INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

On the Bonsall cone spectral radius and the approximate point spectrum

Vladimír Müller
Aljoša Peperko

Preprint No. 58-2016

PRAHA 2016

ON THE BONSALL CONE SPECTRAL RADIUS AND THE APPROXIMATE POINT SPECTRUM

VLADIMIR MÜLLER, ALJOŠA PEPERKO

Abstract. We study the Bonsall cone spectral radius and the approximate point spectrum of (in general non-linear) positively homogeneous, bounded and supremum preserving maps, defined on a max-cone in a given normed vector lattice. We prove that the Bonsall cone spectral radius of such maps is always included in its approximate point spectrum. Moreover, the approximate point spectrum always contains a (possibly trivial) interval. Our results apply to a large class of (nonlinear) max-type operators.

We also generalize a known result that the spectral radius of a positive (linear) operator on a Banach lattice is contained in the approximate point spectrum. Under additional generalized compactness type assumptions our results imply Krein-Rutman type results.

Math. Subj. Classification (2010): 47H07, 47J10, 47H10, 47H08, 47B65, 47A10.
Key words: Bonsall's cone spectral radius; local spectral radii; approximate point spectrum; supremum preserving maps; max kernel operators; normed vector lattices; normed spaces; cones

1. Introduction

Max-type operators (and corresponding max-plus type operators and their tropical versions known also as Bellman operators) arise in a large field of problems from the theory of differential and difference equations, mathematical physics, optimal control problems, discrete mathematics, turnpike theory, mathematical economics, mathematical biology, games and controlled Markov processes, generalized solutions of the Hamilton-Jacobi-Bellman differential equations, continuously observed and controlled quantum systems, discrete and continuous dynamical systems, ... (see e.g. [31], [23], [30], [29], [4] and the references cited there). The eigenproblem of such operators obtained so far substantial attention due to its applicability in the above mentioned problems (see e.g. [31], [23], [4], [3] [25], [2], [32], [13], [14], [36], [16], [35], [40] and the references cited there). However, there seems to be a lack of more general treatment of spectral theory for such operators, eventhough the spectral theory for nonlinear operators on Banach spaces is already quite well developed (see e.g. [11], [10], [12], [17], [18], [19], [20], [21], [39] and the references

Date: November 14, 2016.

cited there). One of the reasons for this might lie in the fact that these operators behave nicely on a suitable subcone (or subsemimodule), but less nicely on the whole (Banach) space. Therefore it appears, that it is not trivial to directly apply this known non-linear spectral theory to obtain satisfactory information on a restriction to a given cone of a max-type operator. The Bonsall cone spectral radius plays the role of the spectral radius in this theory (see e.g. [31], [32], [4], [24], [22], [36] and the references cited there).

In this article we study the Bonsall cone spectral radius and the approximate point spectrum of positively homogeneous, bounded and supremum preserving maps, defined on a max-cone in a given normed vector lattice. We prove that the Bonsall cone spectral radius of such maps is always included in its approximate point spectrum. Moreover, the approximate point spectrum always contains a (possibly trivial) interval. Our results apply to a large class of max-type operators (and their isomorphic versions). Our main interests are results on suitable cones in Banach spaces and Banach lattices. However, since the completeness of the norm does not simplify our proofs, we state our results in the setting of normed spaces and normed vector lattices. Under suitable generalized compactness type assumptions our results imply Krein-Rutman type results.

The paper is organized as follows. In Section 2 we recall basic definitions and facts that we will need in our proofs. In Section 3 we prove our results in the setting of max-cones in normed vector lattices, while in Section 4 we apply our techniques in the setting of normal convex cones in normed spaces. The main results of Section 3 are Theorem 3.6 and its generalization Theorem 3.7 and the main result of Section 4 are Theorems 4.1 and 4.2.

2. Preliminaries

A subset C of a real vector space X is called a cone (with vertex 0) if $tC \subset C$ for all $t \geq 0$, where $tC = \{tx : x \in C\}$. A map $T : C \to C$ is called positively homogeneous (of degree 1) if $T(tx) = tT(x)$ for all $t \geq 0$ and $x \in C$. We say that the cone C is pointed if $C \cap (-C) = \{0\}$.

A convex pointed cone C of X induces on X a partial ordering $\leq$, which is defined by $x \leq y$ if and only if $y - x \in C$. In this case C is denoted by X_+ and X is called an ordered vector space. If, in addition, X is a normed space then it is called an ordered normed space. If, in addition, the norm is complete, then X is called an ordered Banach space.

A convex cone C of X is called a wedge. A wedge induces on X (by the above relation) a vector preordering $\leq$ (which is reflexive, transitive, but not necessary antisymmetric).

We say that the cone C is proper if it is closed, convex and pointed. A cone C of a normed space X is called normal if there exists a constant M such that $\|x\| \leq M\|y\|$ whenever $x \leq y$, $x, y \in C$. A convex and pointed cone $C = X_+$ of an ordered normed space X is normal if and only if there exists an equivalent monotone norm $|||\cdot|||$ on X, i.e., $|||x||| \leq |||y|||$ whenever $0 \leq x \leq y$ (see e.g. [9, Theorem 2.38]). Every proper cone C in a finite dimensional Banach space is necessarily normal.

If X is a normed linear space, then a cone C in X is said to be complete if it is a complete metric space in the topology induced by X. In the case when X is a Banach space this is equivalent to C being closed in X.

If X is an ordered vector space, then a cone $C \subset X_+$ is called a max-cone if for every pair $x, y \in C$ there exists a supremum $x \vee y$ (least upper bound) in C. We consider here on C an order inherited from X_+. A map $T : C \to C$ preserves finite suprema on C if $T(x \vee y) = Tx \vee Ty \quad (x, y \in C)$. If $T : C \to C$ preserves finite suprema, then it is monotone (order preserving) on C, i.e., $Tx \le Ty$ whenever $x \le y$, $x, y \in C$.

An ordered vector space X is called a vector lattice (or a Riesz space) if every two vectors $x, y \in X$ have a supremum and infimum (greatest lower bound) in X. A positive cone X_+ of a vector lattice X is called a lattice cone.

Note that by [9, Corollary 1.18] a pointed convex cone $C = X_+$ of an ordered vector space X is a lattice cone for the vector subspace $C - C$ generated by C in X, if and only if C is a max cone (in this case a supremum of $x, y \in C$ exists in C if only if it exists in X; and suprema coincide). Moreover, if $x, y, z, u \in C$, then

$$(x - y) \vee (z - u) = (x + u) \vee (y + z) - (y + u)$$

holds in $C - C$.

If X is a vector lattice, then the absolute value of $x \in X$ is defined by $|x| = x \vee (-x)$. A vector lattice is called a normed vector lattice (a normed Riesz space) if $|x| \le |y|$ implies $\|x\| \le \|y\|$. A complete normed vector lattice is called a Banach lattice. A positive cone X_+ of a normed vector lattice X is proper and normal.

In a vector lattice X the following Birkhoff's inequality for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ holds:

$$|\bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j| \le \sum_{j=1}^{n} |x_j - y_j|. \tag{1}$$

For the theory of vector lattices, Banach lattices, cones, wedges, cone preserving operators and applications e.g. in financial mathematics we refer the reader to the books [1], [9], [7], [41], [6], [26], [5] and the references cited there.

