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CIRCLES IN THE SPECTRUM AND THE GEOMETRY OF ORBITS: A NUMERICAL RANGES APPROACH

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Abstract. We prove that a bounded linear Hilbert space operator has the unit circle in its essential approximate point spectrum if and only if it admits an orbit satisfying certain orthogonality and almost-orthogonality relations. This result is obtained via the study of numerical ranges of operator tuples where several new results are also obtained. As consequences of our numerical ranges approach, we derive in particular wide generalizations of Arveson's theorem as well as show that the weak convergence of operator powers implies the uniform convergence of their compressions on an infinite-dimensional subspace. Several related results have been proved as well.

1. Introduction

It is well-known that in the study of invariant subspaces of a bounded linear operator T on a Hilbert space, the presence of the unit circle $\mathbb{T}$ in the spectrum $\sigma(T)$ of T plays a special role. According to one of the strongest results in this direction due to Brown, Chevreau and Pearcy [10], see also [5] and [31, p.156 -157], if T is a Hilbert space contraction having the unit circle in its spectrum, then T has a non-trivial invariant subspace. The statement was extended to Banach spaces and to polynomially bounded operators, [1]. For this and related statements one may also consult the recent survey [6], and the books [27, Chapter 5] and [12]. However the spectral condition $\sigma(T) \supset \mathbb{T}$ appeared to be again crucial. Thus, it is of substantial interest to clarify its interplay with the behavior of orbits of T.

That issue has not received an adequate attention in the literature. Curiously enough, known results on the implications of the circle structure of the spectrum for the geometry of orbits have been noted in an area somewhat distant from the classical operator theory. A long time ago, Arveson proved in [3] that the spectrum of a unitary operator T on H is precisely the unit circle $\mathbb{T}$ if and only if for every $n \in \mathbb{N}$ there exists a nonzero $x \in H$ such

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that the elements $x, Tx, ..., T^n x$ are mutually orthogonal. His motivation for such kind of results originated from the intent to identify the maximal ideals space $\mathcal{M}$ of the C^*-algebra generated by an abelian group of unitary operators G. He proved that $\mathcal{M}$ is homeomorphic to the character group $\hat{G}$ of G (with discrete topology) if for every finite subset F of G there is $x \in H$ such that $Ux \perp Vx$ for all U, V from F. For $G = (U^n)_{n\in\mathbb{Z}}$ this is precisely the statement stated above.

Arveson's nice result can be considered as an operator-theoretical version of the well-known Rokhlin Lemma, a basic tool in ergodic theory. Recall that one of the most common variants of the Rokhlin Lemma says that if $\mathcal{P} = (\Omega, \mathcal{B}, \mu)$ is an atomless Lebesgue probability measure space and S is an aperiodic measure-preserving transformation of $\mathcal{P}$, then for all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a measurable set $B \subset \Omega$ such that the sets $B, S(B), ..., S^{n-1}(B)$ are disjoint and $\mu(B \cup S(B) \cup ... \cup S^{n-1}(B)) \geq 1 - \epsilon$, see e.g. [21] and [32]. (A discussion of a similar but weaker statement can also be found in [3].) If the operator U_S on $L^2(\mathcal{P})$ is defined as

$$(U_S f)(\omega) = f(S\omega), \qquad f \in L^2(\mathcal{P}), \qquad \text{a.e.}\, \omega \in \Omega,$$

then as a direct consequence of the Rokhlin Lemma one gets $\sigma(U_S) = \mathbb{T}$, an important fact in ergodic theory. Note that the orthogonal vectors in the Arveson's statement arise naturally here as the corresponding characteristic functions of $B, S(B), ..., S^{n-1}(B)$. (A lattice-theoretical version of Arveson's theorem has been considered in [34]. An alternative approach to the property $\sigma(U_S) = \mathbb{T}$ has been proposed in [19].)

The Rokhlin Lemma admits a number of generalizations in various directions, in particular in the setting of automorphisms of von Neumann algebras. The noncommutative version of it due to Connes (with a much more complicated proof) can be formulated as follows. Let $\mathcal{W}$ be a finite von Neumann algebra with a normal trace μ and unit $\mathbf{1}$, $\mu(\mathbf{1}) = 1$, and let S be an aperiodic *-automorphism of $\mathcal{W}$ such that $\mu \circ S = \mu$. Then for all $\epsilon > 0$ and $n \in \mathbb{N}$ there is a family of mutually orthogonal and nonvanishing projections $\{p_1, ..., p_n\}$ in $\mathcal{W}$ such that $\sum_{i=1}^n p_i = \mathbf{1}$, and

$$\|S(p_1) - p_2\|_2 < \epsilon, \|S(p_2) - p_3\|_2 < \epsilon, ..., \|S(p_n) - p_1\|_2 < \epsilon,$$

where $\|x\|_2 = \mu(x^*x)^{1/2}$, see e.g. [13, Section I.2] and [30, Chapter 17]. Observe that under the assumptions above one has $\sigma(S) = \mathbb{T}$, and this fact played a role in the proof of the noncommutative Rokhlin Lemma in [13] (see also [30, Chapter 17]).

The Arveson's (and Connes') result has been put in a much broader setting here. In particular, within our framework, we are able to treat *arbitrary* bounded operators under Arveson's spectral assumption, and to obtain the next result on this way, proved as Theorem 5.1 in Section 5 below. Related statements can be found in [26].

Theorem 1.1. *Let T be a bounded linear operator on H. The following statements are equivalent.*

(i) $\mathbb{T}$ *belongs to the essential approximate point spectrum of* T;
(ii) *for all* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *there exists* $x \in H$ *such that*

$$|\langle T^m x, T^j x\rangle| < \varepsilon, \qquad 0 \le m, j \le n-1, m \ne j,$$

and

$$\frac{1}{2} \le \|T^j x\| \le 2, \qquad 0 \le j \le n-1;$$

(iii) *for all* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *there exists* $x \in H$ *such that*

$$x \perp T^j x, \qquad 1 \le j \le n-1,$$
$$|\langle T^m x, T^j x\rangle| < \varepsilon, \qquad 1 \le m, j \le n-1, m \ne j,$$
$$1-\varepsilon < \|T^j x\| < 1+\varepsilon, \qquad 0 \le j \le n-1,$$

and

$$\|T^n x - x\| < \varepsilon.$$

If T is unitary then Theorem 1.1 can be reduced formally to Arveson's theorem, however, our condition (iii) says much more. Moreover, in this case, we give also a related statement in the spirit of Connes, see Section 5 below.

Orthogonality relations for orbits of general bounded operators have also been studied before, in terms of a concept of weakly wandering vectors originating from ergodic theory. Recall that a vector x is called weakly wandering for a bounded operator T on H if there is a strictly increasing subsequence (n_k) (depending on x) such that the elements $(T^{n_k}x)$ are mutually orthogonal. Generalizing classical results due to Krengel and more recent ones (see e.g. [11] and [4]), one may prove for example that if T is power bounded, $\sigma(T) \cap \mathbb{T}$ is infinite, and $\sigma_p(T) \cap \mathbb{T}$ is empty, then the set of weakly wandering vectors of T is dense in H, see [28]. (In fact, one can omit the condition of power boundedness of T here.) While in contrast to Theorem 1.1 the sequence (n_k) is infinite, one has, in general, no control on it. Thus the results of this kind are essentially different from the ones in the present paper, although the technique has some points in common.

Another motivation to the study of the circles structure of the spectrum stems again from ergodic theory, namely from the research on mixing dynamical systems. Recently, using harmonic analysis arguments, Hamdan proved in [18] that if a unitary operator T is such that $T^n \to 0$ in the weak operator topology, then $\sigma(T) = \mathbb{T}$ if and only if for every $\epsilon > 0$ there exists a unit vector $x \in H$ satisfying

$$\sup_{n \ge 1} |\langle T^n x, x\rangle| < \epsilon.$$

The result was inspired by recent results due to Bashtanov and Ryzhikov on fine structure of mixing transformations, see for a relevant discussion [18]. Towards a sufficiency direction, our approach allows one to get rid of the unitarity assumption on T in Hamdan's theorem, to essentially weaken the spectral assumptions on T there, and to replace the vector x above by an infinite-dimensional subspace L of H. In particular, the following surprising result is proved (see Corollary 6.3 below).

Theorem 1.2. *Let T be a bounded linear operator on H be such that $T^n \to 0$ in the weak operator topology. If $\sigma(T) \supset \mathbb{T}$, then for every $\epsilon > 0$ there exists an infinite-dimensional subspace L of H such that*

$$\lim_{n\to\infty} \|P_L T^n P_L\| = 0 \qquad \textit{and} \qquad \sup_{n\geq 1} \|P_L T^n P_L\| < \epsilon,$$

where P_L is the orthogonal projection on L.

In other words, for each T with weakly vanishing powers there is a nontrivial block-decomposition of T^n in $H = L \oplus L^\perp$:

$$T^n = \begin{pmatrix} P_L T^n P_L & * \\ * & * \end{pmatrix}$$

such that the left upper corner vanishes *uniformly* !

Of course, as far as we consider *arbitrary* bounded operators, our approach is necessarily more delicate and involved than the ones in e.g. [3] and [18]. A particular novelty is that in our studies of orbits we rely on the numerical ranges methodology. The condition of orthogonality of elements from an orbit of a bounded operator T can be recasted in terms of the joint numerical range of the tuple $\mathcal{T} = (T, ..., T^n)$. (See Section 3 for more on that and related notions.) On the other hand, as we prove below, the joint numerical range $W(\mathcal{T})$ of $\mathcal{T}$ contains the interior of the essential joint numerical range $W_e(\mathcal{T})$ of $\mathcal{T}$. This and similar facts allow us to construct the desired elements from the (essential) approximate eigenvalues using inductive arguments. Other instances of these inductive arguments can be found e.g. in [27, Chapter 5]. The constructions are far from being straightforward, and we have to overcome several technical difficulties. To give a flavor of the results on numerical ranges proved in this paper, we formulate the following statement, proved in Section 4 (see Corollary 4.2 and Theorem 4.9 below). It is a heart matter for subsequent considerations.

