

Proof. As T is a bijective (m,q)-isometry, T^{-1} is also an (m,q)-isometry (Proposition 2.4), hence the equation (3.11) holds for $n = 0, \pm 1, \pm 2...$ If the leading coefficient in (3.11) is γ_k with k odd, then

$$\lim_{n\to\infty} d(T^n x, T^n y) = \infty \qquad \text{and} \qquad \lim_{n\to -\infty} d(T^n x, T^n y) = -\infty \ .$$

Since $d(T^n x, T^n y) \geq 0$, k is even, and so $\gamma_{m-1} = 0$. Consequently, T is an $(m-1,q)$-isometry. □

Definition 3.12. *A map T is called* power bounded *if*

$$\sup\{d(T^n x, T^n y) : n = 0, 1, \dots\} < \infty$$

for all $x,y \in E$.

Proposition 3.13. *Let $T : E \longrightarrow E$ be an (m, q)-isometry. If T is power bounded, then T is an isometry.*

Proof. Let $x, y \in E$ and $K = \sup\{d(T^n x, T^n y) : n = 0, 1, \dots\}$. By (3.11) we have

$$0 \leq \sup\Big\{\sum_{k=0}^{m-1} \gamma_k(x, y)n^k : n = 0, 1, \dots\Big\} \leq K^q$$

for all n. So $\gamma_{m-1}(x, y) = \cdots = \gamma_1(x, y) = 0$. Hence $d(T^n x, T^n y) = d(x, y)$ and T is an isometry. □

A product of (m, q)-isometries is not necessarily an (m, q)-isometry (see, for example, [13, Example 3.1]). In [34, Theorem 2.2], Sid Ahmed proved that if X is a normed space and T and S commuting bounded linear operators on X such that T is a 2-isometry and S is an m-isometry, then ST is an $(m + 1)$-isometry. This result was improved in [13, Theorem 3.3]: if $TS = ST$, T is an (m, q)-isometry and S is an (n, q)-isometry, then ST is an $(m + n - 1, q)$-isometry. Now we generalize it to metric spaces. As the proof is very similar to [13, Theorem 3.3], we omit it.

Theorem 3.14. *Let $T : E \longrightarrow E$ be an (m, q)-isometry and $S : E \longrightarrow E$ an (n, q)-isometry. If $TS = ST$, then TS is an $(m + n - 1, q)$-isometry.*

It is clear that if T is an isometry, then T^r is also an isometry. Patel in [31, Theorem 2.1] proves that any power of a $(2, 2)$-isometry on a Hilbert space is again a $(2, 2)$-isometry. In [10] it was showed that any power of a Banach space (m, q)-isometry is again an (m, q)-isometry. Now we give this result in the setting of metric spaces. We omit the proof because of it is analogous to [10].

Theorem 3.15. *Let T an (m, q)-isometry. Then any power T^r is also an (m, q)-isometry.*

In general the converse of Theorem 3.15 is false, see [10, Example 3.5]. However, if we assume that two suitable different powers of T are (m, q)-isometries, then we obtain that T is (m, q)-isometry. Again we omit the proof since is very similar to [10, Theorem 3.6].

Theorem 3.16. *Let T be a map, r, s, m, l positive integers and $q > 0$ real. If T^r is an (m, q)-isometry and T^s an (l, q)-isometry, then T^t is an (h, q)-isometry, where t is the greatest common divisor of r and s, and h the minimum of m and l.*

In the following result we consider some particular cases of Theorem 3.16.

Corollary 3.17. *Let T be a map, r, s, m positive integers and $q > 0$ real.*

(1) If T is an (m, q)-isometry and T^s is an isometry, then T is an isometry.

(2) If T^r and T^{r+1} are (m,q)-isometries, then T is an (m,q)-isometry.

(3) If T^r is an (m,q)-isometry and T^{r+1} is an (n,q)-isometry with $m<n$, then T is an (m,q)-isometry.

(4) If T is a strict (m,q)-isometry, then any power T^r of T is a strict (m,q)-isometry.

Proposition 3.18. *For $i=1,2,...,n$, let E_i be a metric space with distance d_i, and let $T_i : E_i \longrightarrow E_i$ be a map, $m_i \geq 1$ integer and $q \geq 1$ real. Denote by $E := E_1 \times E_2 \times \cdots \times E_n$ the product space endowed with the product distance*

$$d((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) := \left(\sum_{i=1}^{n} d_i(x_i,y_i)^q\right)^{1/q} .$$

Let $T := T_1 \times T_2 \times \cdots \times T_n : E \longrightarrow E$ be defined by $T(x_1,x_2,...,x_n) := (T_1x_1, T_2x_2, ..., T_nx_n)$. If T_i is an (m_i,q)-isometry for $i=1,2,...,n$, then T is an (m,q)-isometry, where $m = \max\{m_1,m_2,...,m_n\}$.