Let X be a normed space and $C \subset X$ a non-zero cone. Let $T : C \to C$ be positively homogeneous and bounded, i.e.,

$$\|T\| := \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in C, x \neq 0 \right\} < \infty.$$

It is easy to see that $\|T\| = \sup\{\|Tx\| : x \in C, \|x\| \le 1\}$ and $\|T^{m+n}\| \le \|T^m\| \cdot \|T^n\|$ for all $m, n \in \mathbb{N}$. It is well known that this implies that the limit $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and is equal to $\inf_n \|T^n\|^{1/n}$. The limit $r(T) := \lim_{n\to\infty} \|T^n\|^{1/n}$ is called the Bonsall cone spectral radius of T. The approximate point spectrum $\sigma_{ap}(T)$ of T is defined as the set of all $s \ge 0$ such that $\inf\{\|Tx - sx\| : x \in C, \|x\| = 1\} = 0$.

For $x \in C$ define the local cone spectral radius by $r_x(T) := \limsup_{n\to\infty} \|T^n x\|^{1/n}$. Clearly $r_x(T) \le r(T)$ for all $x \in C$. It is known that the equality

$$\sup\{r_x(T) : x \in C\} = r(T) \tag{2}$$

is not valid in general. In [31] there is an example of a proper cone C in a Banach space X and a positively homogeneous and continuous (hence bounded) map $T : C \to C$ such that $\sup\{r_x(T) : x \in C\} < r(T)$. A recent example of such kind, where T is in addition monotone, is obtained in [22, Example 3.1]. However, if C is a normal, complete, convex and pointed cone in a normed space X and $T : C \to C$ is positively homogeneous, monotone and continuous, then [32, Theorem 3.3], [31, Theorem 2.2] and [22, Theorem 2.1] ensure that (2) is valid.

If X is a Banach lattice, $C \subset X_+$ a max-cone and $T : C \to C$ a mapping which is bounded, positively homogeneous and preserves finite suprema, then the equality (2) is not necessary valid as the following example shows.

Example 2.1. Let $X = l^2$ with a standard orthonormal basis $\{e_1, e_2, \dots\}$. Let $C = \{\bigvee_{j=1}^n \alpha_j e_j : n \ge 1, \alpha_1, \dots \alpha_n \ge 1\}$. Define $T : C \to C$ by $T(\bigvee_{j=1}^n \alpha_j e_j) = \bigvee_{j=1}^{n-1} \alpha_{j+1} e_j$ (the backward shift). It is easy to see that $r(T) = 1$ and $r_x(T) = 0$ for each $x \in C$. It also holds that $\sigma_{ap}(T) = [0, 1]$.

Some additional examples of maps for which (2) is not valid can be found in [22].

Let C be a cone in a normed space X and $T : C \to C$. Then T is called Lipschitz if there exists $L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in C$.

3. Results on max-cones in normed vector lattices

As noted in Example 2.1 it may happen that $\sup\{r_x(T) : x \in C\} < r(T)$ in the case when C is a max-cone. We will prove in Theorem 3.6 that the Bonsall cone spectral radius of a bounded, positively homogeneous, finite suprema preserving mapping $T : C \to C$, defined on a max-cone C in a normed vector lattice, is contained in its approximate point spectrum. Moreover, we will show that the interval $[\sup\{r_x(T) : x \in C\}, r(T)]$ is included in $\sigma_{ap}(T)$ for such maps T.

We shall need the following three lemmas.

Lemma 3.1. *Let X be a normed vector lattice and let $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Then*

$$\Big\|\bigvee_{j=1}^n x_j - \bigvee_{j=1}^n y_j\Big\| \le \sum_{j=1}^n \|x_j - y_j\|.$$

Proof. By (1) we have

$$\Big\|\bigvee_{j=1}^n x_j - \bigvee_{j=1}^n y_j\Big\| = \Big\|\,\Big|\bigvee_{j=1}^n x_j - \bigvee_{j=1}^n y_j\Big|\,\Big\| \le \Big\|\sum_{j=1}^n |x_j - y_j|\,\Big\|$$

$$\leq \sum_{j=1}^{n} \| \, |x_j - y_j| \, \| = \sum_{j=1}^{n} \|x_j - y_j\|,$$

which completes the proof. □

Lemma 3.2. *Let X be a vector lattice and $x_j, y_j \in X$ for $j = 1, \ldots, n$. Then*

$$\bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \leq \bigvee_{j=1}^{n} (x_j - y_j). \tag{3}$$

If, in addition, X is a normed vector lattice and $x_j \geq y_j \geq 0$ for $j = 1, \ldots, n$, then

$$\| \bigvee_{j=1}^{n} x_j - \bigvee_{j=1}^{n} y_j \| \leq \| \bigvee_{j=1}^{n} (x_j - y_j) \|. \tag{4}$$

Proof. We have

$$\bigvee_{j=1}^{n} x_j = \bigvee_{j=1}^{n} (x_j - y_j + y_j) \leq \bigvee_{j=1}^{n} (x_j - y_j) + \bigvee_{j=1}^{n} y_j.$$

which proves (3). If X is a normed vector lattice and $x_j \geq y_j \geq 0$ for $j = 1, \ldots, n$, then $\bigvee_{j=1}^{n} x_j \geq \bigvee_{j=1}^{n} y_j$ and this implies (4). □

Lemma 3.3. *Let X be a normed space and let $C \subset X$ be a non-zero cone. If $T : C \to C$ is positively homogeneous and Lipschitz, then $r(T) \geq t$ for all $t \in \sigma_{ap}(T)$.*

Proof. Since $T(0) = 0$ and T is Lipschitz it follows that T is also bounded and so $r(T)$ is well defined. If $t \in \sigma_{ap}(T)$, then there exists a sequence (x_k) of unit vectors such that $\lim_{k\to\infty} \|Tx_k - tx_k\| = 0$. By induction it follows that $\lim_{k\to\infty} \|T^j x_k - t^j x_k\| = 0$ for all $j \in \mathbb{N}$. Indeed,

$$\begin{aligned} \|T^j x_k - t^j x_k\| &\leq \|T^j x_k - T^{j-1}(tx_k)\| + \|T^{j-1}(tx_k) - t^j x_k\| \\ &\leq L^{j-1} \|Tx_k - tx_k\| + t\|T^{j-1} x_k - t^{j-1} x_k\| \to 0 \end{aligned}$$

as $k \to \infty$, by the induction assumption. Here L denotes the Lipschitzity constant of T.

It follows that $\|T^j\| \geq \lim_{k\to\infty} \|T^j x_k\| = t^j$ and so $r(T) \geq t$. □

The following example shows that in Lemma 3.3 we can not replace the property that "T is Lipschitz" by a weaker property that "T is bounded".

Example 3.4. Let $X = \ell^\infty$ with the standard basis $x_n, y_n, z_n \quad (n = 1, 2, \ldots)$. More precisely, the elements of X are formal sums

$$x = \sum_{n=1}^{\infty} (\alpha_n x_n + \beta_n y_n + \gamma_n z_n)$$

with real coefficients $\alpha_n, \beta_n, \gamma_n$ such that

$$\|x\| := \sup\{|\alpha_n|, |\beta_n|, |\gamma_n| : n = 1, 2, \ldots\} < \infty.$$

Then X is a Banach lattice with the natural order.