Theorem 1.3. *Let $\mathcal{T} = (T_1, \dots, T_n), n \in \mathbb{N}$, be an n-tuple of bounded linear operators on H. Then $W(\mathcal{T})$ contains the interior of $W_e(\mathcal{T})$. If $\mathcal{T} = (T, \dots, T^n)$ for some bounded linear operator T on H, then the interior of $W_e(T, \dots, T^n)$ contains any tuple $(\lambda, \dots, \lambda^n)$ with λ from the interior of the polynomial convex hull of $\sigma(T)$.*

Theorem 1.3 can be considered as a partial generalization of the main result in [36, Theorem 2.2] dealing with numerical ranges of operators on Banach spaces. Note that while the result in [36] allows to find parts of the spectrum of $\mathcal{T}$ in the closure $\overline{W(\mathcal{T})}$ of $W(\mathcal{T})$, we may replace $\overline{W(\mathcal{T})}$ by a smaller and more transparent set $W(\mathcal{T})$, and this has direct implications for orbits orthogonality. Theorem 1.3 extends also a result from [2] to the setting of operator tuples.

We stress that while our results mentioned above can formally be considered as generalizations of a previous work, they are of a different nature since we are dealing with subspaces rather than elements of the Hilbert space here, and the generality of our setting necessitates the use of new ideas.

We remark finally that there are standard and related notions of wandering subspaces (and vectors) of $T \in B(H)$, that is subspaces L (or merely vectors) of H such that their orbits $T^n(L), n \in \mathbb{N}$, under T consist of mutually orthogonal elements. However, they seem to be too strong for any characterization in mere spectral terms. For a characterization of wandering vectors of unitary operators one may consult e.g. [33] and [15]. Wandering subspaces and related concepts are discussed in e.g. [23] where more references can be found.

2. Notation

It will be convenient to fix some of the notations in a separate section. In particular, we let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and $B(H)$ the space of all bounded linear operators on H. For a bounded linear operator T on H we denote by $\sigma(T)$ its spectrum, by $r(T)$ its spectral radius and by $N(T)$ its kernel.

For a closed set $K \subset \mathbb{C}^n$ we denote by ∂K the topological boundary of K, by $\overline{K}$ the closure of K, by $\operatorname{Int} K$ the interior of K, by $\operatorname{conv} K$ the convex hull of K, and by $\hat{K}$ the polynomial convex hull of K. If $K \subset \mathbb{C}$ then $\hat{K}$ is the union of K with all bounded components of the complement $\mathbb{C} \setminus K$. (In that case taking $\hat{K}$ can be viewed as filling "holes" that might exist in K.)

Finally, we let $\mathbb{T}$ stand for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

3. Preliminaries

We start with recalling certain basic notions and facts from the spectral theory of operator tuples on Hilbert spaces. They can be found e.g. in [27, Chapters II.9,10 and III.18,19]. See also [20] and [14].

In the following we consider an n-tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, $n \in \mathbb{N}$. Note that we do not in general assume that the operators T_j commute. For $x, y \in H$ we write shortly $\langle \mathcal{T}x, y\rangle = (\langle T_1 x, y\rangle, \dots, \langle T_n x, y\rangle) \in \mathbb{C}^n$ and $\mathcal{T}x = (T_1 x, \dots, T_n x) \in H^n$. Similarly for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ we write $\mathcal{T} - \lambda = (T_1 - \lambda_1, \dots, T - \lambda_n)$ and $\|\lambda\| = \max\{|\lambda_1|, \dots, |\lambda_n|\}$.

If $T_i, 1 \le i \le n$, mutually commute then for $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ we denote by $\sigma(\mathcal{T})$ its joint (Harte) spectrum. Recall that $\sigma(\mathcal{T})$ can be defined as the complement to the set of those $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for which

$$\sum_{i=1}^n L_i(T_i - \lambda_i) = \sum_{i=1}^n (T_i - \lambda_i) R_i = I$$

for some $L_i, R_i, 1 \le i \le n$, from the algebra $B(H)$. If $n = 1$ then the joint spectrum as above reduces to the usual spectrum of a single operator.

We will also be using a finer and somewhat more transparent notion of the approximate point spectrum $\sigma_\pi(\mathcal{T})$ of $\mathcal{T}$ described by

$$\sigma_\pi(\mathcal{T}) := \Big\{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n : \inf_{x\in H, \|x\|=1} \sum_{j=1}^n \|(T_j - \lambda_j)x\| = 0\Big\}.$$

It is well-known that $\sigma(\mathcal{T})$ and $\sigma_\pi(\mathcal{T})$ are non-empty compact subsets of $\mathbb{C}^n$ and $\sigma_\pi(\mathcal{T}) \subset \sigma(\mathcal{T})$.

There are also other joint spectra of n-tuples of commuting operators studied in the literature, for example the Taylor spectrum. However, in this paper we speak only about the polynomial convex hull of the joint spectrum which coincides for all reasonable joint spectra.

For $n \in \mathbb{N}$ let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ be an n-tuple of commuting operators. One can define the joint essential spectrum $\sigma_e(\mathcal{T})$ as the (Harte) spectrum of the n-tuple $(T_1 + \mathcal{K}(H), \dots, T_n + \mathcal{K}(H))$ in the Calkin algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of all compact operators on H.

For our purpose more important than the joint essential spectrum is the essential approximate point spectrum $\sigma_{\pi e}(\mathcal{T})$ which is the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that

$$\inf_{x\in M, \|x\|=1} \sum_{j=1}^n \|(T_j - \lambda_j)x\| = 0$$

for every subspace $M \subset H$ of finite codimension. Again $\sigma_{\pi e}(\mathcal{T}) \subset \sigma_e(\mathcal{T})$ and the polynomial convex hulls $\hat{\sigma}_e(\mathcal{T})$ and $\hat{\sigma}_{\pi e}(\mathcal{T})$ coincide (see [27, Corollary III.19.16]).

If $n = 1$ then $\sigma_e(T_1) = \{\lambda_1 \in \mathbb{C} : T_1 - \lambda_1 \text{ is not Fredholm}\}$ and $\sigma_{\pi e}(T_1) = \{\lambda_1 \in \mathbb{C} : T_1 - \lambda_1 \text{ is not upper semi-Fredholm}\}$. It is important that the topological boundary of $\partial\sigma(T_1)$ is contained in $\sigma_\pi(T_1)$. Analogously, $\partial\sigma_e(T_1) \subset \sigma_{\pi e}(T_1)$. (Such inclusions are not true any more for $n \geq 2$, see e.g. [35, Section 2.5]). Moreover, $\partial\sigma(T_1) \setminus \sigma_{\pi e}(T_1)$ and $\sigma(T_1) \setminus \hat{\sigma}_e(T_1)$ consist of isolated points of $\sigma(T_1)$ (in fact of eigenvalues of T_1 of finite multiplicity), see e.g. [20, p. 184] and [27, Theorem III.19.18]. Thus, in particular,

$$\mathbb{T} \subset \sigma(T_1),\ r(T_1) \leq 1 \quad \Longrightarrow \quad \mathbb{T} \subset \sigma_{\pi e}(T_1). \tag{3.1}$$

If $T \in B(H)$ and $\mathcal{T} = (T, T^2, \dots, T^n) \in B(H)^n$, then $\sigma(\mathcal{T}) = \{(\lambda, \dots, \lambda^n) : \lambda \in \sigma(T)\}$ and $\sigma_\pi(\mathcal{T}) = \{(\lambda, \dots, \lambda^n) : \lambda \in \sigma_\pi(T)\}$. Similar relations are true for the essential spectrum σ_e and essential approximate point spectrum $\sigma_{\pi e}$. For the essential spectrum theory in the realm of Hilbert spaces one may also consult [20] for the case of single operators and [14] for the case of n-tuples.

As in the case of a single operator, it is often useful to relate $\sigma(\mathcal{T})$ to a larger and easier computable set $W(\mathcal{T}) \subset \mathbb{C}^n$ called the joint numerical range of $\mathcal{T}$ and defined as

$$W(\mathcal{T}) = \{(\langle T_1x, x\rangle, ..., \langle T_nx, x\rangle) : x \in H, \|x\| = 1\}.$$

(Note that the definition of $W(\mathcal{T})$ makes sense for non-commuting tuples as well). The set $W(\mathcal{T})$ can be identified with a subset of $\mathbb{R}^{2n}$ if one identifies $\mathcal{T}$ with the $2n$-tuple of selfadjoint operators $(\operatorname{Re} T_1, \operatorname{Im} T_1, ..., \operatorname{Re} T_n, \operatorname{Im} T_n)$. Unfortunately, if $n > 1$, then $W(\mathcal{T})$ is not in general convex, see e.g [25].

As in the spectral theory, there is also a notion of the joint essential numerical range $W_e(\mathcal{T})$ associated to $\mathcal{T}$. For $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ we define $W_e(\mathcal{T})$ as the set of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exist an orthonormal sequence $(x_k) \subset H$ with

$$\lim_{k\to\infty} \langle T_j x_k, x_k\rangle = \lambda_j, \qquad j = 1, \dots, n.$$

Equivalently, $\lambda \in W_e(\mathcal{T})$ if for every subspace $M \subset H$ of finite codimension and every $\delta > 0$ there exists a unit vector $x \in M$ such that $\|\langle \mathcal{T}x, x\rangle - \lambda\| < \delta$.

Alternatively, $W_e(\mathcal{T})$ can be defined as

$$W_e(\mathcal{T}) := \bigcap \overline{W(T_1 + K_1, \dots, T_n + K_n)},$$

where the intersection is taken over all n-tuples $K_1, \dots, K_n$ of compact operators on H. Recall that $W_e(\mathcal{T})$ is a compact and, in contrast to $W(\mathcal{T})$, *convex* subset of $\overline{W(\mathcal{T})}$. Moreover, if $\mathcal{T}$ consists of commuting operators, then since $W_e(\mathcal{T})$ is convex, $\sigma_{\pi e}(\mathcal{T}) \subset W_e(\mathcal{T})$ and the convex hulls of $\sigma_e(\mathcal{T})$ and $\sigma_{\pi e}(\mathcal{T})$ coincide (see the proof of Corollary 4.4), one has $W_e(\mathcal{T}) \supset \operatorname{conv} \sigma_e(\mathcal{T})$.

For a comprehensive account of essential numerical ranges one may consult [25] and the references therein.

4. Spectra and numerical ranges for tuples

In the following we consider an n-tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, $n \in \mathbb{N}$. Note that we do not, in general, assume that the operators $T_j, 1 \le j \le n$, commute.

The next proposition will be instrumental in approximating numerical ranges by spectra, and in relating spectra to orthogonality relations.