Proof. Since T_i is an (m_i,q)-isometry, it is also an (m,q)-isometry ($i=1,2,...,n$). So for $x := (x_1,x_2,...,x_n)$ and $y := (y_1,y_2,...,y_n)$ in E, we have

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} d(T^kx,T^ky)^q =$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left(\sum_{i=1}^{n} d_i(x_i,y_i)^q\right) =$$

$$= \sum_{i=1}^{n}\left(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} d_i(x_i,y_i)^q\right) = 0 .$$

Hence T is an (m,q)-isometry. □

4. Continuity of (m,q)-isometries

In the next example we show that (m,q)-isometries are in general neither continuous nor linear.

Example 4.1. Let $E = \mathbb{R}$ with the usual distance $d(x,y) = |x-y|$. Consider the map $T : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$Tx = \begin{cases} x-1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ x+1 & \text{if } x>0 \end{cases}$$

It is easy verify that T is a $(2,1)$-isometry, but T is neither continuous nor linear.

Proposition 4.2. *Let E be a complete metric space. If T is a continuous (m,q)-isometry, then T is injective and its range $R(T)$ is closed.*

Proof. It is clear that T is injective (see Remark 2.3).

We prove that $R(T)$ is closed. Let $(x_n)_{n\geq 1}$ be a sequence in E such that the sequence $(Tx_n)_{n\geq 1}$ is convergent to some $y \in E$, hence $(T^k x_n)_{n\geq 1}$ converges to $T^{k-1}y$ for $k = 2, 3...$, since T is continuous. By definition we have

$$d(x_r, x_s)^q = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} d(T^k x_r, T^k x_s)^q \ ,$$

for $r, s \geq 1$. As $(T^k x_n)_{n\geq 1}$ is a Cauchy sequence for $k = 1, 2, ..., m$, we have that $(x_n)_{n\geq 1}$ is a Cauchy sequence, hence it is convergent to some $x \in M$. Consequently, $(Tx_n)_{n\geq 1}$ converges to $Tx = y$ and $y \in R(T)$. So $R(T)$ is closed. □

Proposition 4.3. *Let E be a complete metric space. If T is a $(2,q)$-isometry, then the restriction $T_{|R(T)}$ of T to its range $R(T)$ is Lipschitz, so uniformly continuous.*

Proof. For all $x, y \in E$ we have

$$d(T^2x, T^2y)^q \leq d(T^2x, T^2y)^q + d(x,y)^q = 2d(Tx, Ty)^q \ ,$$

hence

$$d(T^2x, T^2y) \leq 2^{1/q} d(Tx, Ty) \ .$$

Consequently, T is Lipschitz on $R(T)$ with constant $2^{1/q}$. □

Given a normed space X and an (m,q)-isometry $T : X \longrightarrow X$, it is natural to investigate the relations between the linearity and continuity of T. The celebrated theorem of Mazur-Ulam affirms that if X, Y are real normed spaces and $T : X \longrightarrow Y$ is a surjective isometry (hence continuous) such that $T0 = 0$, then T is linear. The situation in the realm of the (m,q)-isometries is more complicated. In the next example we prove that there exists a linear $(2,2)$-isometry on a Hilbert space which is not continuous. Later we give an example of a continuous $(2,2)$-isometry on ℓ_2 which is not linear, but it is not surjective. We do not know if a version of the Mazur-Ulam theorem is valid for (m,q)-isometries whose range is an affine subspace (see Open problem 1).

Example 4.4. *There exists a Hilbert space Y and a linear unbounded $(2,2)$-isometry $T : Y \longrightarrow Y$.*

Let $(H, \|\cdot\|_0)$ be an infinite-dimensional Hilbert space and we consider

$$(\tilde{K}, \|\cdot\|_1) := (H, \|\cdot\|_0) \oplus (H, \|\cdot\|_0) \ ,$$

where $\tilde{K} := H \times H = \{(h, h') : h, h' \in H\}$ and $\|(h,h')\|_1 := (\|h\|_0^2 + \|h'\|_0^2)^{1/2}$, for any $h, h' \in H$.