Let

$$C=\Big\{\sum_{j=1}^\infty(\alpha_jx_j+\beta_jy_j+\gamma_jz_j)\in X:\alpha_j,\gamma_j\ge 0,\beta_j=j\alpha_j+j\gamma_j\ \text{ for all } j\Big\}.$$

Then C is a closed max-cone (moreover convex and normal). Let $T:C\to C$ be defined by

$$T\Big(\sum_{j=1}^\infty(\alpha_jx_j+\beta_jy_j+\gamma_jz_j)\Big)=\sum_{j=1}^\infty(j\alpha_jy_j+\alpha_jz_j).$$

Clearly $\|T\|\le 1$, T is positively homogeneous and preserves (all) suprema.

For $k\in\mathbb{N}$ let $u_k=k^{-1}x_k+y_k$. Then $\|u_k\|=1$ and $Tu_k=y_k+k^{-1}z_k$. So $\|Tu_k-u_k\|=k^{-1}$ and $1\in\sigma_{ap}(T)$. On the other hand $T^2=0$ and so $r(T)=0$.

Note that T is not Lipschitz, since $\|Tu_k-u_k\|=k^{-1}$ but $\|T^2u_k-Tu_k\|=\|Tu_k\|=1$.

The following technical lemma is essentially needed in the proofs of our main results Theorem 3.6, Theorem 3.7, Theorem 4.1 and Theorem 4.2.

Lemma 3.5. *Let $\varepsilon>0$ and $K\ge 1$. Then there exists $n\in\mathbb{N}$ with the following property: if $(\alpha_k)_{k=0}^\infty$ is a sequence of real numbers such that $0\le\alpha_k\le K^k$ for all k, $\alpha_n\ge 1/2$ and $\limsup_{k\to\infty}\alpha_k^{1/k}\le 1$, then there exist $m\in\mathbb{N}$ and nonnegative numbers $\beta_k,(k=0,1,\dots)$, such that*

$$\beta_0\le\varepsilon,$$
$$|\beta_{k+1}-\beta_k|\le 2\varepsilon,$$
$$\beta_k<\beta_{k+1}\quad (\textit{for all } k=0,1,\dots,m-1),$$
$$\beta_k>\beta_{k+1}\quad (\textit{for all } k=m,m+1,\dots),$$
$$\alpha_m\beta_m=1,\alpha_k\beta_k\le 1\quad (\textit{for all } k),$$
$$\lim_{k\to\infty}\alpha_k\beta_{k+1}=0.$$

Proof. Choose $m_0\in\mathbb{N}$ such that $(1+\varepsilon)^{m_0}>2\varepsilon^{-1}$. Choose $n>m_0$ such that $(1+\varepsilon)^n>2K^{m_0}(1+\varepsilon)^{m_0}$. Let (α_k) be a sequence of nonnegative numbers satisfying the assumptions.

Set $\gamma_k=\alpha_n(1+\varepsilon)^{|k-n|}$. Then $\gamma_n=\alpha_n\ge 1/2$. Let $m\in\mathbb{N}$ satisfy

$$\frac{\alpha_m}{\gamma_m}=\max_k\frac{\alpha_k}{\gamma_k}$$

(such an m exists since $\lim_{k\to\infty}\frac{\alpha_k}{\gamma_k}=\alpha_n^{-1}\lim_{k\to\infty}\frac{\alpha_k}{(1+\varepsilon)^{k-n}}=0$). In particular, we have $\frac{\alpha_m}{\gamma_m}\ge\frac{\alpha_n}{\gamma_n}=1$ and $\alpha_m\ge\gamma_m\ge\alpha_n\ge 1/2$.

We show that $m\ge m_0$. Suppose the contrary that $m<m_0$. Then $n\ge m$, since otherwise $n<m<m_0$ provides a contradiction. We have $\alpha_m\le K^m$ and $\gamma_m=\alpha_n(1+\varepsilon)^{n-m}$. So

$$\frac{\alpha_m}{\gamma_m}\le\frac{K^m}{\alpha_n(1+\varepsilon)^{n-m}}<\frac{2K^{m_0}(1+\varepsilon)^{m_0}}{(1+\varepsilon)^n}<1,$$

a contradiction. So $m\ge m_0$.

Set $\beta_k = \alpha_m^{-1}(1+\varepsilon)^{-|k-m|}$. Clearly $\alpha_m\beta_m = 1$ and $\beta_m = \alpha_m^{-1} \le 2$. We have $\beta_0 = \alpha_m^{-1}(1+\varepsilon)^{-m} \le 2(1+\varepsilon)^{-m_0} < \varepsilon$. Clearly $\lim_{k\to\infty} \alpha_k\beta_{k+1} = 0$.

For $k = 0, 1, \dots, m-1$ we have $\beta_k < \beta_{k+1}$ and

$$\beta_{k+1} - \beta_k = \alpha_m^{-1}\big((1+\varepsilon)^{-(m-k-1)} - (1+\varepsilon)^{-(m-k)}\big) = \alpha_m^{-1}(1+\varepsilon)^{k-m}(1+\varepsilon-1) \le \varepsilon\alpha_m^{-1} \le 2\varepsilon.$$

Similarly, for $k = m, m+1, \dots$ we have $\beta_k > \beta_{k+1}$ and

$$\beta_k - \beta_{k+1} = \alpha_m^{-1}\big((1+\varepsilon)^{-(k-m)} - (1+\varepsilon)^{-(k+1-m)}\big) = \alpha_m^{-1}(1+\varepsilon)^{m-k-1}\varepsilon \le 2\varepsilon.$$

Finally, for each k we have $\frac{\alpha_k}{\gamma_k} \le \frac{\alpha_m}{\gamma_m}$. So

$$\alpha_k\beta_k \le \frac{\alpha_m\gamma_k\beta_k}{\gamma_m} = \frac{\alpha_n(1+\varepsilon)^{|k-n|}}{\alpha_n(1+\varepsilon)^{|m-n|}(1+\varepsilon)^{|k-m|}} \le 1$$

and the proof is complete. □

The following theorem is one of the main results of this section.

Theorem 3.6. *Let X be a normed vector lattice, let $C \subset X_+$ be a non-zero max-cone. Let $T : C \to C$ be a mapping which is bounded, positively homogeneous and preserves finite suprema. Let $\sup\{r_x(T) : x \in C\} \le t \le r(T)$. Then $t \in \sigma_{ap}(T)$.*

In particular, $r(T) \in \sigma_{ap}(T)$.

Proof. If $t = 0$, then for each $x \in C$, $\|x\| = 1$ we have $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$. For each $\varepsilon > 0$ there exists $k \ge 0$ such that $\|T^{k+1}x\| < \varepsilon\|T^k x\|$. If $u = \frac{T^k x}{\|T^k x\|}$ then $\|u\| = 1$ and $\|Tu\| < \varepsilon$.

So without loss of generality we may assume that $t = 1$.

We distinguish two cases:

I. Suppose that $\sup\{\|\bigvee_{j=0}^n T^j x\| : x \in C, \|x\| \le 1, n \in \mathbb{N}\} = \infty$.

Let $k \in \mathbb{N}$. Find $x_k \in C$, $\|x_k\| \le 1$ and $n_k \in \mathbb{N}$ such that $\|\bigvee_{j=0}^{n_k} T^j x_k\| > k$. Find $t_k \in (1, 1+k^{-1})$ such that $t_k^{-n_k-1} > 1/2$. Find $r_k > n_k$ such that $\frac{\|T^{r_k+1}x_k\|}{t_k^{r_k+1}} < 1$. Set $y_k := \bigvee_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j+1}}$. Then $\|y_k\| \ge \|t_k^{-n_k-1}\bigvee_{j=0}^{n_k} T^j x_k\| \ge k/2$.