Proposition 4.1. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, $k \in \mathbb{N} \cup \{0\}$, $r > 0$, and*

$$\{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) : \|\varepsilon\| \le r\} \subset W_e(\mathcal{T}).$$

Suppose $M \subset H$ is a subspace of a finite codimension, and $x \in M$ satisfies

$$\|x\|^2 = 1 - 2^{-k} \qquad \text{and} \qquad |\langle T_j x, x\rangle| \le r2^{-k-1}, \qquad j = 1, \dots, n.$$

Then there exists $x' \in M$ such that

$$\|x'\|^2 = 1 - 2^{-k-1}, \qquad \|x' - x\|^2 = 2^{-k-1} \quad \text{and} \quad |\langle T_j x', x'\rangle| \le r2^{-k-2}$$

for all $j = 1, \dots, n$,.

Consequently, there exists $w \in M$ such that

$$\|w\| = 1, \qquad \|w - x\| \le 3 \cdot 2^{-\frac{k}{2}-1} \qquad \text{and} \qquad \langle T_j w, w\rangle = 0$$

for all $j = 1, \dots, n$.

Proof. Let $\varepsilon = \langle \mathcal{T}x, x\rangle$, so that $\|\varepsilon\| \le \frac{r}{2^{k+1}}$ and $-2^{k+1}\varepsilon \in W_e(\mathcal{T})$. Note that

$$L := \bigvee\{x, T_jx, T_j^*x : j = 1, \dots, n\}$$

is a finite-dimensional subspace of H. Thus, by assumption, there exists a unit vector $u \in M \cap L^\perp$ such that

$$\|\langle T_ju, u\rangle + \varepsilon 2^{k+1}\| < \frac{r}{2}, \qquad j = 1, \dots, n.$$

Set $x' = x + 2^{-\frac{k+1}{2}}u$. Then

$$\|x'\|^2 = \|x\|^2 + \frac{1}{2^{k+1}} = 1 - 2^{-k-1},$$
$$\|x' - x\|^2 = 2^{-k-1},$$

and

$$|\langle T_jx', x'\rangle| = \left|\langle T_jx, x\rangle + \frac{1}{2^{k+1}}\langle T_ju, u\rangle\right| = \left|\varepsilon_j + \frac{1}{2^{k+1}}\langle T_ju, u\rangle\right| \le \frac{r}{2^{k+2}}$$

for all $1 \le j \le n$. This finishes the proof of the first part of the proposition.

To prove its second part, we construct w as the limit of an appropriate sequence $(x_m), m \ge k$. To construct the sequence, set $x_k = x \in M$. We have

$$\|x_k\|^2 = 1 - 2^{-k} \qquad \text{and} \qquad \|\langle \mathcal{T}x_k, x_k\rangle\| \le \frac{r}{2^{k+1}}.$$

If we put $x_{k+1} = x'$, then by the first part of the proposition,

$$\|x_{k+1}\|^2 = 1 - \frac{1}{2^{k+1}}, \qquad \|x_{k+1} - x_k\|^2 = \frac{1}{2^{k+1}} \quad \text{and} \quad \|\langle \mathcal{T}x_{k+1}, x_{k+1}\rangle\| \le \frac{r}{2^{k+2}}.$$

Thus, repeating the procedure above, we construct inductively vectors $x_m \in M, m \ge k$, such that

$$\|x_m\|^2 = 1 - \frac{1}{2^m}, \qquad \|x_{m+1} - x_m\|^2 = \frac{1}{2^{m+1}} \quad \text{and} \quad \|\langle \mathcal{T}x_m, x_m\rangle\| \le \frac{r}{2^{m+1}}.$$

Clearly the sequence (x_m) is Cauchy. Let w be its limit. By construction,

$$w \in M, \qquad \|w\| = 1, \qquad \langle T_jw, w\rangle = 0,$$

for all $1 \le j \le n$, and

$$\|w - x\| \le \sum_{m=k}^{\infty} \|x_{m+1} - x_m\| = 2^{-k/2}\frac{1}{\sqrt{2} - 1} < 3 \cdot 2^{-\frac{k}{2}-1}.$$

□

Proposition 4.1 implies in particular that points from the interior of $W_e(T_1, \dots, T_n)$ belong to $W(T_1, \dots, T_n)$ and, moreover, can be attained on any subspace of H of finite codimension.

Corollary 4.2. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$\operatorname{Int} W_e(\mathcal{T}) \subset W(\mathcal{T}).$$

Moreover, if $\lambda = (\lambda_1, \dots, \lambda_n) \in \operatorname{Int} W_e(\mathcal{T})$ then for every subspace $M \subset H$ of a finite codimension there exists $x \in M$ such that $\|x\| = 1$ and

$$\big(\langle T_1 x, x\rangle, \dots, \langle T_n x, x\rangle\big) = \lambda.$$

Proof. Without loss of generality we may assume that $\lambda = (0, \dots, 0)$ (by considering the n-tuple $(T_1 - \lambda_1, \dots, T_n - \lambda_n)$ instead of $\mathcal{T}$).

Let $k = 0$ and $x = 0$. Then Proposition 4.1 yields the statement. □

Another interesting consequence of Proposition 4.1 allows one to find a *joint* diagonal compression for $T_1, \dots, T_n$ to an infinite-dimensional subspace of H. This can be considered as non-commutative generalization of the technique (and statements) employed in e.g. [2], [8] and other papers dealing with compressions with help of essential numerical ranges. The statement below was proved in [2, p.440] for $n = 1$. (For $T \in B(H)$ the problem of characterizing $\lambda \in \mathbb{C}$ such that $PTP = \lambda P$ for an infinite rank projection P was posed in [20, p.190].)

Corollary 4.3. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \operatorname{Int} W_e(\mathcal{T})$. Then there exists an infinite-dimensional subspace L of H such that*

$$P_L T_j P_L = \lambda_j P_L, \qquad j = 1, \dots, n,$$

where P_L is the orthogonal projection on L.

Proof. Using Corollary 4.2, find a unit vector $x_1 \in H$ such that $\langle \mathcal{T} x_1, x_1\rangle = \lambda$. Construct inductively a sequence $(x_k) \subset H$ of unit vectors such that

$$x_{k+1} \perp \{x_m, T_j x_m, T_j^* x_m : 1 \le m \le k, 1 \le j \le n\}$$

and

$$\langle \mathcal{T} x_k, x_k\rangle = \lambda$$

for all $k \in \mathbb{N}$ using the fact that

$$\bigvee \{x_m, T_j x_m, T_j^* x_m : 1 \le m \le k, 1 \le j \le n\}$$

is a subspace of finite dimension. Let $L = \bigvee_{k=1}^{\infty} x_k$. Clearly L is an infinite-dimensional subspace with an orthonormal basis (x_k). Let $y \in L$. Then, in view of our construction of (x_k), it is easy to see that

$$\langle \mathcal{T} y, y\rangle = \lambda \|y\|^2.$$

Hence $P_L T_j P_L = \lambda_j P_L$ for all $1 \le j \le n$. □

The next result allows one also to describe a "large" subset of $W(\mathcal{T})$ in purely spectral terms.

Corollary 4.4. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ be a n-tuple of commuting operators. Then*

$$\operatorname{Int} \operatorname{conv} \sigma_e(\mathcal{T}) \subset W(\mathcal{T}).$$

Proof. Since $\partial\sigma_e(T) \subset \sigma_{\pi e}(T)$ and $\sigma_{\pi e}(T) \subset \sigma_e(T)$ (see [27, Proposition III.19.1]), we conclude that $\sigma_e(T)$ is the union of $\sigma_{\pi e}(T)$ and those bounded components ("holes") of $\sigma_{\pi e}$ that meet $\sigma_e(T)$ (cf. [20, p.183]). Taking into account that $W_e(\mathcal{T})$ is convex and it contains $\sigma_{\pi e}(\mathcal{T})$, we have

$$\operatorname{conv}\, \sigma_e(\mathcal{T}) = \operatorname{conv}\, \sigma_{\pi e}(\mathcal{T}) \subset W_e(\mathcal{T}).$$

So the statement follows from Corollary 4.2. □

To clarify further the interplay between joint spectra and numerical ranges, we will need a statement on approximation of points from polynomial hulls by powers of complex numbers formulated in Proposition 4.8. It relies on the two auxiliary lemmas given below.

Lemma 4.5. *Let $K \subset \mathbb{C}$ be a compact subset, $n \in \mathbb{N}$, $u \in \operatorname{Int} \hat{K}$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in K$ and $c_1, \dots, c_m \geq 0$ such that $\sum_{j=1}^m c_j = 1$ and*

$$\Big|\sum_{j=1}^m c_j \lambda_j^k - u^k\Big| < \delta, \qquad k = 1, \dots, n.$$

Proof. Consider the Banach space $C(K)$ of continuous functions on K with the sup-norm. Let $\mathcal{P}(K) \subset C(K)$ be the linear space of all polynomials. Define a linear functional $f : \mathcal{P}(K) \to \mathbb{C}$ by

$$f(p) = p(u), \qquad p \in \mathcal{P}(K).$$

Then $\|f\| = f(1_K) = 1$, where 1_K denotes the constant function equal to 1 on K. By the Hahn-Banach theorem f can be extended to a bounded linear functional $F : C(K) \to \mathbb{C}$ such that $\|F\| = F(1_K) = 1$. Since the norm of F is 1 and it is attained at 1_K, we infer by e.g. [9, p.80-81] that the functional F is positive.

By the Riesz theorem there exists a positive measure ν on K such that $\nu(K) = \|F\| = 1$ and $F(g) = \int_K g \,\mathrm{d}\,\nu$ for all $g \in C(K)$ (see also Remark 4.6). In particular,

$$\int_K z^k \mathrm{d}\,\nu(z) = F(z^k) = f(z^k) = u^k, \qquad k = 1, \dots, n.$$

Let

$$0 < \delta' < \frac{\delta}{n \max\{1, |w|^n : w \in K\}}.$$

Write $K = \bigcup_{j=1}^m K_j$ as a disjoint union of nonempty Borel sets K_j such that $\operatorname{diam} K_j < \delta'$. Choose $\lambda_j \in K_j$ and let $c_j = \nu(K_j)$ so that $\sum_{j=1}^m c_j = 1$.

For $z \in K_j$ and $1 \leq k \leq n$, we have

$$|z^k - \lambda_j^k| = |z - \lambda_j| \cdot |z^{k-1} + z^{k-2}\lambda_j + \dots + \lambda_j^{k-1}| \leq \delta' k \cdot \max\{|w|^{k-1} : w \in K_j\} < \delta.$$

Hence for every k, $1 \leq k \leq n$,

$$\Big|\sum_{j=1}^m c_j \lambda_j^k - u^k\Big| \leq \sum_{j=1}^m \Big| c_j \lambda_j^k - \int_{K_j} z^k d\nu(z)\Big| \leq \sum_{j=1}^m \Big|\int_{K_j} (\lambda_j^k - z^k) d\nu(z)\Big| < \sum_{j=1}^m c_j \delta = \delta.$$

□

Remark 4.6. In fact, one can construct a representing measure (in general, different from the measure ν) for the functional F above concentrated on the set ∂K, and it is a harmonic measure associated with the set $\mathbb{C} \setminus K$. However, we do not need these fine considerations, and we refer to e.g. [24, p.101-102] and [29, p.284-285, 321] for more details on that.