Let $Z : (H, \|\cdot\|_0) \to (H, \|\cdot\|_0)$ be a linear unbounded mapping and define $S : H \longrightarrow \tilde{K}$ by $Sh = (h, Zh)$ for $h \in H$. Then $S : (H, \|\cdot\|_0) \longrightarrow (\tilde{K}, \|\cdot\|_1)$ is a linear unbounded mapping and $\|Sh\|_1 \geq \|h\|_0$, for $h \in H$. Moreover, $\|Sh\|_1^2 - \|h\|_0^2 = \|Zh\|_0^2$, so $h \mapsto (\|Sh\|_1^2 - \|h\|_0^2)^{1/2}$ defines a seminorm on H.

Let $M := SH$ and $K = \overline{M}$, the closure of M in $(\tilde{K}, \|\cdot\|_1)$. Then $(K, \|\cdot\|_1)$ is a Hilbert space.

For $k \geq 2$ define a norm $\|\cdot\|_k$ on M by

$$\|Sh\|_k = (k\|Sh\|_1^2 - (k-1)\|h\|_0^2)^{1/2} =$$

$$(\|Sh\|_1^2 + (k-1)(\|Sh\|_1^2 - \|h\|_0^2))^{1/2} = (\|Sh\|_1^2 + (k-1)\|Zh\|_0^2)^{1/2}.$$

Clearly $\|\cdot\|_k$ is a seminorm on M and

$$\|Sh\|_1 \leq \|Sh\|_k \leq \sqrt{k}\|Sh\|_1 \qquad (h \in H).$$

So $\|\cdot\|_k$ is a norm equivalent to $\|\cdot\|_1$ on M and can be extended continuously to K. Clearly $(K, \|\cdot\|_k\|)$ is a Hilbert space for each $k \geq 1$.

Let

$$Y := (H, \|\cdot\|_0) \oplus \bigoplus_{k=1}^{\infty} (K, \|\cdot\|_k) \ ;$$

that is, $(h, k_1, k_2, ...) \in Y$ if $h \in H$, $k_i \in K$ $(i = 1, 2, ...)$ and $\sum \|k_i\|_i^2 < \infty$. Then Y is a Hilbert space with the norm

$$\|(h, k_1, k_2, ...)\| := \left(\|h\|_0^2 + \sum_{i=1}^{\infty} \|k_i\|_i^2 \right)^{1/2} .$$

Define $T : Y \longrightarrow Y$ in the following way: for $x = (h, k_1, k_2, \dots) \in Y$ let

$$Tx = T(h, k_1, k_2, \dots) := (0, Sh, k_1, k_2, \dots).$$

So $T^2 x = (0, 0, Sh, k_1, k_2, \dots)$. Clearly T is a linear unbounded mapping.

To show that T is a $(2,2)$-isometry, we need to prove that

$$\|x\|^2 - 2\|Tx\|^2 + \|T^2 x\|^2 = 0 \tag{4.14}$$

for all $x \in Y$. For $x = (h, k_1, k_2, \dots) \in Y$ we have

$$\|x\|^2 = \|h\|_0^2 + \sum_{i=1}^{\infty} \|k_i\|_i^2 \ ,$$

hence

$$\|Tx\|^2 = \|Sh\|_1^2 + \sum_{i=1}^{\infty} \|k_i\|_{i+1}^2$$

and

$$\|T^2x\|^2 = \|Sh\|_2^2 + \sum_{i=1}^{\infty} \|k_i\|_{i+2}^2.$$

So

$$\|x\|^2 - 2\|Tx\|^2 + \|T^2x\|^2 = (\|h\|_0^2 - 2\|Sh\|_1^2 + \|S^h\|_2^2) + \sum_{i=1}^{\infty}(\|k_i\|_i^2 - 2\|k_i\|_{i+1}^2 + \|k_i\|_{i+2}^2).$$

So it is sufficient to show that

$$\|h\|_0^2 - 2\|Sh\|_1^2 + \|S^h\|_2^2 = 0 \qquad (h \in H)$$

and

$$\|k\|_i^2 - 2\|k\|_{i+1}^2 + \|k\|_{i+2}^2 \qquad (i \geq 1, k \in K).$$

For $h \in H$ we have by definition $\|Sh\|_2^2 = 2\|Sh\|_1^2 - \|h\|_0^2$, which proves the first equality.