Set $u_k = \frac{y_k}{\|y_k\|}$. Then $\|u_k\| = 1$ and by Lemma 3.1 it follows

$$\|Tu_k - u_k\| \le \|Tu_k - t_k u_k\| + (t_k - 1)\|u_k\| = \|y_k\|^{-1}\Big\|\bigvee_{j=0}^{r_k} \frac{T^{j+1}x_k}{t_k^{j+1}} - \bigvee_{j=0}^{r_k} \frac{T^j x_k}{t_k^j}\Big\| + (t_k - 1)$$

$$\le \|y_k\|^{-1}\Big\|\frac{T^{r_k+1}x_k}{t_k^{r_k+1}} - x_k\Big\| + k^{-1} \le 2\|y_k\|^{-1} + k^{-1} \le \frac{5}{k} \to 0$$

as $k \to \infty$. Hence $1 \in \sigma_{ap}(T)$.

II. Suppose that

$$M_0 := \sup\Big\{\Big\|\bigvee_{j=0}^n T^j x\Big\| : x \in C, \|x\| \le 1, n \in \mathbb{N}\Big\} < \infty.$$

Let $\varepsilon > 0$ and $K := \|T\|$. Let n be the number constructed in Lemma 3.5. We have $\|T^n\| \ge r(T^n) = r(T)^n \ge 1$. Find $x \in C$ such that $\|x\| = 1$ and $\|T^n x\| \ge 1/2$. Let $\alpha_k = \|T^k x\|$.

By Lemma 3.5, there exist $m \in \mathbb{N}$ and nonnegative numbers β_k such that $\beta_0 \le \varepsilon$, $|\beta_{k+1} - \beta_k| \le 2\varepsilon$, $\beta_k < \beta_{k+1}$ for all $k = 0, 1, \dots, m-1$, $\beta_k > \beta_{k+1}$ for all $k = m, m+1, \dots$, $\alpha_m \beta_m = 1$, $\alpha_k \beta_k \le 1$ for all k and $\lim_{k\to\infty} \alpha_k \beta_{k+1} = 0$.

Fix $r > m$ such that $\beta_r < \varepsilon \|T^{r-1}x\|^{-1}$.

Set $u = \bigvee_{k=0}^r \beta_k T^k x$. Since $u \ge \beta_m T^m x$, we have $\|u\| \ge \beta_m \|T^m x\| = 1$. By Lemma 3.1 and Lemma 3.2 we have

$$\|Tu - u\| = \Big\| \bigvee_{k=0}^{r} \beta_k T^{k+1} x - \bigvee_{k=0}^{r} \beta_k T^k x \Big\|$$

$$\le \Big\|\beta_0 x\Big\| + \Big\| \bigvee_{k=1}^{m} T^k x \beta_k - \bigvee_{k=1}^{m} T^k x \beta_{k-1} \Big\| + \Big\| \bigvee_{k=m+1}^{r} T^k x \beta_{k-1} - \bigvee_{k=m+1}^{r} T^k x \beta_k \Big\| + \Big\| \beta_r T^{r+1} x \Big\|$$

$$\le \varepsilon + \Big\| \bigvee_{k=1}^{m} T^k x(\beta_k - \beta_{k-1}) \Big\| + \Big\| \bigvee_{k=m+1}^{r} T^k x(\beta_{k-1} - \beta_k) \Big\| + \varepsilon K^2 \le \varepsilon + 4\varepsilon M_0 + \varepsilon K^2 \to 0$$

as $\varepsilon \to 0$. Hence $1 \in \sigma_{ap}(T)$. ☐

The proof above shows more. Namely, the following more general result is proved in the same way.

Theorem 3.7. *Let X be a normed vector lattice, let $C \subset X_+$ be a non-zero max-cone. Let $T : C \to C$ be a mapping which is bounded, positively homogeneous and preserves finite suprema. Let $C' \subset C$ be a bounded subset satisfying $\|T^n\| = \sup\{\|T^n x\| : x \in C'\}$ for all n. Then*

$$[\sup\{r_x(T) : x \in C'\}, r(T)] \subset \sigma_{ap}(T).$$

Theorem 3.6 and Lemma 3.3 imply the following result.

Corollary 3.8. *Let X be a normed vector lattice and let $C \subset X_+$ be a non-zero max-cone. If $T : C \to C$ is a Lipschitz, positively homogeneous mapping which preserves finite suprema, then $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$.*

From Theorem 3.6 also the following corollary follows.

Corollary 3.9. *Let X be a normed vector lattice and let $C \subset X_+$ be a non-zero max-cone. Let $T : C \to C$ be a bounded, positively homogeneous mapping which preserves finite suprema. Then $r_x(T) \in \sigma_{ap}(T)$ for each $x \in C$, $x \ne 0$.*

Proof. Let $x \in C$, $x \ne 0$. Let

$$K = \{\bigvee_{j=0}^{n} \alpha_j T^j x : n \in \mathbb{N}, \alpha_j \ge 0 \quad (j = 0, 1, \dots, n)\}.$$

Clearly K is a non-zero max-cone, $TK \subset K$. Let $y \in K$, $y = \bigvee_{j=0}^{n} \alpha_j T^j x$ for some $n, \alpha_0, \ldots, \alpha_n$.

Let $k \in \mathbb{N}$. We have

$$\|T^k y\| \le \sum_{j=0}^{n} \alpha_j \|T^{k+j}x\| \le \max_j \alpha_j \cdot (n+1) \max\{\|T^{k+j}x\| : j = 0, 1, \ldots, n\}$$

and

$$\|T^k y\|^{1/k} \le (\max_j \alpha_j)^{1/k} \cdot (n+1)^{1/k} \max\{\|T^{k+j}x\| : j = 0, 1, \ldots, n\}^{1/k} \to r_x(T)$$

as $k \to \infty$. So $r_y(T) \le r_x(T)$. Thus $\sup\{r_y(T) : y \in K\} = r_x(T) \le r(T|_K)$. By Theorem 3.6, $r_x(T) \in \sigma_{ap}(T|_K) \subset \sigma_{ap}(T)$. □

Remark 3.10. (i) As shown in Example 3.4, in Corollary 3.8 the assumption "T is Lipschitz" can not be replaced by a weaker assumption "T is bounded".

(ii) An inspection of the proofs, or an application of the above results, show that Theorems 3.6 and 3.7 and Corollaries 3.8 and 3.9 hold under slightly weaker assumptions on X. It suffices that X is a ordered normed space, $C \subset X_+$ a non-zero max-cone and $X_+ - X_+$ a normed vector lattice (equivalently X_+ is a max-cone and there exists a lattice norm on $X_+ - X_+$).

Let us consider the following example from [31].

Example 3.11. Let $X = C[0,1]$, $C = X_+$ and let $T : X \to X$ be a bounded linear operator defined by $T(x)(s) = sx(s)$. The map $T : C \to C$ also preserves finite suprema (maxima) (and is Lipschitz and positively homogeneous) on C. As pointed out in [31], $r(T) = 1$ and $T(x) \ne x$ for all $x \in C$, $x \ne 0$. However, $1 \in \sigma_{ap}(T)$ and the approximate sequence of vectors $(x_k)_{k\in\mathbb{N}} \subset C$, $\|x_k\| = 1$, is given by $x_k(s) = s^k$, since

$$\|Tx_k - x_k\| = \frac{k^k}{(k+1)^{k+1}} \to 0$$

as $k \to \infty$.