For $\rho > 0$ denote $\mathbb{T}_\rho = \{\lambda \in \mathbb{C} : |\lambda| = \rho\}$, so that $\mathbb{T}_1 = \mathbb{T}$.

Lemma 4.7. *Let $\rho > 0$ be fixed, and let $n \in \mathbb{N}$. There exists $r > 0$ such that for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$, $\|\varepsilon\| \le r$, there are $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \mathbb{T}_\rho$ and $c_1, \dots, c_m \ge 0$ satisfying $\sum_{j=1}^m c_j = 1$ and*

$$\sum_{j=1}^{m} c_j \lambda_j^k = \varepsilon_k, \qquad k = 1, \dots, n.$$

Proof. Let us prove first the next claim.

There exists $b_n > 0$ with the following property: If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$ then there are $s \in \mathbb{N}$, $\mu_1, \dots, \mu_s \in \mathbb{T}_\rho$ and $a_1, \dots, a_s \ge 0$ such that

$$\sum_{j=1}^{s} a_j \le b_n \|\varepsilon\| \qquad \text{and} \qquad \sum_{j=1}^{s} a_j \mu_j^k = \varepsilon_k, \qquad 1 \le k \le n.$$

We prove the claim by induction on n. For $n = 1$ set $b_1 = \rho^{-1}$. If $\varepsilon_1 \in \mathbb{C}$, $\varepsilon_1 = |\varepsilon_1| \cdot e^{2\pi i \varphi}$ for some $\varphi \in [0, 1)$, then take $\lambda_1 = \rho e^{2\pi i \varphi}$ and $c_1 = \frac{|\varepsilon_1|}{\rho}$. We have

$$c_1 \lambda_1 = |\varepsilon_1| \cdot e^{2\pi i \varphi} = \varepsilon_1 \qquad \text{and} \qquad c_1 = b_1 |\varepsilon_1|,$$

hence the claim clearly holds for $n = 1$.

Suppose that the claim is true for some integer $n - 1 \ge 1$, and prove it for n. Set

$$b_n = 2b_{n-1} + \rho^{-n}.$$

By the induction assumption, there exist $l \in \mathbb{N}$, $z_1, \dots, z_l \in \mathbb{T}_\rho$ and $\alpha_1, \dots, \alpha_l \ge 0$ such that

$$\sum_{j=1}^{l} \alpha_j \le b_{n-1} \|\varepsilon\|$$

and

$$\sum_{j=1}^{l} \alpha_j z_j^k = \varepsilon_k, \qquad k = 1, \dots, n - 1.$$

Let

$$\tilde{\varepsilon}_n = \varepsilon_n - \sum_{j=1}^{l} \alpha_j z_j^n.$$

Then

$$|\tilde{\varepsilon}_n| \leq |\varepsilon_n| + \rho^n \sum_{j=1}^{l} \alpha_j \leq \|\varepsilon\| + \rho^n b_{n-1} \|\varepsilon\|.$$

Write $\tilde{\varepsilon}_n = |\tilde{\varepsilon}_n| \cdot e^{2\pi i \varphi}$ for some $\varphi \in [0, 1)$ and set

$$\xi_j = \rho e^{2\pi i(\varphi + j)/n} \qquad \text{and} \qquad \beta_j = \frac{|\tilde{\varepsilon}_n|}{n\rho^n}, \qquad 1 \leq j \leq n.$$

If $1 \leq k \leq n-1$ then

$$\sum_{j=1}^{n} \beta_j \xi_j^k = \frac{|\tilde{\varepsilon}_n|}{n\rho^n} \rho^k e^{2\pi i k \varphi/n} \sum_{j=1}^{n} e^{2\pi i jk/n} = 0.$$

Similarly,

$$\sum_{j=1}^{n} \beta_j \xi_j^n = \frac{|\tilde{\varepsilon}_n|}{n\rho^n} \rho^n e^{2\pi i \varphi} \cdot n = \tilde{\varepsilon}_n.$$

Thus for every k, $1 \leq k \leq n-1$, we have

$$\sum_{j=1}^{l} \alpha_j z_j^k + \sum_{j=1}^{n} \beta_j \xi_j^k = \varepsilon_k$$

and

$$\sum_{j=1}^{l} \alpha_j z_j^n + \sum_{j=1}^{n} \beta_j \xi_j^n = \varepsilon_n - \tilde{\varepsilon}_n + \tilde{\varepsilon}_n = \varepsilon_n.$$

Finally,

$$\sum_{j=1}^{l} \alpha_j + \sum_{j=1}^{n} \beta_j \leq b_{n-1} \|\varepsilon\| + \frac{|\tilde{\varepsilon}_n|}{\rho^n} \leq \|\varepsilon\| \big(2b_{n-1} + \rho^{-n}\big) = b_n \|\varepsilon\|,$$

and the proof of the claim is finished.

Now let

$$r = b_n^{-1},$$

where b_n has been constructed in the claim above. If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$, $\|\varepsilon\| \leq r$, then from the claim it follows that there exist $s \in \mathbb{N}$, $\mu_1, \dots, \mu_s \in \mathbb{T}_\rho$ and $a_1, \dots, a_s \geq 0$ such that $\sum_{j=1}^{s} a_j \leq 1$ and

$$\sum_{j=1}^{s} a_j \mu_j^k = \varepsilon_k, \qquad k = 1, \dots, n.$$

Let $d = 1 - \sum_{j=1}^{s} a_j$. For $j = 1, \dots, n+1$ set

$$\mu_j' = \rho e^{2\pi i j/(n+1)} \in \mathbb{T}_\rho \qquad \text{and} \qquad a_j' = \frac{d}{n+1}.$$

Then

$$\sum_{j=1}^{s} a_j + \sum_{j=1}^{n+1} a_j' = 1,$$

and

$$\sum_{j=1}^{s} a_j \mu_j^k + \sum_{j=1}^{n+1} a_j'(\mu_j')^k = \varepsilon_k, \qquad 1 \le k \le n.$$

Thus we can take the $(s+n+1)$-tuple

$$(\mu_1, \dots, \mu_s, \mu_1', \dots, \mu_{n+1}')$$

as λ_j's, and the $(s+n+1)$-tuple

$$(a_1, \dots, a_s, a_1', \dots, a_{n+1}')$$

as c_j's. □

Proposition 4.8. *Let $K \subset \mathbb{C}$ be a compact set and $0 \in \operatorname{Int} \hat{K}$. Let $n \in \mathbb{N}$ and $\delta > 0$. Then there exists $r > 0$ such that the following is true. For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$ with $\|\varepsilon\| \le r$, there are $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in K$ and $c_1, \dots, c_m \ge 0$ satisfying $\sum_{j=1}^m c_j = 1$ and*

$$\Big| \sum_{j=1}^{m} c_j \lambda_j^k - \varepsilon_k \Big| < \delta, \qquad k = 1, \dots, n.$$

Proof. Let $\rho > 0$ be such that $\mathbb{T}_\rho \subset \operatorname{Int} \hat{K}$. Let $r > 0$ be the number constructed in Lemma 4.7.

If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$, $\|\varepsilon\| \le r$, then by Lemma 4.7 there exist $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \mathbb{T}_\rho$ and $c_1, \dots, c_m \ge 0$ such that $\sum_{j=1}^m c_j = 1$ and

$$\sum_{j=1}^{m} c_j \lambda_j^k = \varepsilon_k, \qquad k = 1, \dots, n.$$

By Lemma 4.5, for every $j, 1 \le j \le m$, there exist $m_j \in \mathbb{N}$, $\lambda_{j,1}, \dots, \lambda_{j,m_j} \in K$ and $c_{j,1}, \dots, c_{j,m_j} \ge 0$ such that

$$\sum_{l=1}^{m_j} c_{j,l} = 1$$

and

$$\Big| \sum_{l=1}^{m_j} c_{j,l} \lambda_{j,l}^k - \lambda_j^k \Big| < \delta, \qquad k = 1, \dots, n.$$

Hence

$$\sum_{j=1}^{m} \sum_{l=1}^{m_j} c_j c_{j,l} = 1$$

and

$$\Big| \sum_{j=1}^{m} \sum_{l=1}^{m_j} c_j c_{j,l} \lambda_{j,l}^k - \varepsilon_k \Big| \le \sum_{j=1}^{m} c_j \Big| \sum_{l=1}^{m_j} c_{j,l} \lambda_{j,l}^k - \lambda_j^k \Big| < \delta.$$

□

Now we are ready to prove the statement which will also be basic for constructions of orbits with orthogonality properties, and will complement Proposition 4.1.

Theorem 4.9. *Let $T \in B(H)$ and let $\lambda \in \operatorname{Int} \hat{\sigma}(T)$. Then*

$$(\lambda, \lambda^2, \dots, \lambda^n) \in \operatorname{Int} W_e(T, T^2, \dots, T^n).$$

for all $n \in \mathbb{N}$.

Proof. Assume first that $\lambda = 0$. Then

$$0 \in \operatorname{Int} \hat{\sigma}(T) = \operatorname{Int} \hat{\sigma}_e(T) = \operatorname{Int} \hat{\sigma}_{\pi e}(T)$$

(see Section 3 or [27, Corollary III. 19.16 and Theorem III.19.18]).