Let $i \geq 1$ and $k \in K$. Then there exists a sequence $(x_r) \subset H$ such that $\lim_{r\to\infty} \|Sx_r - k\|_1 = 0$ (and so $\lim_{r\to\infty} \|Sx_r - k\|_j = 0$ for all $j \geq 1$). Thus

$$\|k\|_i^2 - 2\|k\|_{i+1}^2 + \|k\|_{i+2}^2 = \lim_{r\to\infty}(\|Sx_r\|_i^2 - 2\|Sx_r\|_{i+1}^2 + \|Sx_r\|_{i+2}^2)$$

$$= \lim_{r\to\infty}(i\|Sx_r\|_1^2 - (i-1)\|x_r\|_0^2 - 2(i+1)\|Sx_r\|_1^2 + 2i\|x_r\|_0^2 + (i+2)\|Sx_r\|_1^2 - (i+1)\|x_r\|_0^2) = 0.$$

Hence the equality (4.14) holds, and so T is a $(2,2)$-isometry.

Example 4.5. *There exists a non-affine continuous $(2,2)$-isometry $T : \ell_2 \longrightarrow \ell_2$ on the space ℓ_2.*

Let $T : \ell_2 \longrightarrow \ell_2$ be defined by

$$T(x_1, x_2, x_3, ...) := (x_1x_2, 1, x_1, x_2, x_3, ...) ,$$

so

$$T^2(x_1, x_2, x_3, ...) := (x_1x_2, 1, x_1x_2, 1, x_1, x_2, x_3, ...) .$$

We will show that, for all $x, y \in \ell_2$,

$$\|x - y\|^2 - 2\|Tx - Ty\|^2 + \|T^2x - T^2y\|^2 = 0 . \tag{4.15}$$

We have

$$\|x - y\|^2 - 2\|Tx - Ty\|^2 + \|T^2x - T^2y\|^2 =$$
$$\sum_{n=1}^{\infty} |x_n - y_n|^2 - 2\left(|x_1x_2 - y_1y_2|^2 + \sum_{n=1}^{\infty} |x_n - y_n|^2\right) +$$
$$\left(2|x_1x_2 - y_1y_2|^2 + \sum_{n=1}^{\infty} |x_n - y_n|^2\right) = 0 .$$

Therefore, T verifies (4.15), so T is a $(2,2)$-isometry. It is clear that T is continuous but not affine.

We finish this section with the following problems:

Open problem 1. *[Mazur-Ulam theorem for (m,q)-isometries] Let X be a real normed space and $T : X \longrightarrow X$ a continuous (m,q)-isometry ($m \geq 1$, $q > 0$ real) such that its range $R(T)$ is an affine subspace. Is then T necessarily affine?*

Open problem 2. *Let X be a real normed space and $T : X \longrightarrow X$ a surjective (m,q)-isometry ($m \geq 3$, $q > 0$ real). Is then T necessarily affine?*

5. Distances associated to (m,q)-isometries

Every (m,q)-isometry $T : E \longrightarrow E$ becomes an isometry for an adequate distance on E. The following results are analogous to those of Bayart [9].

Proposition 5.1. *Let T be an (m,q)-isometry. For $x, y \in E$ define*

$$\rho_T(x,y) := f_T(m-1, q; x, y)^{1/q} .$$

Then ρ_T is a semi-distance and moreover,

$$\rho_T(x,y)^q = (m-1)! \lim_{n\to\infty} \frac{d(T^n x, T^n y)^q}{n^{m-1}} . \tag{5.16}$$

Proof. Write for short $f(k) := f_T(k, q; x, y)$. By Corollary 3.9, we have

$$d(T^n x, T^n y)^q = \sum_{k=0}^{m-1} \binom{n}{k} f(k) .$$

Notice that the coefficient $\binom{n}{k}$ at $f(k)$ is a polynomial in n of degree k and $f(k) = 0$ if $k > m-1$. Therefore

$$f(m-1) = \lim_{n\to\infty} \frac{d(T^n x, T^n y)^q}{\binom{n}{m-1}} = (m-1)! \lim_{n\to\infty} \frac{d(T^n x, T^n y)^q}{n^{m-1}} .$$

We show that ρ_T is a semi-metric. By (5.16) it is clear that $\rho \geq 0$. Clearly $\rho(x,x) = 0$ and $\rho(x,y) = \rho(y,x)$ for all $x, y \in E$. It remains to show the triangular inequality. Let $x, y, z \in E$. Then

$$\rho(x,y) = f_T(m-1, q; x, y)^{1/q} = [(m-1)!]^{1/q} \lim_{n\to\infty} \frac{d(T^n x, T^n y)}{n^{\frac{m-1}{q}}} \leq$$

$$\leq [(m-1)!]^{1/q} \lim_{n\to\infty} \frac{d(T^n x, T^n z)}{n^{\frac{m-1}{q}}} + [(m-1)!]^{1/q} \lim_{n\to\infty} \frac{d(T^n z, T^n y)}{n^{\frac{m-1}{q}}} = \rho(x,z) + \rho(z,y) .$$