Remark 3.12. Under additional compactness type assumptions on T, Theorem 3.6 implies Krein-Rutman type results. As is well known, and also shown by Example 3.11, some additional assumptions are necessary to obtain such results. Let X, C and T be as in Theorem 3.6, where C is also closed. If, in addition, T is compact (and continuous) and $r(T) > 0$, then there exists $y \in C$, $y \ne 0$ such that $Ty = r(T)y$. Moreover, for each nonzero $t \in \sigma_{ap}(T)$ there exists an eigenvector in C.

Indeed, there exists a sequence $(x_k) \subset C$, $\|x_k\| = 1$, with $Tx_k - tx_k \to 0$. Passing to a subsequence if necessary one can assume that $Tx_k \to y$ for some $y \in C$. Clearly $tx_k \to y$, $y \ne 0$ and $Ty = ty$.

It is not hard to see that the same holds if we replace the assumption that "T is compact" by the assumption that "T is power compact" (i.e., that there exists $m \in \mathbb{N}$ such that T^m is compact).

The results on the existence an eigenvector $x \in C$ for a non-linear operator T corresponding to $r(T)$ are known also under more general compactness type assumptions on T (see e.g. [31, Theorem 3.4, Theorem 3.10]), [32, Theorem 4.4], [11, Theorem 10.6]). We illustrate the usefulness of our Theorem 3.7 by giving an alternative proof of [31, Theorem 3.4] in the case of max-cones in Banach lattices and providing additional information in this case (Theorem 3.14). Moreover, we do not need to assume the completeness of the norm. In our proof we apply some of the ideas from [17]. On the other hand, the proof of [31, Theorem 3.4] was based on a lemma from fixed point index theory ([31, Lemma 3.2], [37, Theorem 2.1]).

To do this, we firstly recall some notions from [31]. If X is a normed space, let ν denote a homogeneous generalized measure of non-compactness on X (as defined in [31, Section 3]), i.e., ν is a map which assigns to each bounded subset of X a non-negative, finite number $\nu(A)$ and satisfies the following five conditions:
(i) $\nu(A) = 0$ if and only if $\overline{A}$ is compact,
(ii) $\nu(A + B) \le \nu(A) + \nu(B)$,
(iii) $\nu(\overline{co(A)}) = \nu(A)$,
(iv) $\nu(A \cup B) = \max\{\nu(A), \nu(B)\}$,
(v) $\nu(\lambda A) = \lambda \nu(A)$ if $\lambda \ge 0$.
Here we denote $A + B = \{a + b : a \in A, b \in B\}$, and $\overline{co(A)}$ denotes the smallest closed convex set containing A.

Let X be normed vector lattice, let $C \subset X_+$ be a non-zero closed max-cone and assume that $T : C \to X$ is a continuous and positively homogeneous mapping (and thus bounded). Let

$$\nu_C(T) = \inf\{\lambda > 0 : \nu(T(A)) \le \lambda \nu(A) \text{ for every bounded set } A \subset C\}$$

and

$$w_C(T) = \sup\{\lambda > 0 : \nu(T(A)) \ge \lambda \nu(A) \text{ for every bounded set } A \subset C\},$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. Note that $w_C(I) = \nu_C(I) = 1$ if $\dim(X) = \infty$. In this case we also have

$$w_C(tI - T) \ge t - \nu_C(T) \tag{5}$$

for $t \ge 0$. Indeed,

$$t = w_C(tI) = w_C(tI - T + T) \le w_C(tI - T) + \nu_C(T),$$

which establishes (5).

If $T : C \to C$, let

$$\beta_\nu(T) = \lim_{n\to\infty} \nu_C(T^n)^{1/n} = \inf_{n\in\mathbb{N}} \nu_C(T^n)^{1/n}, \tag{6}$$

if $\nu_C(T^n) < \infty$ except for finitely many n. (If $\nu_C(T^n) = \infty$ for infinitely many n one may define $\beta_\nu(T) = \infty$.) The quantity $\beta_\nu(T)$ was called the cone essential radius of T in [31], but this terminology was changed in [32] due to its imperfections.

Example 3.13. [31, Examples on p. 14 and 15] Let X be a normed space and A a bounded subset of X. By $\alpha(A)$ we denote the classical Kuratowski-Darbo generalized measure of noncompactness, i.e.,

$$\alpha(A) = \inf\{\delta > 0 : \text{there exist } k \in \mathbb{N} \text{ and } S_i \subset X, i = 1, \ldots, k \\ \text{with } \operatorname{diam}(S_i) \le \delta \text{ such that } A = \cup_{i=1}^k S_i\},$$

where diam denotes the diameter of the set. In the case $X = C(W)$, where (W, d) is a metric space, let us denote for $\delta > 0$

$$\gamma_\delta(A) = \sup\{|x(t) - x(s)| : x \in A, \text{with } t, s \in W \text{ satisfying } d(t,s) \le \delta\}.$$

Then

$$\gamma(A) = \inf_{\delta > 0} \gamma_\delta(A) = \lim_{\delta \to 0^+} \gamma_\delta(A)$$

defines a generalized measure of noncompactness that satisfies $\alpha(A) \le \gamma(A) \le 2\alpha(A)$. Consequently, $\beta_\alpha = \beta_\gamma$. However, there exist nonequivalent measures of non-compactness (see [33], [34]) and this is one of the flaws of the quantity $\beta_\nu(T)$.

The following result is a version of [31, Theorem 3.4] for max-cones in normed vector lattices, which provides more information than [31, Theorem 3.4] even for e.g. max-cones in Banach lattices.

Theorem 3.14. *Let X be a normed vector lattice with $\dim(X) = \infty$ and let $C \subset X_+$ be a non-zero closed max-cone. Let $T : C \to C$ be a mapping which is continuous, positively homogeneous and preserves finite suprema and let $C' \subset C$ be a bounded subset satisfying $\|T^n\| = \sup\{\|T^n x\| : x \in C'\}$ for all n. Further, assume that ν is a homogeneous generalized measure of non-compactness on X.*

If $t \in [\sup\{r_x(T) : x \in C'\}, r(T)]$ satisfies $t > \beta_\nu(T)$, then there exists a nonzero $x \in C$ such that $Tx = tx$.

Proof. By Theorem 3.7 we have $t \in \sigma_{ap}(T)$, so there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset C$, $\|x_k\| = 1$, such that $\|(tI - T)x_k\| \to 0$ as $k \to \infty$. Denote $A = \{x_k : k \in \mathbb{N}\}$.

First assume that $\nu_C(T) < t$. By (5) we have $w_C(tI - T) \ge t - \nu_C(T) > 0$. It follows from

$$w_C(tI - T)\nu(A) \le \nu((tI - T)A) = 0$$

that $\nu(A) = 0$ and so A has a compact closure. Therefore, there exist $z \in C$, $\|z\| = 1$ and a subsequence $(x_{k_j}) \subset C$ such that $x_{k_j} \to z$ as $j \to \infty$ and so $Tz = tz$.

Since $\beta_\nu(T) < t$, there exists $m \in \mathbb{N}$ such that $\nu_C(T^m) < t^m$ by (6). We also have

$$\sup\{r_x(T^m) : x \in C'\} = (\sup\{r_x(T) : x \in C'\})^m \le t^m \le r(T)^m = r(T^m)$$

(see e.g. the proof of [31, Proposition 2.1]). By the above proved assertion there exists $y \in C$, $\|y\| = 1$ such that $T^m y = t^m y$. Define $S = t^{-1}T$. Now the nonzero vector

$$x = y \vee Sy \vee \cdots \vee S^{m-1}y \in C$$

satisfies $Sx = x$ and so $Tx = tx$. □

We call a max cone $C \subset X_+$ σ-order complete if for any $(x_n)_{n\in\mathbb{N}} \subset C$, such that $x_n \le y$ for some $y \in X_+$ and all $n \in \mathbb{N}$, there exists $\vee_{n=1}^{\infty} x_n \in C$. In the following result we give some sufficient conditions for the existence of x in a max-cone C such that $r_x(T) = r(T)$.