We apply Proposition 4.8 to the compact set $K = \sigma_{\pi e}(T)$. Let $r > 0$ be given by Proposition 4.8. Let $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{C}$, $\max_{1 \le j \le n} |\varepsilon_j| \le r$ and $\delta > 0$. Let $M \subset H$ be a subspace of a finite codimension. We show that there exists a unit vector $x \in M$ such that

$$|\langle T^j x, x\rangle - \varepsilon_j| < \delta, \qquad j = 1, \dots, n.$$

By Proposition 4.8, there exist $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \sigma_{\pi e}(T)$ and numbers $c_i \ge 0$ with $\sum_{i=1}^m c_i = 1$ such that

$$\Big|\sum_{i=1}^m c_i \lambda_i^j - \varepsilon_j\Big| < \delta, \qquad j = 1, \dots, n.$$

Let

$$0 < \delta' < \frac{\delta}{n \cdot \max\{1, \|T\|^n\}}.$$

Since $\lambda_i \in \sigma_{\pi e}(T), 1 \le i \le m$, we can find inductively unit vectors $x_i \in M$ such that

$$\begin{aligned} x_i \perp x_k, &\qquad i \ne k,\ 1 \le i, k \le m, \\ x_i \perp T^j x_k, &\qquad j = 1, \dots, n,\ i \ne k,\ 1 \le i, k \le m, \\ \|T x_i - \lambda_i x_i\| \le \delta', &\qquad 1 \le i \le m. \end{aligned}$$

Note that for every $n \in \mathbb{N}$ and all j such that $1 \le j \le n$ we then have

$$\|T^j x_i - \lambda_i^j x_i\| \le \delta' n \max\{1, \|T\|^n\} < \delta.$$

Set $x = \sum_{i=1}^m c_i^{1/2} x_i$. Then $x \in M$ and $\|x\|^2 = \sum_{i=1}^m c_i = 1$. If $1 \le j \le n$ then

$$\begin{aligned}|\langle T^j x, x\rangle - \varepsilon_j| =& \Big|\sum_{i=1}^m c_i \langle T^j x_i, x_i\rangle - \varepsilon_j\Big| \\ \le& \sum_{i=1}^m c_i \|T^j x_i - \lambda_i^j x_i\| + \Big|\sum_{i=1}^m c_i \langle \lambda_i^j x_i, x_i\rangle - \varepsilon_j\Big| \\ \le& \delta + \Big|\sum_{i=1}^m c_i \lambda_i^j - \varepsilon_j\Big| \\ \le& 2\delta.\end{aligned}$$

Since $\delta > 0$ and $M \subset H$, $\operatorname{codim} M < \infty$ were arbitrary, we have $(0, \dots, 0) \in \operatorname{Int} W_e(T, T^2, \dots, T^n)$.

Let now $\lambda \in \operatorname{Int} \hat\sigma(T)$ be arbitrary, and set $S = T - \lambda$. Thus, $0 \in \operatorname{Int} \hat\sigma(S)$, and then, as we have proved above,

$$(0, \dots, 0) \in \operatorname{Int} W_e(S, S^2, \dots, S^n).$$

Observe that for each n,

$$(T, T^2, \dots, T^n) = \big(S + \lambda, S^2 + 2\lambda S + \lambda^2, \dots, \sum_{j=0}^n \binom{n}{j} S^j \lambda^{n-j}\big).$$

Let the mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ be defined by

$$G(z_1, \dots, z_n) = (z_1 + \lambda, z_2 + 2\lambda z_1 + \lambda^2, \dots, \sum_{j=1}^n \binom{n}{j} z_j \lambda^{n-j} + \lambda^n).$$

Note that the mapping

$$(z_1, \dots, z_n) \mapsto G(z_1, \dots, z_n) - (\lambda, \lambda^2, \dots, \lambda^n)$$

is linear and invertible (since it is determined by an upper triangular matrix with non-zero diagonal). So G maps any neighbourhood of $(0, \dots, 0)$ onto a neighbourhood of $(\lambda, \lambda^2, \dots, \lambda^n)$. Using the definition of $W_e(T, T^2, \dots, T^n)$, it is easy to see that

$$W_e(T, T^2, \dots, T^n) = \big\{G(z_1, \dots, z_n) : (z_1, \dots, z_n) \in W_e(S, S^2, \dots, S^n)\big\}.$$

(Note that a similar relation holds also for $W(T, \dots, T^n)$). Hence, we infer that

$$(\lambda, \lambda^2, \dots, \lambda^n) \in \operatorname{Int} W_e(T, \dots, T^n),$$

and the theorem follows. □

The following corollary is an immediate consequence of Theorem 4.9 and Corollary 4.2.

Corollary 4.10. *Let $T \in B(H)$ and let $\lambda \in \operatorname{Int} \hat{\sigma}(T)$. Then*

$$(\lambda, \lambda^2, \dots, \lambda^n) \in W(T, T^2, \dots, T^n)$$

for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$ there exists an infinite-dimensional subspace $L \subset H$ such that

$$P_L T^j P_L = \lambda^j P_L, \qquad j = 1, \dots, n,$$

where P_L is the orthogonal projection on L.

5. Circles in the spectrum and orthogonality

In this section we characterize operators having the unit circle in their spectra by means of orthogonality (and "almost orthogonality") properties of their orbits. The next statement is a very general result of this kind. Its proof is based on numerical ranges consideration from the previous section.

Theorem 5.1. *Let $T \in B(H)$. The following statements are equivalent.*
(i) $\mathbb{T} \subset \sigma_{\pi e}(T)$;
(ii) for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists $x \in H$ such that

$$|\langle T^m x, T^j x\rangle| < \varepsilon, \qquad 0 \le m, j \le n-1, m \ne j,$$

and

$$\frac{1}{2} \le \|T^j x\| \le 2, \qquad 0 \le j \le n-1;$$

(iii) for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists a unit vector $x \in H$ such that

$$\begin{aligned} x \perp T^j x, &\qquad 1 \le j \le n-1, \\ |\langle T^m x, T^j x\rangle| < \varepsilon, &\qquad 1 \le m, j \le n-1, m \ne j, \\ 1-\varepsilon < \|T^j x\| < 1+\varepsilon, &\qquad 0 \le j \le n-1, \end{aligned}$$

and

$$\|T^n x - x\| < \varepsilon.$$

Proof. The implication (iii)⇒(ii) is obvious.

(ii)⇒(i): Let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$. For $k \in \mathbb{N}$ let $\varepsilon = k^{-3}$ and $n = (k+1)^2$. Let x_k be a vector x satisfying (ii) for such ϵ and n.

Set

$$y_{k,0} = x_k + \lambda^{-1} T x_k + \cdots + \lambda^{-k+1} T^{k-1} x_k.$$

Then

$$\|(T-\lambda) y_{k,0}\| = \left\| -\lambda x_k + \lambda^{-k+1} T^k x_k \right\| \le 4$$

and

$$\begin{aligned}\|y_{k,0}\|^2 &= \sum_{j,m=0}^{k-1} \langle \lambda^{-j} T^j x_k, \lambda^{-m} T^m x_k \rangle \\ &\geq \sum_{j=0}^{k-1} \|T^j x_k\|^2 - \sum_{\substack{0\leq j,m \leq k-1 \\ j\neq m}} \left|\langle \lambda^{-j} T^j x_k, \lambda^{-m} T^m x_k \rangle\right| \\ &\geq \frac{k}{4} - k^2\varepsilon \\ &= \frac{k}{4} - k^{-1}.\end{aligned}$$

Let $u_{k,0} = \frac{y_{k,0}}{\|y_{k,0}\|}$. Then $\|u_{k,0}\| = 1$ and $\lim_{k\to\infty} \|(T-\lambda)u_{k,0}\| = 0$. Hence $\lambda \in \sigma_\pi(T)$.

Suppose on the contrary that $\lambda \notin \sigma_{\pi e}(T)$. Then, by [27, Theorem III.16.8], the operator $T-\lambda$ is upper semi-Fredholm, that is $\dim N(T-\lambda) < \infty$ and $T \restriction_{N(T-\lambda)^\perp}$ is bounded below. Let P be the orthogonal projection onto $N(T-\lambda)$. Let $x_k, y_{k,0}$ and $u_{k,0}$ be as above. Then $(T-\lambda)u_{k,0} \to 0, k \to \infty$. Since $(T-\lambda)P = 0$, we also have

$$(T-\lambda)(I-P)u_{k,0} \to 0, \quad k \to \infty,$$

and so $(I-P)u_{k,0} \to 0, k \to \infty$. Since the unit ball in $N(T-\lambda)$ is compact, we can assume (by passing to a subsequence if necessary) that $Pu_{k,0} \to v_0, k \to \infty$, and $v_0 \in N(T-\lambda)$. Hence

$$u_{k,0} \to v_0, \quad k \to \infty, \qquad \text{and} \qquad \|v_0\| = 1.$$

For $j = 1, \dots, k$ set

$$y_{k,j} = T^{kj} y_{k,0} \qquad \text{and} \qquad u_{k,j} = \frac{y_{k,j}}{\|y_{k,j}\|}.$$

In the same way as for $u_{k,0}$ one can show that

$$\lim_{k\to\infty} (T-\lambda)u_{k,j} = 0$$

for all $j \in \mathbb{N}$. As above one can assume that

$$\lim_{k\to\infty} u_{k,j} = v_j \in N(T-\lambda),$$

where $\|v_j\| = 1$ for all j. Moreover, if $j \neq m$ then

$$|\langle u_{k,j}, u_{k,m} \rangle| \leq \sum_{s,s'=0}^{k-1} |\langle T^{s+kj} x_k, T^{s'+km} x_k \rangle| \leq k^2 \varepsilon = k^{-1}.$$

So $\langle v_j, v_m \rangle = 0$ for all $j, m \in \mathbb{N}, j \neq m$. Hence $\dim N(T-\lambda) = \infty$, a contradiction.

(i)⇒(iii): Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be fixed. Note that $\|T\| \geq 1$.

Using the assumption that $\mathbb{T} \subset \sigma_{\pi e}(T)$, find inductively unit vectors $u_0, u_1, \dots, u_{n-1}$ such that

$$\langle T^j u_k, T^{j'} u_{k'} \rangle = 0, \qquad 0 \le k, k' \le n-1, k \ne k', 0 \le j, j' \le n-1,$$

and

$$\|Tu_k - e^{2\pi ik/n} u_k\| < \frac{\varepsilon}{4n^{3/2}\|T\|^{2n}}, \qquad 0 \le k \le n-1.$$

For $1 \le j \le n-1$ and $0 \le k \le n-1$ we have

$$\begin{aligned}
&\|T^j u_k - e^{2\pi ikj/n} u_k\| \\
&\le \left\|T^{j-1} + T^{j-2} e^{2\pi ik/n} + \cdots + e^{2\pi ik(j-1)/n}\right\| \cdot \|Tu_k - e^{2\pi ik/n} u_k\| \\
&\le \frac{\varepsilon}{4n^{1/2}\|T\|^n}.
\end{aligned}$$

Set

$$v := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} u_k.$$

Then $\|v\| = 1$.