$\square$

Theorem 5.2. *Let $T : E \to E$ be a mapping. Then T is an (m,q)-isometry if and only if $T : (E, \rho_T) \longrightarrow (E, \rho_T)$ is an isometry.*

Proof. By Proposition 2.5, T is an (m,q)-isometry if and only if $f_T(m-1, q; x, y) = f_T(m-1, q; Tx, Ty)$, if and only if $\rho_T(Tx, Ty) = \rho_T(x, y)$. $\square$

Given an (m,q)-isometry T, consider the semidistance ρ_T and the set

$$N(\rho_T) := \{(x,y) \in E \times E : \rho_T(x,y) = 0\} \ .$$

By Theorem 5.2, we have

$$(T \times T)N(\rho_T) := \{(Tx, Ty) : (x,y) \in N(\rho_T)\} \subset N(\rho_T) \ .$$

If T is bijective, then we have $(T \times T)N(\rho_T) = N(\rho_T)$. Of course, ρ_T is a distance if and only if $N(\rho_T) = \{(x,x) : x \in E\}$.

Proposition 5.3. *Let T be an (m,q)-isometry Lipschitz with constant c. Then*

$$\rho_T \leq (c-1)^{m-1} d \ .$$

Hence the topology generated by d is stronger than the topology generated by ρ_T.

Proof. For all $k = 0, 1, 2, ..., m-1$ and $x, y \in E$, we have

$$d(T^k x, T^k y) \leq c d(T^{k-1} x, T^{k-1} y) \leq \cdots \leq c^k d(x,y) \ ,$$

hence

$$\rho_T(x,y) = f_T(m-1, q; x, y) = \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} d(T^k x, T^k y) \leq$$
$$\leq \left(\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} c^k \right) d(x,y) = (c-1)^{m-1} d(x,y).$$

This finishes the proof. $\square$

Let $T : E \longrightarrow E$ be a map. Suppose that $q \geq 1$. Define on E the distances $d_{T,1}$ and $d_{T,2}$ in the following way, for $x, y \in E$:

$$d_{T,1}(x,y) := \left[\sum_{0 \leq j \leq m, \ j \text{ even}} \binom{m}{j} d(T^j x, T^j y)^q \right]^{1/q}$$

$$d_{T,2}(x,y) := \left[\sum_{0 \leq j \leq m, \ j \text{ odd}} \binom{m}{j} d(T^{j-1} x, T^{j-1} y)^q \right]^{1/q}$$

Notice that

$$|f_T(m,q;x,y)| = \left|d_{T,1}(x,y)^q - d_{T,2}(Tx,Ty)^q\right|.$$

Consequently, we obtain the following result:

Theorem 5.4. *Let T be a map. The following assertions are equivalent:*

(1) $T:(E,d)\longrightarrow(E,d)$ is an (m,q)-isometry

(2) $T:(E,d_{T,1})\longrightarrow(E,d_{T,2})$ is an isometry

Acknowledgements: The first author is partially supported by grant of Ministerio de Ciencia e Innovación, Spain, proyect no. MTM2011-26538. The last author was supported by grant No. IAA10190903 of GA AV ČR and RVO:67985840.

References

[1] R. P. Agarwal. *Difference equations and inequalities. Theory, methods, and applications*, Monographs and Textbooks in Pure and Applied Mathematics, 155. Marcel Dekker, Inc., New York, 1992.

[2] J. Agler: A disconjugacy theorem for Toeplitz operators. Amer. J. Math. 112 (1990) 1-14.

[3] J. Agler, W. Helton, M. Stankus. Classification of Hereditary Matrices. Linear Algebra App. 274 (1998) 125-160.

[4] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. I. Integral Equat. Ope. Theory 21 (1995) 383-429.

[5] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. II. Integral Equat. Ope. Theory 23 (1995) 1-48.

[6] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. III. Integral Equat. Ope. Theory 24 (1996) 379-421.

[7] D. Alpay, H. T. Kaptanoglu. Toeplitz Operators on Arveson and Dirichlet Spaces. Integral Equat. Ope. Theory 58 (2007) 1-33.