Proposition 3.15. *Let X be a normed ordered space such that X_+ is complete. Let $C \subset X_+$ be a σ-order complete normal max-cone and let $T : C \to C$ be a bounded, positively homogeneous, monotone mapping. Then there exists $x \in C$ such that $r_x(T) = r(T)$.*

Proof. The statement is trivial if $r(T) = 0$. So without loss of generality we may assume that $r(T) = 1$. Then for each $k \in \mathbb{N}$ we have $\|T^k\| \ge r(T^k) = 1$. Find $x_k \in C$ such that $\|x_k\| = 1$ and $\|T^k x_k\| \ge 1/2$. Set $x = \bigvee_{k=1}^{\infty} k^{-2} x_k$. Then $x \in C$, $T^k x \ge k^{-2} T^k x_k$ and so

$$M\|T^k x\| \ge k^{-2}\|T^k x_k\| \ge \frac{1}{2k^2},$$

where M is the constant from the definition of a normal cone. So
$r_x(T) = \limsup_k \|T^k x\|^{1/k} \ge 1 = r(T)$. Hence $r_x(T) = r(T)$. □

Our results can be applied to various max-type operators (and to the corresponding max-plus type operators and their tropical versions known also as Bellman operators) arising in diverse areas of mathematics and related applications (see e.g. [31], [23], [30], [29], [4] and the references cited there). We point out the following example that was studied in detail in [31] and [32].

Example 3.16. Given $a > 0$, consider the following max-type kernel operators $T : C[0, a] \to C[0, a]$ of the form

$$(T(x))(s) = \max_{t\in[\alpha(s),\beta(s)]} k(s, t)x(t),$$

where $x \in C[0, a]$ and $\alpha, \beta : [0, a] \to [0, a]$ are given continuous functions satisfying $\alpha \le \beta$. The kernel $k : \mathcal{S} \to [0, \infty)$ is a given non-negative continuous function, where $\mathcal{S}$ denotes the compact set

$$\mathcal{S} = \{(s, t) \in [0, a] \times [0, a] : t \in [\alpha(s), \beta(s)]\}.$$

It is clear that for $C = C_+[0, a]$ it holds $TC \subset C$. We will denote the restriction $T|_C$ again by T. The eigenproblem of these operators arises in the study of periodic solutions of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t - \tau)), \quad \tau = \tau(y(t)),$$

with state-dependent delay (see e.g. [31]).

By [31, Proposition 4.8] and its proof the operator $T : C \to C$ is a positively homogeneous, Lipschitz map that preserves finite suprema. Hence by Corollary 3.8 it follows that $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$. By [31, Theorem 4.3] it also holds that $r(T) = \lim_{n\to\infty} b_n^{1/n} = \inf_{n\geq 1} b_n^{1/n}$, where $b_n = \|T^n\| = \max_{\sigma\in\mathcal{S}_n} k_n(\sigma)$,

$$k_n(\sigma) = k(s_0, s_1)k(s_1, s_2)\cdots k(s_{n-1}, s_n)$$

and

$$\mathcal{S}_n = \{(s_0, s_1, s_2, \ldots, s_n) : s_0 \in [0, a], s_i \in [\alpha(s_{i-1}), \beta(s_{i-1})], i = 1, 2, \ldots, n\}$$

Recall that certain Krein-Rutman type results were proved for $T : C \to C$ in [31, Theorems 4.1, 4.2 , 4.4, Corollaries 4.21, 4.22], i.e., under suitable additional conditions on α, β and k (suitable generalized compactness type conditions on T), there exists $x \in C$, $x \neq 0$, such that $Tx = r(T)x$. However, it was also shown in [31, Proposition 4.23] that there are also reasonable conditions on α, β and k for which such an eigenvector x does not exist.

We also consider the following related example.

Example 3.17. Let M be a nonempty set and let X be the set of all bounded real functions on M. With the norm $\|f\|_\infty = \sup\{|f(t)| : t \in M\}$ and natural operations X is a normed vector lattice. Let $C = X_+$ be the positive cone.

Let $k : M \times M \to [0, \infty)$ satisfy $\sup\{k(t, s) : t, s \in M\} < \infty$.

Let $T : C \to C$ be defined by $(Tf)(s) = \sup\{k(s, t)f(t) : t \in M\}$ and so $\|T\| = \sup\{k(t, s) : t, s \in M\}$. Clearly C is a max-cone, T is bounded, positive homogeneous and preserves finite maxima. So Theorem 3.6 applies. Moreover, T is Lipschitz. So by Corollaries 3.8 and 3.9 we have that $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$ and $r_x(T) \in \sigma_{ap}(T)$ for each $x \in C$, $x \neq 0$. Note also that $r(T) = r_e(T)$, where $e(t) = 1$ for all $t \in M$.

In particular, if M is the set of all natural numbers $\mathbb{N}$, our results apply to infinite bounded non-negative matrices $A = [a(i, j)]$ (i.e., $a(i, j) \geq 0$ for all $i, j \in \mathbb{N}$ and $\|A\| = \sup_{i,j\in\mathbb{N}} a(i, j) < \infty$). In this case, $X = l^\infty$ and $C = l^\infty_+$. We denote $T_A = T$ and we have

$$\|T_A\| = \|A\| = \sup_{j\in\mathbb{N}} \|T_A e_j\|,$$

where $\{e_j : j \in \mathbb{N}\}$ is the set of standard basis vectors. By Theorem 3.7 the following result follows.

Corollary 3.18. *Let A be an infinite bounded non-negative matrix and let $\sup\{r_{e_j}(T_A) : j \in \mathbb{N}\} \leq t \leq r(T_A)$. Then $t \in \sigma_{ap}(T_A)$.*

Proof. The set $C' = \{e_j : j \in \mathbb{N}\}$ satisfies the conditions of Theorem 3.7, which gives the result. □

The following example shows that in general $\sup\{r_{e_j}(T_A) : j \in \mathbb{N}\} \neq r(T_A)$.

Example 3.19. Consider the left (backward) shift $T_A : C \to C$, $C = l^\infty_+$,

$$T_A(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

i.e., $T_A e_1 = 0$ and $T_A e_j = e_{j-1}$ for all $j \geq 2$. Then $r_{e_j}(T_A) = 0$ for each basis element e_j, but $r(T_A) = 1$. We also have $\sigma_{ap}(T_A) = [0, 1] = \sigma_p(T_A)$, where

$$\sigma_p(T_A) = \{t \geq 0 : T_A x = tx \text{ for some } x \in C, \|x\| = 1\}.$$

On the other hand, for its restriction $T_A|_{c_0^+}$, to the positive cone of the space of null convergent sequences c_0^+, we have $r(T_A|_{c_0^+}) = 1$, $\sigma_p(T_A|_{c_0^+}) = [0, 1)$ and $\sigma_{ap}(T_A|_{c_0^+}) = [0, 1]$. Note that this again shows, in particular, that some compactness type assumptions in Remark 3.12 and in Theorem 3.14 are necessary.