If $0 \le j \le n$ then

$$\left\|T^j v - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ikj/n} u_k\right\| \le \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \|T^j u_k - e^{2\pi ikj/n} u_k\| \le \frac{\varepsilon}{4\|T\|^n} \le \varepsilon/4.$$

So for $0 \le j, m \le n-1, j \ne m$ it follows that

$$\begin{aligned}
|\langle T^j v, T^m v\rangle| \le &\left\|T^j v - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ijk/n} u_k\right\| \cdot \|T^m v\| \\
+&\left\|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ijk/n} u_k\right\| \cdot \left\|T^m v - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi imk/n} u_k\right\| + \frac{1}{n}\left|\sum_{k=0}^{n-1} e^{2\pi i(j-m)k/n}\right| \\
\le &\|T^m\| \cdot \frac{\varepsilon}{4\|T\|^n} + \frac{\varepsilon}{4\|T\|^n} \\
\le &\varepsilon/2.
\end{aligned}$$

Similarly,

$$\|T^n v - v\| = \left\|T^n v - \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ik} u_k\right\| < \frac{\varepsilon}{4\|T\|^n} \le \varepsilon/4.$$

Finally,

$$\left\|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi ijk/n} u_k\right\| = 1, \qquad j = 0, 1, \dots, n-1,$$

and so for every j such that $1 \le j \le n-1$,

$$1 - \varepsilon/4 \le \|T^j v\| \le 1 + \varepsilon/4, .$$

Choose now $c \in \mathbb{N}$ such that

$$32 \cdot 2^{-c/2} \|T\|^n < \varepsilon,$$

and let $w = (1-2^{-c})^{1/2} v$. Then

$$\|w\|^2 = 1 - 2^{-c} \qquad \text{and} \quad |\langle T^j w, w\rangle| \le |\langle T^j v, v\rangle| < \varepsilon/2$$

for each $1 \le j \le n-1$. Since $\mathbb{T} \subset \sigma_{\pi e}(T)$, we have

$$0 \in \operatorname{Int} \hat{\sigma}_e(T).$$

Hence, by Theorem 4.9, it follows that

$$(0, \dots, 0) \in \operatorname{Int} W_e(T, \dots, T^{n-1}).$$

Then Proposition 4.1 implies that there exists a unit vector $x \in H$ such that

$$\langle T^j x, x\rangle = 0, \qquad j = 1, \dots, n-1, \quad \text{and} \quad \|x - w\| = 3 \cdot 2^{-c/2}.$$

So

$$\|x-v\| \le \|x-w\| + \|w-v\| \le 3 \cdot 2^{-c/2} + (1 - \sqrt{1-2^{-c}}) \le 4 \cdot 2^{-c/2} < \frac{\varepsilon}{8\|T\|^n}.$$

If $0 \le j \le n-1$ then

$$\|T^j x\| \le \|T^j v\| + \|T^j x - T^j v\| \le 1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < 1 + \varepsilon,$$

and similarly,

$$\|T^j x\| \ge \|T^j v\| - \|T^j x - T^j v\| \ge 1 - \frac{\varepsilon}{4} - \frac{\varepsilon}{8} > 1 - \varepsilon.$$

Moreover,

$$\|T^n x - x\| \le \|T^n x - T^n v\| + \|T^n v - v\| + \|v - x\| \le (\|T\|^n + 1)\|x - v\| + \varepsilon/4 < \varepsilon.$$

Finally, for $1 \le j, m \le n-1, j \ne m$,

$$\begin{aligned} |\langle T^j x, T^m x\rangle| &\le \|T^j x - T^j v\| \cdot \|T^m x\| + \|T^j v\| \cdot \|T^m x - T^m v\| + |\langle T^j v, T^m v\rangle| \\ &\le 4\|T\|^n \cdot \|x - v\| + \varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

Hence x satisfies all conditions of (iii). □

The following result shows that under mild assumptions one can replace essential spectrum by spectrum in Theorem 5.1.

Theorem 5.2. *Let $T \in B(H)$ satisfy $r(T) \le 1$. The following statements are equivalent.*

(i) $\mathbb{T} \subset \sigma(T)$.

(ii) *for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists a unit vector $x \in H$ such that*

$$|\langle T^m x, T^j x\rangle| < \varepsilon, \qquad 0 \le m, j \le n-1, m \ne j,$$

and

$$\|T^n x - x\| < \varepsilon.$$

(iii) *for every* $\varepsilon > 0$ *and every* $n \in \mathbb{N}$ *there exists a unit vector* $x \in H$ *such that*

$$\begin{aligned} x \perp T^j x, \qquad & 1 \le j \le n-1, \\ |\langle T^m x, T^j x\rangle| < \varepsilon, \qquad & 1 \le m, j \le n-1, m \ne j, \\ 1-\varepsilon < \|T^j x\| < 1+\varepsilon, \qquad & 1 \le j \le n-1, \end{aligned}$$

and

$$\|T^n x - x\| < \varepsilon.$$

Proof. The implication (iii)⇒(ii) is obvious.

(i)⇒(iii): Since $\mathbb{T} \subset \sigma(T)$ and $r(T) \le 1$, we infer by (3.1) that $\mathbb{T} \subset \sigma_{\pi e}(T)$. So (iii) follows from Theorem 5.1.

(ii)⇒(i): Let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$. For $k \in \mathbb{N}$ fix any $n \ge k$ such that $|\lambda^n - 1| < k^{-1}$, and let $\varepsilon = n^{-3}$. Let x_k be the vector x satisfying (ii) for n and ϵ as above.

Set

$$y_k = x_k + \lambda^{-1} T x_k + \dots + \lambda^{-n+1} T^{n-1} x_k.$$

Then

$$\begin{aligned} \|(T-\lambda)y_k\| &= \|-\lambda x_k + \lambda^{-n+1} T^n x_k\| \\ &= \left\|-x_k + \lambda^{-n} T^n x_k\right\| \\ &\le \|-x_k + \lambda^{-n} x_k\| + \left\|-\lambda^{-n} x_k + \lambda^{-n} T^n x_k\right\| \\ &\le 2k^{-1}, \end{aligned}$$

and

$$\begin{aligned} \|y_k\|^2 &= \sum_{j,j'=0}^{n-1} \langle \lambda^{-j} T^j x_k, \lambda^{-j'} T^{j'} x_k\rangle \\ &= \sum_{j=0}^{n-1} \|T^j x_k\|^2 + \sum_{\substack{0 \le j,j' \le n-1 \\ j \ne j'}} \langle \lambda^{-j} T^j x_k, \lambda^{-j'} T^{j'} x_k\rangle \\ &\ge \|x_k\|^2 - n^2 \varepsilon \\ &\ge 1 - 1/k. \end{aligned}$$

Hence $\lambda \in \sigma_\pi(T) \subset \sigma(T)$. □

Now we turn to the case of unitary T. The next corollary of Theorem 5.2 is a strengthening of Arveson's theorem from [3] (for the discrete group $(T^n), n \in \mathbb{Z}$) formulated in the introduction.

Theorem 5.3. *Let T be a unitary operator on H. The following statements are equivalent.*

(i) $\sigma(T) = \mathbb{T}$.

(ii) *for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists a unit vector $x \in H$ such that*

$$x \perp T^j x, \qquad 1 \le j \le n-1,$$
$$|\langle T^m x, T^j x\rangle| < \varepsilon, \qquad 1 \le m, j \le n-1, i \neq j,$$

and

$$\|T^n x - x\| < \varepsilon.$$

(iii) *For every $n \in \mathbb{N}$ there exists a unit vector $x \in H$ such that the vectors $x, Tx, T^2, \dots, T^n x$ are mutually orthogonal.*

(iv) *For every $n \in \mathbb{N}$,*

$$(0, \dots, 0) \in \overline{W(T, T^2, \dots, T^n)}.$$

Proof. The equivalence (i)⇔(ii) was proved in the previous theorem.

The implication (ii)⇒(iii) follows from the fact that T is unitary and (iii)⇒(iv) is obvious.

(iv)⇒(i): See Theorem 5.1, implication (ii)⇒(i). □

We finish this section with our operator analogue of Connes' version of the Rokhlin Lemma. Note that we do not assume that T below is unitary.

Theorem 5.4. *Let $T \in B(H)$. The following statements are equivalent.*

(i) $\mathbb{T} \subset \sigma_{\pi e}(T)$;

(ii) for all $\varepsilon > 0$, $n > \max(4\|T\|^2\varepsilon^{-2}, 1)$ and $u \in H$, $\|u\| = 1$ there exist orthonormal vectors $w_0, \dots, w_{n-1} \in H$ such that

$$\|Tw_j - w_{j+1}\| < \varepsilon \qquad 0 \le j \le n-2, \qquad \|Tw_{n-1} - w_0\| < \varepsilon,$$

and

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w_j = u.$$

Proof. (ii)⇒(i): Let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$, and let $u \in H$, $\|u\| = 1$ be arbitrary. For $k \in \mathbb{N}$ choose any $n > \max(4\|T\|^2\varepsilon^{-2}, k^2)$. Suppose that $w_0, \dots, w_{n-1}$ satisfy (ii) with $\varepsilon = k^{-1}$, and set

$$y_{k,0} = w_0 + \lambda^{-1} w_1 + \cdots + \lambda^{-k+1} w_{k-1}.$$

Then $\|y_{k,0}\| = \sqrt{k}$ and

$$\begin{aligned}
&\|(\lambda - T)y_{k,0}\| \\
=&\left\|\lambda w_0 + (w_1 - Tw_0) + \cdots + \lambda^{-k+2}(w_{k-1} - Tw_{k-2}) + \lambda^{-k+1} T w_{k-1}\right\| \\
\le&\|w_0\| + (k-1)\varepsilon + \|Tw_{k-1}\| \\
\le&1 + (k-1)\varepsilon + \|Tw_{k-1} - w_k\| + \|w_k\| \\
\le&2 + k\varepsilon \\
=&3,
\end{aligned}$$

so that

$$\lim_{k\to\infty}\Big\|(T-\lambda)\frac{y_{k,0}}{\|y_{k,0}\|}\Big\|=0.$$

Consider then vectors $y_{k,1},\dots,y_{k,k}, k\in\mathbb{N}$, where

$$y_{k,m}=w_{mk}+\lambda^{-1}w_{mk+1}+\dots+\lambda^{-k+1}w_{mk+k-1},\qquad 1\le m\le k.$$

Analogously to the above one can show that

$$\lim_{k\to\infty}\Big\|(T-\lambda)\frac{y_{k,m}}{\|y_{k,m}\|}\Big\|=0,$$

for all m such that $1\le m\le k$. Moreover, $y_{k,m}\perp y_{k,m'}$ for all k,m and m' such that $0\le m,m'\le k, m\ne m'$. Thus, arguing as in the proof of Theorem 5.1, (ii)⇒(i), one assumes that $\lambda\notin\sigma_{\pi e}(T)$, and arrives at a contradiction.