[8] A. Athavale. Some operator theoretic calculus for positive definite kernels. Proc. Amer. Math. Soc. 112(3) (1999) 701-708.

[9] F. Bayart. m-isometries on Banach spaces. Math. Nachr. 284 (2011) 2141-2147.

[10] T. Bermúdez, C. Díaz-Mendoza, A. Martinón. Powers of m-isometries. Studia Math. 208 (2012) 249-255.

[11] T. Bermúdez, I. Marrero, A. Martinón. On the orbit of an m-isometry. Integral Equat. Ope. Theory 64 (2009) 487-494.

[12] T. Bermúdez, A. Martinón, E. Negrín. Weighted shift operators which are m-isometries. Integral Equat. Ope. Theory 68 (2010) 301-312.

[13] T. Bermúdez, A. Martinón, J. Noda. Products of m-isometries. Linear Algebra App. 438 (2013) 80-86.

[14] T. Bermúdez, A. Martinón, J. Noda. An isometry plus an nilpotent operator is an m-isometry. Preprint.

[15] F. Botelho. On the existence of n-isometries on ℓ_p spaces. Acta Sci. Math. (Szeged) 76 (2010) 183-192.

[16] F. Botelho, J. Jamison. Isometric properties of elementary operators. Linear Algebra App. 432 (2010) 357-365.

[17] F. Botelho, J. Jamison, B. Zeng. Strict isometries of arbitrary order. Linear Algebra Appl. 436 (2012) 3303-3314.

[18] M. Cho, S. Ota, K. Tanahashi. Invertible weighted shift operators which are m-isometries. Proc. Amer. Math. Soc. (to appear).

[19] A. Devika, M. Malarvizhi. Spectral Properties of 2-Isometric Operators. Int. J. Contemp. Math. Sciences 5 (2010) 1233-1240.

[20] B. P. Duggal. Tensor product of n-isometries. Linear Algebra Appl. 436 (2012) 307-318.

[21] B. P. Dugal. Tensor product of m-isometries II. Functional Analysis, Approx. Comput. 4 (2012) 27-32.

[22] M. Faghih Ahmadi, K. Hedayatian. Hypercyclicity and supercyclicity of m-isometric operators. Rocky Mountain J. Math. 42 (2012) 15-24.

[23] M. Faghih Ahmadi, K. Hedayatian: m-isometric weighted shifts and reflexivity of some operators. Rocky Mountain J. Math. (to appear).

[24] J. Gleason, S. Richter. m-Isometric Commuting Tuples of Operators on a Hilbert space. Integral Equat. Ope. Theory 56 (2006) 181-196.

[25] K. Hedayatian. A class of four-isometries on function spaces. Italian J. Pure Appl. Math. 16 (2004) 193-200.

[26] C. Helleings. Two-Isometries on Pontryagin Spaces. Integral Equat. Ope. Theory 61 (2008) 211-239.

[27] P. Hoffmann, M. Mackey, M. . Searcid. On the second parameter of an (m, p)-isometry. Integral Equat. Ope. Theory 71 (2011) 389-405.

[28] W. G. Kelley, A. C. Peterson. *Difference Equations. An Introduction with Applications.* Academic Press, 1991.

[29] A. Olofsson. A von Neumann Wold decomposition of two-isometries. Acta Sci. Math. (Szeged) 70 (2004) 715-726.

[30] S. Panayappan, S. K. Latha. Some isometric composition operators. Int. J. Contemp. Math. Sciences 5 (2010) 615-621.

[31] S. M. Patel. 2-isometric operators. Glasnik Mat. 37 (2002) 143-147.

[32] L. J. Patton, M. E. Robbins. Composition operators that are m-isometries. Houston J. Math. 31 (2005) 255-266.

[33] S. Richter. A representation theorem for cyclic analytic two-isometries. Tran. Amer. Math. Soc. 328 (1991) 325-349.

[34] O. A. M. Sid Ahmed. m-isometric operators on Banach spaces. Asian-European J. Math. 3 (2010) 1-19.

[35] O. A. M. Sid Ahmed, A. Saddi. A-m-Isometric operators in semi-Hilbertian spaces. Linear Algebra Appl. 436 (2012) 3930-3942.

E-mail address: tbermude@ull.es

E-mail address: anmarce@ull.es

E-mail address: muller@math.cas.cz

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna (Tenerife), Spain

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna (Tenerife), Spain

Mathematical Institute, Czech Academy of Sciences, 115 67 Prague, Czech Republic