Remark 3.20. The special case of Example 3.17 when $M = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ is well known and studied under the name max-algebra (an analogue of linear algebra). Together with its isomorphic versions (max-plus algebra and min-plus algebra also known as tropical algebra) it provides an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, DNA analysis, ... (see e.g. [14], [15], [36], [35] [13], [16], [38] and the references cited there).

In particular, for a non-negative $n \times n$ matrix A it holds (see e.g. [36, Theorem 2.7]) that

$$\sigma_{ap}(T_A) = \sigma_p(T_A) = \{t : \text{ there exists } j \in \{1, \ldots, n\}, t = r_{e_j}(T_A)\}.$$

However, as Example 3.19 shows, an analogue of this result is not valid for infinite bounded non-negative matrices.

4. Results on normal convex cones in normed spaces

Next we prove the analogues of Theorems 3.6 and 3.7 for positively homogeneous, additive and Lipschitz maps defined on normal wedges in normed spaces. This result generalizes and extends an implicitly known result that for a positive linear operator T on a Banach lattice X there exists a positive sequence of approximative vectors for the usual spectral radius (see e.g. the proof of Krein-Rutman's theorem [1, Theorem 7.10]). We do not assume the completeness of the norm, we do not assume that the space X is a lattice and we also do not need to assume that the wedge C is closed. In the proof we apply a technique of the proof of Theorem 3.6 and we include it for the sake of completeness.

Theorem 4.1. *Let X be a normed space, $C \subset X$ a non-zero normal wedge and let $T : C \to C$ be positively homogeneous, additive and Lipschitz. If*

$$\sup\{r_x(T) : x \in C\} \leq t \leq r(T),$$

then $t \in \sigma_{ap}(T)$.

In particular, $r(T) \in \sigma_{ap}(T)$. Moreover, $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$.

Proof. By Lemma 3.3 we have $r(T) \geq t$ for all $t \in \sigma_{ap}(T)$.

Let $\sup\{r_x(T) : x \in C\} \leq t \leq r(T)$. Similarly as in the proof of Theorem 3.6 we may assume without loss of generality that $t = 1$.

We distinguish two cases:

I. Suppose that $\sup\{\|\sum_{j=0}^{n} T^j x\| : x \in C, \|x\| = 1, n \in \mathbb{N}\} = \infty$.

Let $k \in \mathbb{N}$. Find n_k and $x_k \in C$ such that $\|x_k\| = 1$ and $\|\sum_{j=0}^{n_k} T^j x_k\| > k$. Find $t_k \in (1, 1 + k^{-1})$ such that $t_k^{-n_k-1} > 1/2$. Find $r_k > n_k$ such that $\frac{\|T^{r_k} x_k\|}{t_k^{r_k}} < 1$. Set $y_k := \sum_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j+1}}$. Then $M\|y_k\| \geq \|t_k^{-n_k-1} \sum_{j=0}^{n_k} T^j x_k\| \geq k/2$, where M is the constant from the definition of a normal cone.

Set $u_k = \frac{y_k}{\|y_k\|}$. Then $\|u_k\| = 1$. Since T is additive, positively homogeneous and Lipschitz with a Lipschitz constant L, we have

$$\|Tu_k - u_k\| \leq \|Tu_k - t_k u_k\| + (t_k - 1)\|u_k\| = \|y_k\|^{-1} \left\| T\left(\sum_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j+1}}\right) - \sum_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j}} \right\| + (t_k - 1)$$

$$\leq \|y_k\|^{-1} \left(\left\| T\left(\sum_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j+1}}\right) - T\left(\sum_{j=1}^{r_k} \frac{T^{j-1} x_k}{t_k^{j}}\right) \right\| + 1 \right) + (t_k - 1)$$

$$\leq \frac{2M}{k} \left(L \left\| \sum_{j=0}^{r_k} \frac{T^j x_k}{t_k^{j+1}} - \sum_{j=1}^{r_k} \frac{T^{j-1} x_k}{t_k^{j}} \right\| + 1 \right) + k^{-1} = \frac{2M}{k} \left(L \left\| \frac{T^{r_k} x_k}{t_k^{r_k+1}} \right\| + 1 \right) + k^{-1}$$

$$< \frac{2M(L+1)+1}{k} \to 0$$

as $k \to \infty$. Hence $1 \in \sigma_{ap}(T)$.

II. Suppose that $M_0 := \sup\{\|\sum_{j=0}^{n} T^j x\| : x \in C, \|x\| = 1, n \in \mathbb{N}\} < \infty$.

Let $\varepsilon > 0$ and $K = \|T\|$. Let $n \in \mathbb{N}$ be the number constructed in Lemma 3.5. We have $\|T^n\| \geq r(T^n) = r(T)^n \geq 1$, so there exists $x \in C$ such that $\|x\| = 1$ and $\|T^n x\| \geq 1/2$. Write $\alpha_k = \|T^k x\|$.

Let $m \in \mathbb{N}$ and β_k $(k \geq 0)$ be the numbers constructed in Lemma 3.5 , i.e., $\beta_k \geq 0$, $\beta_0 \leq \varepsilon$, $|\beta_{k+1} - \beta_k| \leq 2\varepsilon$, $\beta_k < \beta_{k+1}$ for all $k = 0, 1, \dots, m-1$, $\beta_k > \beta_{k+1}$ for all $k = m, m+1, \dots$, $\alpha_k \beta_k \leq 1$, $\alpha_m \beta_m = 1$, $\lim_{k\to\infty} \alpha_k \beta_{k+1} = 0$.

Fix $r > m$ such that $\beta_r < \varepsilon \|T^{r-1} x\|^{-1}$.

Set $u = \sum_{k=0}^{r} \beta_k T^k x$. Since $u \geq \beta_m T^m x$, we have $M\|u\| \geq \beta_m \|T^m x\| = 1$. We have

$$\|Tu - u\| = \left\| T\left(\sum_{k=0}^{r} \beta_k T^k x\right) - \sum_{k=0}^{r} \beta_k T^k x \right\|$$

$$\leq \left\| T\left(\sum_{k=0}^{r} \beta_k T^k x\right) - \sum_{k=1}^{r} \beta_k T^k x \right\| + \left\| \beta_0 x \right\|$$

$$\leq L \left\| \sum_{k=0}^{r} \beta_k T^k x - \sum_{k=1}^{r} \beta_k T^{k-1} x \right\| + \varepsilon$$

$$\leq L\left(\Big\|\beta_r T^r x\Big\| + \Big\|\sum_{k=0}^{m-1} T^k x(\beta_{k+1}-\beta_k)\Big\| + \Big\|\sum_{k=m}^{r-1} T^k x(\beta_k-\beta_{k+1})\Big\|\right) + \varepsilon$$
$$\leq L(K\varepsilon + 4\varepsilon M_0 M) + \varepsilon \to 0$$

as $\varepsilon \to 0$. Hence $1 \in \sigma_{ap}(T)$. □

Similarly as Theorem 3.7, the following more general result is proved in a similar way.

Theorem 4.2. *Let X be a normed space, $C \subset X$ a non-zero normal wedge and let $T : C \to C$ be positively homogeneous, additive and Lipschitz. Let $C' \subset C$ be a bounded subset satisfying $\|T^n\| = \sup\{\|T^n x\| : x \in C'\}$ for all n. Then*

$$[\sup\{r_x(T) : x \in C'\}, r(T)] \subset \sigma_{ap}(T)$$

and $r(T) = \max\{t : t \in \sigma_{ap}(T)\}$.