(i)⇒(ii): Let $\varepsilon>0$ and $n>\max(4\|T\|^2\varepsilon^{-2},1)$. Fix $u\in H$ with $\|u\|=1$ and set $\lambda=e^{2\pi i/n}$. Choose

$$\varepsilon'\in\Big(0,\frac{1}{\sqrt{n}}\Big(\varepsilon-\frac{2\|T\|}{\sqrt{n}}\Big)\Big),$$

and let $u_0=u$. Using the fact that $\sigma_{\pi e}(T)\supset\mathbb{T}$, choose inductively orthonormal vectors $u_1,\dots,u_{n-1}$ such that $u_k\perp u$ and $\|(T-\lambda^k)u_k\|<\varepsilon'$ for $k=1,\dots,n-1$.

For $j=0,\dots,n-1$ set

$$w_j=\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\lambda^{jk}u_k.$$

Then the vectors $w_0,\dots,w_{n-1}$ are orthonormal and

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}w_j=u.$$

For $0\le j\le n-2$ we have

$$\begin{aligned}\|Tw_j-w_{j+1}\|&\le\frac{1}{\sqrt{n}}\Big(\|Tu_0-u_0\|+\sum_{k=1}^{n-1}\|\lambda^{jk}(Tu_k-\lambda^ku_k)\|\Big)\\&\le\frac{1}{\sqrt{n}}(1+\|T\|+n\varepsilon')\\&<\varepsilon.\end{aligned}$$

Similarly, $\|Tw_{n-1}-w_0\|<\varepsilon$. □

In view of (3.1) the following corollary of Theorem 5.4 is immediate.

Corollary 5.5. *Let $T\in B(H)$ be such that $r(T)\le 1$. Then the following statements are equivalent.*

(i) $\mathbb{T}\subset\sigma(T)$.

(ii) for all $\varepsilon > 0$, $n > \max(4\|T\|^2\varepsilon^{-2}, 1)$ *and* $u \in H$, $\|u\| = 1$ *there exist orthonormal vectors* $w_0, \dots, w_{n-1} \in H$ *such that*

$$\|Tw_j - w_{j+1}\| < \varepsilon, \qquad 0 \le j \le n-2, \qquad \|Tw_{n-1} - w_0\| < \varepsilon$$

and

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w_j = u.$$

Remark 5.6. Note that if we do not require the property $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w_j = u$ then there is an alternative direct approach to the construction of vectors w_j as above. If T is unitary, $\varepsilon > 0$, $n > \max(2\pi\varepsilon^{-1}, 1)$, then for every $w_0 \in H$, $\|w_0\| = 1$, we construct vectors $w_1, \dots, w_{n-1}$ such that

$$\|Tw_j - w_{j+1}\| < \varepsilon, \qquad j = 0, 1, \dots, n-2, \qquad \|Tw_{n-1} - w_0\| < \varepsilon,$$

and, moreover,

$$\sum_{j=0}^{n-1} w_j = 0.$$

Without loss of generality we may assume that T is cyclic. Thus we may assume that $H = L^2(\mathbb{T}, \nu)$ for some probability measure ν and $T \in B(H)$ is defined by

$$(Tx)(z) = zx(z), \qquad x \in L^2(\mathbb{T}, \nu), \quad z \in \mathbb{T} \quad \text{a.e.}$$

For $z = e^{is}$ with $s \in [0, 2\pi)$ and $n \in \mathbb{N}, n \ge 2$, fix $t(z) \in [0, 2\pi/n]$ such that $s + t(z) \in \{\frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{2\pi(n-1)}{n}\}$ (note that 0 and 2π are not elements of this set). Clearly the function $z \mapsto t(z)$ is measurable (it has only a finite number of discontinuity points).

For $j = 1, \dots, n$ define $w_j(z) = w_0(z) \cdot z^j e^{ijt(z)}$, $z \in \mathbb{T}$. Clearly $w_j \in H$, $\|w_j\| = 1$ and $w_n = w_0$. For $j = 0, \dots, n-1$ we have

$$\begin{aligned} \|Tw_j - w_{j+1}\|^2 &= \int_{\mathbb{T}} |w_0(z) z^{j+1}(e^{ijt(z)} - e^{i(j+1)t(z)})|^2 d\nu(z) \\ &\le \int_{\mathbb{T}} |w_0(z)|^2 \cdot |1 - e^{it(z)}|^2 \, d\nu(z) \le \Big(\frac{2\pi}{n}\Big)^2. \end{aligned}$$

So

$$\|Tw_j - w_{j+1}\| \le \frac{2\pi}{n} < \varepsilon.$$

If $z = e^{is}, s \in [0, 2\pi)$, then

$$\sum_{j=0}^{n-1} w_j(z) = \sum_{j=0}^{n-1} w_0(z) e^{ijs} e^{ijt(z)} = 0.$$

6. Asymptotics of powers compressions

In this section we apply our numerical ranges ideology to study of the interplay between the circle structure of the spectrum and the asymptotic properties of orbits. It appears that the property $\sigma(T) \supset \mathbb{T}$ strengthens the convergence of powers of a bounded linear operator T on H on an appropriate, "large" subspace of H. In particular, we recover and complement the main result from [18] and prove much more general statements. The statements allow one to pass from the convergence of (T^n) to zero in the weak operator topology to the convergence to zero in the uniform operator topology of compressions of (T^n) to an infinite-dimensional subspace. At the same subspace, the norms of the compressions are as small as we please. This property generalizes a similar property just for a single weak orbit of (T^n) introduced in [18].

First, we will need the following numerical ranges lemma.

Lemma 6.1. *Let $T \in B(H)$ be such that $T^n \to 0$ in the weak operator topology. Suppose that for all $n \in \mathbb{N}$,*

$$(0, \dots, 0) \in W_e(T, \dots, T^n). \tag{6.1}$$

Let $A \subset H$ be a finite set, $\varepsilon > 0$ and $M \subset H$ be a subspace of a finite codimension. Then there exists a unit vector $x \in M$ such that

$$\sup_{n\ge1} |\langle T^n x, x\rangle| \le \varepsilon, \quad \sup_{n\ge1} |\langle T^n x, a\rangle| \le \varepsilon, \quad \textit{and} \quad \sup_{n\ge1} |\langle T^{*n} x, a\rangle| \le \varepsilon,$$

for all $a \in A$.

Proof. Clearly T is power bounded by the uniform boundedness principle. Let $K = \sup\{\|T^n\| : n = 0, 1, \dots\}$. It is apparent that also $T^{*n} \to 0$ in the weak operator topology. Without loss of generality we may assume that $\max\{\|a\| : a \in A\} \le 1$.

Choose $s \in \mathbb{N}$ such that $s > 16K^2\varepsilon^{-2}$, and set formally $n_0 = 0$.

Choose $u_1 \in M$ with $\|u_1\| = 1$ arbitrarily. Choose $n_1 > n_0$ such that

$$|\langle T^n u_1, u_1\rangle| < \frac{\varepsilon}{4s}, \qquad n \ge n_1,$$
$$|\langle T^n u_1, a\rangle| < \frac{\varepsilon}{4s}, \qquad n \ge n_1, a \in A,$$

and

$$|\langle T^{*n} u_1, a\rangle| < \frac{\varepsilon}{4s}, \qquad n \ge n_1, a \in A.$$

We construct unit vectors $u_2, \dots, u_s \in M$ in the following way: Let $1 \le r \le s-1$ and suppose that the unit vectors $u_1, \dots, u_r \in M$ and numbers $n_0 < n_1 < \dots < n_r$ have already been constructed.

By assumption (6.1), there exists a unit vector $u_{r+1} \in M$ such that

$$u \perp \{T^n u_k, T^{*n} u_k, T^n a, T^{*n} a : 0 \le n \le n_r, 1 \le k \le r, a \in A\}$$

and

$$|\langle T^n u_{r+1}, u_{r+1}\rangle| < \frac{\varepsilon}{4}, \qquad 1 \le n \le n_r.$$

Find $n_{r+1} > n_r$ such that

$$|\langle T^n u_{r+1}, u_k\rangle| < \frac{\varepsilon}{4s},$$
$$|\langle T^{*n} u_{r+1}, u_k\rangle| < \frac{\varepsilon}{4s},$$
$$|\langle T^n u_{r+1}, a\rangle| < \frac{\varepsilon}{4s},$$

and

$$|\langle T^{*n} u_{r+1}, a\rangle| < \frac{\varepsilon}{4s}$$

for all $n \geq n_{r+1}$, $1 \leq k \leq r+1$ and $a \in A$.

Let $u_1, \dots, u_s$ and $n_0, \dots, n_s$ be constructed in this way. Set

$$x = \frac{1}{\sqrt{s}} \sum_{k=1}^{s} u_k.$$

Clearly $x \in M$. Moreover, $\|x\| = 1$ since the vectors u_k are orthonormal.

For $n \geq n_s$ we have

$$|\langle T^n x, x\rangle| \leq s^{-1} \sum_{k,k'=1}^{s} |\langle T^n u_k, u_{k'}\rangle| \leq s^{-1} s^2 \frac{\varepsilon}{4s} < \varepsilon.$$

Let $0 \leq r \leq s-1$ and $n_r < n \leq n_{r+1}$. Then

$$|\langle T^n x, x\rangle| = s^{-1} \Big| \sum_{k,k'=1}^{s} \langle T^n u_k, u_{k'}\rangle \Big|$$

$$\leq s^{-1} \sum_{k,k'=1}^{r} |\langle T^n u_k, u_{k'}\rangle| + s^{-1} \sum_{k=1}^{r+1} |\langle T^n u_{r+1}, u_k\rangle| + s^{-1} \sum_{k=1}^{r} |\langle T^n u_k, u_{r+1}\rangle|$$

$$+ s^{-1} \sum_{k=r+2}^{s} |\langle T^n u_k, u_k\rangle| + s^{-1} \sum_{\substack{1 \leq k,k' \leq s, k \neq k' \\ \max\{k,k'\} \geq r+2}} |\langle T^n u_k, u_{k'}\rangle|,$$

where the last term is equal to 0 by the construction. So

$$\begin{aligned} |\langle T^n x, x\rangle| \leq & s^{-1} r^2 \frac{\varepsilon}{4s} + s^{-1} \|T^n u_{r+1}\| \cdot \|\sum_{k=1}^{r+1} u_k\| \\ & + s^{-1} \|T^{*n} u_{r+1}\| \cdot \|\sum_{k=1}^{r} u_k\| + s^{-1}(s-r-1)\frac{\varepsilon}{4} \\ \leq & \frac{\varepsilon}{4} + s^{-1} K \sqrt{r+1} + s^{-1} K \sqrt{r} + \frac{\varepsilon}{4} \\ \leq & \varepsilon. \end{aligned}$$

Hence

$$\sup_{n \geq 1} |\langle T^n x, x\rangle| \leq \varepsilon.$$

Let $a \in A$. For $n \geq n_s$ we have

$$|\langle T^n x, a\rangle| \leq \frac{1}{\sqrt{s}} \sum_{k=1}^{s} |\langle T^n u_k, a\rangle| \leq \frac{1}{\sqrt{s}} \cdot s \cdot \frac{\varepsilon}{4s} < \varepsilon.$$

Let $0 \leq r \leq s-1$ and $n_r \leq n < n_{r+1}$. Then

$$\begin{aligned}|\langle T^n x, a\rangle| &\leq \frac{1}{\sqrt{s}} \sum_{k=1}^{r} |\langle T^n u_k, a\rangle| + \frac{1}{\sqrt{s}} |\langle T^n u_{r+1}, a\rangle| + \frac{1}{\sqrt{s}} \sum_{k=r+2}^{s} |\langle T^n u_k, a\rangle| \\ &\leq \frac{1}{\sqrt{s}} \cdot r \cdot \frac{\varepsilon}{4s} + \frac{1}{\sqrt{s}} \cdot K \\ &< \varepsilon.\end{aligned}$$

So

$$\sup_{n\geq 1} |\langle T^n x, a\rangle| \leq \varepsilon$$

for all $a \in A$.

The property $\sup_{n\geq 1} |\langle T^{*n} x, a\rangle| \leq \varepsilon$ for all $a \in A$ can be proved similarly. □

Now we are ready to use essential numerical ranges for the study of operator norm convergence. The following theorem is one of the main results of the paper.

Theorem 6.2. *Let $T \in B(H)$ and let $T^n \to 0$ in the weak operator topology. Suppose that $(0, \dots, 0) \in W_e(T, \dots, T^n)$ for all $n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists an infinite-dimensional subspace L of H such that*

$$\sup_{n\geq 1} \|P_L T^n P_L\| \leq \varepsilon \qquad \textit{and} \qquad \lim_{n\to\infty} \|P_L T^n P_L\| = 0,$$

where P_L is the orthogonal projection on L.

Proof. Let $n_0 = 1$. By Lemma 6.1, there exists $y_1 \in H$, $\|y_1\| = 1$, such that

$$\sup_{n\geq 1} |\langle T^n y_1, y_1\rangle| < \frac{\varepsilon}{2}.$$

Find $n_1 > n_0$ satisfying

$$|\langle T^n y_1, y_1\rangle| < \frac{\varepsilon}{4}, \qquad n \geq n_1.$$

We construct inductively unit vectors $y_2, y_3, \dots \in H$ and numbers $n_1 < n_2 < \cdots$ in the following way:

Let $r \in \mathbb{N}$ and suppose that the unit vectors $y_1, \dots, y_r$ and numbers $n_1, \dots, n_r$ have already been constructed. By Lemma 6.1, there exists $y_{r+1} \in$

H such that

$$\begin{aligned} \|y_{r+1}\| &= 1, \\ y_{r+1} &\perp \{T^n y_k, T^{*n} y_k : 0 \le n < n_r, 1 \le k \le r\}, \\ \sup_{n\ge 1} |\langle T^n y_{r+1}, y_{r+1}\rangle| &< \frac{\varepsilon}{2^{r+3}(r+1)}, \\ \sup_{n\ge 1} |\langle T^n y_{r+1}, y_k\rangle| &< \frac{\varepsilon}{2^{r+3}(r+1)}, \qquad 1 \le k \le r, \\ \sup_{n\ge 1} |\langle T^{*n} y_{r+1}, y_k\rangle| &< \frac{\varepsilon}{2^{r+3}(r+1)}, \qquad 1 \le k \le r. \end{aligned}$$

Find $n_{r+1} > n_r$ satisfying

$$|\langle T^n y_k, y_{k'}\rangle| \le \frac{\varepsilon}{2^{r+4}(r+1)}, \qquad 1 \le k, k' \le r+1, n \ge n_{r+1}.$$

Now suppose that the vectors y_k and numbers n_k have been constructed as above. Let

$$L = \bigvee_{r=1}^{\infty} y_r.$$

Clearly L is an infinite-dimensional subspace with an orthonormal basis (y_r).

Let $y \in L$, $\|y\| = 1$. Then

$$y = \sum_{k=1}^{\infty} \alpha_k y_k \qquad \text{with} \qquad \sum_{k=1}^{\infty} |\alpha_k|^2 = 1.$$

Note that $\sum_{k=1}^{r} |\alpha_k| \le \sqrt{r}$ for all $r \in \mathbb{N}$.

Let $r \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and $n_r \le n < n_{r+1}$. Then

$$\langle T^n y_k, y_{k'}\rangle = 0$$

if $k \ne k'$ and $\max\{k, k'\} \ge r+2$. So

$$\begin{aligned} |\langle T^n y, y\rangle| \le & \sum_{k,k'=1}^{r} |\alpha_k \bar{\alpha}_{k'}| \cdot |\langle T^n y_k, y_{k'}\rangle| + \sum_{k=1}^{r+1} |\alpha_{r+1}\bar{\alpha}_k| \cdot |\langle T^n y_{r+1}, y_k\rangle| \\ & + \sum_{k=1}^{r} |\alpha_k \bar{\alpha}_{r+1}| \cdot |\langle T^n y_k, y_{r+1}\rangle| + \sum_{k=r+2}^{\infty} |\alpha_k|^2 \cdot |\langle T^n y_k, y_k\rangle| \\ \le & r \cdot \frac{\varepsilon}{2^{r+3} r} + \sqrt{r+1} \cdot \frac{\varepsilon}{2^{r+3}(r+1)} + \sqrt{r} \cdot \frac{\varepsilon}{2^{r+3}(r+1)} + \frac{\varepsilon}{2^{r+4}(r+2)} \\ < & \frac{\varepsilon}{2^{r+1}}. \end{aligned}$$

Thus, if $n_r \le n \le n_{r+1}$ then the numerical radius $w(P_L T^n P_L)$ of $P_L T^n P_L$ satisfies

$$w(P_L T^n P_L) := \sup\bigl\{|\langle P_L T^n P_L y, y\rangle| : y \in H, \|y\| = 1\bigr\} \le 2^{-r-1}\varepsilon.$$

Since for any $T \in B(H)$, one has $\|T\| \le 2w(T)$ (see e.g. [17, p. 33] or [16, Theorem 1.3-1]), we infer that

$$\|P_L T^n P_L\| \le 2^{-r}\varepsilon.$$

Hence

$$\sup_{n\ge 1} \|P_L T^n P_L\| \le \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|P_L T^n P_L\| = 0.$$

□

The next corollary of Theorem 6.2 replaces the numerical ranges condition $(0,\dots,0) \in W_e(T,\dots,T^n), n \in \mathbb{N}$, taking into account *all* powers of T, by the more transparent spectral assumption $0 \in \hat{\sigma}_e(T)$.

Corollary 6.3. *Let $T \in B(H)$, and let $T^n \to 0$ in the weak operator topology. Suppose that $0 \in \hat{\sigma}_e(T)$. Then for every $\varepsilon > 0$ there exists an infinite-dimensional subspace L of H such that*

$$\sup_{n\ge 1} \|P_L T^n P_L\| \le \varepsilon \qquad \textit{and} \qquad \lim_{n\to\infty} \|P_L T^n P_L\| = 0,$$

where P_L is the orthogonal projection on L. In particular, this is true if the assumption $0 \in \hat{\sigma}_e(T)$ is replaced by $\mathbb{T} \subset \sigma(T)$.

Proof. We consider two cases. If $0 \in \operatorname{Int} \hat{\sigma}_e(T)$ then by Theorem 4.9 we have $0 \in W_e(T,\dots,T^n)$ for every $n \in \mathbb{N}$, and the statement follows from Theorem 6.2.

On the other hand, if $0 \in \partial\hat{\sigma}_e(T)$, then using elementary properties of polynomial convex hulls and [27, Proposition III.19.1], we have

$$0 \in \partial\hat{\sigma}_e(T) \subset \partial\sigma_e(T) \subset \sigma_{\pi e}(T),$$

so that

$$(0,\dots,0) \in \sigma_{\pi e}(T,\dots,T^n) \subset W_e(T,\dots,T^n),$$

for every $n \in \mathbb{N}$. (Alternatively, using [27, Proposition III.19.1 and Corollary III.19.16], one may note that

$$0 \in \partial\hat{\sigma}_e(T) = \partial\hat{\sigma}_{\pi e}(T) \subset \sigma_{\pi e}(T), \qquad n \in \mathbb{N},$$

and then $(0,\dots,0) \in W_e(T,\dots,T^n)$ as above.) Thus

$$(0,\dots,0) \in W_e(T,\dots,T^n), \qquad n \in \mathbb{N},$$

again, and we can use Theorem 6.2.

If $\mathbb{T} \subset \sigma(T)$, then (3.1) yields $\mathbb{T} \subset \sigma_e(T)$, so that $0 \in \hat{\sigma}_e(T)$. □

Remark 6.4. Observe that one may replace the assumption $0 \in \hat{\sigma}_e(T)$ in Corollary 6.3 by $0 \in \operatorname{Int} \hat{\sigma}(T)$.

If T is unitary then the above corollary can be sharpened. The result below is an essential generalization of the main result in [18] (with a completely different proof).

Corollary 6.5. *Let T be a unitary operator on H such that $T^n \to 0$ in the weak operator topology. Then the following conditions are equivalent.*

(i) $\sigma(T) = \mathbb{T}$.

(ii) for every $\varepsilon > 0$ there exists $x \in H$, $\|x\| = 1$, with

$$\sup_{n\geq 1} |\langle T^n x, x\rangle| < \varepsilon.$$

(iii) for every $\varepsilon > 0$ there exists an infinite-dimensional subspace $L \subset H$ such that

$$\sup_{n\geq 1} \|P_L T^n P_L\| \leq \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \|P_L T^n P_L\| = 0,$$

where P_L is the orthogonal projection on L.

Proof. The implication (ii)⇒(i) follows from Theorem 5.1.

If $\sigma(T) = \mathbb{T}$ then $\sigma_{\pi e}(T) = \mathbb{T}$ by (3.1). So (i)⇒(iii) follows from the previous corollary. The implication (iii)⇒(ii) is trivial. □

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