Remark 4.3. In fact, in the setting of Theorems 4.1 and 4.2 the map T extends to a bounded linear operator on the normed space $C - C$. Note that in general $C - C$ is not a lattice. Moreover, the obtained approximate eigenvectors are in C.

Corollary 4.4. *Let X be a normed space with a non-zero normal wedge $C \subset X$. If $T : C \to C$ is positively homogeneous, additive and Lipschitz, then $r_x(T) \in \sigma_{ap}(T)$ for each $x \in C$.*

Proof. Let $x \in C$ and let K be the smallest wedge generated by x invariant for T, i.e., $K = \{\sum_{j=0}^n \alpha_j T^j x : n \in \mathbb{N}, \alpha_0, \dots, \alpha_n \geq 0\}$.

Let $y = \sum_{j=0}^n \alpha_j T^j x \in K$. Then $\|T^k y\| \leq \sum_{j=0}^n \alpha_j \|T^{k+j} x\|$ and it is easy to see that $r_y(T|_K) \leq r_x(T) \leq r(T|_K)$. By Theorem 4.1, $r_x(T) \in \sigma_{ap}(T|_K) \subset \sigma_{ap}(T)$. □

Remark 4.5. Let X, C and T be as in Theorem 4.2. Under additional compactness type assumptions from Remark 3.12 or Theorem 3.14, Theorem 4.2 implies Krein-Rutman type results, i.e., the existence of an eigenvector $x \in C$, $x \neq 0$ such that $Tx = tx$. We omit the details. As is well-known and also illustrated by Examples 3.11 and 3.19, such additional assumptions are necessary.

The following result is essentially known, see e.g. [32, Theorem 3.3], [22, Theorem 2.1] and [31, Theorem 2.2 and remarks after it]. The sketch of the proof is included for the sake of completeness.

Proposition 4.6. *Let X be a normed space and $C \subset X$ a non-zero normal complete wedge. Let $T : C \to C$ be bounded and positively homogeneous. If, in addition,*

(i) T is monotone on C or

(ii) T is continuous and additive on C,

then there exists $x \in C$ such that $r_x(T) = r(T)$.

Proof. Without loss of generality we assume that $r(T) = 1$ and for each $k \in \mathbb{N}$ choose $x_k \in C$ as in the proof of Proposition 3.15. Define $x = \sum_{k=1}^{\infty} k^{-2} x_k \in C$. If T satisfies (i) or (ii), it follows that $Tx \geq k^{-2} T x_k$. Conclude the proof as in the proof of Proposition 3.15. □

Remark 4.7. Similarly as in [31, Remark on p.12], a slight generalization of Proposition 4.6 is possible. Namely, if $C_1 \subset C$ are given wedges and T satisfies the conditions of Theorem 4.6 with respect to the wedge C as stated. Additionally, we assume that $TC_1 \subset C_1$, where the wedge C_1 is complete. Then there exists $x \in C_1$ that equals the Bonsall cone spectral radius of T with respect to C_1.

As pointed out (and applied) in [31] and [32, Theorem 3.3], the main reason for this generalization is that it may happen that a non-linear map is monotone with respect to the (pre)ordering $\leq_C$, but it is not monotone with respect to the (pre)ordering $\leq_{C_1}$ (see, for instance, [27] and the "renormalization operators" which occur in discussing diffusion on fractals).

Acknowledgments. The first author was supported by grants No. 14-07880S of GA CR and RVO: 67985840.

The second author was supported in part by the JESH grant of the Austrian Academy of Sciences and by grant P1-0222 of the Slovenian Research Agency. The second author thanks Marko Kandić and Roman Drnovšek for useful comments and to his collegues and staff at TU Graz for their hospitality during his stay in Austria.

References

[1] Y.A. Abramovich and C.D. Aliprantis, *An invitation to operator theory*, American Mathematical Society, Providence, 2002.

[2] M. Akian and S. Gaubert, Policy iteration for perfect information stochastic mean payoff games with bounded first return times is strongly polynomial, E-print: arXiv:1310.4953

[3] M. Akian, S. Gaubert and C. Walsh, Discrete max-plus spectral theory, in *Idempotent Mathematics and Mathematical Physics*, G.L. Litvinov and V.P. Maslov, Eds, vol. 377 of Contemporary Mathematics, pp. 53–77, AMS, 2005. E-print: arXiv:math.SP/0405225.

[4] M. Akian, S. Gaubert and R.D. Nussbaum, A Collatz-Wielandt characterization of the spectral radius of order-preserving homogeneous maps on cones, E-print: arXiv:1112.5968, 2014.

[5] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis, A Hitchhiker's Guide*, Third Edition, Springer, 2006.

[6] C.D. Aliprantis, D.J. Brown and O. Burkinshaw, *Existence and optimality of competitive equilibria*, Springer-Verlag, Berlin, 1990.

[7] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Reprint of the 1985 original, Springer, Dordrecht, 2006.

[8] C.D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*, Second edition, Mathematical Surveys and Monographs 105, American Mathematical Society, Providence, RI, 2003.

[9] C.D. Aliprantis and R. Tourky, *Cones and duality*, American Mathematical Society, Providence, 2007.

[10] J. Appell, E. De Pascale and A. Vignoli, A comparison of different spectra for nonlinear operators, *Nonlinear Anal.* 40 (2000), 73–90.

[11] J. Appell, E. De Pascale and A. Vignoli, *Nonlinear Spectral Theory*, Walter de Gruyter GmbH and Co. KG, Berlin, 2004.
[12] J. Appell, E. Giorgieri and M. Väth, Nonlinear spectral theory for homogeneous operators, *Nonlinear Funct. Anal. Appl.* 7 (2002), 589–618.
[13] F.L. Baccelli, G. Cohen, G.-J. Olsder and J.-P.Quadrat, *Synchronization and linearity*, John Wiley, Chichester, New York, 1992.
[14] R.B. Bapat, A max version of the Perron-Frobenius theorem, *Linear Algebra Appl.* 275-276, (1998), 3–18.
[15] P. Butkovič, *Max-linear systems: theory and algorithms*, Springer-Verlag, London, 2010.
[16] P. Butkovič, S. Gaubert and R.A. Cuninghame-Green, Reducible spectral theory with applications to the robustness of matrices in max-algebra, *SIAM J. Matrix Anal. Appl.* 31(3) (2009), 1412–1431.
[17] W. Feng, A new spectral theory for nonlinear operators and its applications, *Abstr. Appl. Anal.* 2 (1997), 163–183.
[18] M. Furi and A. Vignoli, A nonlinear spectral approach to surjectivity in Banach spaces, *J. Funct. Anal.* 20 (1975), 304–318.
[19] M. Furi, M. Martelli and A. Vignoli, Contributions to the spectral theory for nonlinear operators in Banach spaces, *Ann. Pura Appl.* 118, (1978), 229–294.
[20] K. Georg and M. Martelli, On the spectral theory for nonlinear operators, *J. Funct. Anal.* 24 (1977), 140–147.
[21] E. Giorgieri and M. Väth, A characterization of 0-epi maps with a degree, *J. Funct. Anal.* 187 (2001), 183–199.
[22] G. Gripenberg, On the definition of the cone spectral radius, *Proc. Amer. Math. Soc.* 143 (2015), 1617–1625.
[23] V.N. Kolokoltsov and V.P. Maslov, *Idempotent analysis and its applications*, Kluwer Acad. Publ., 1997.
[24] B. Lemmens and R.D. Nussbaum, Continuity of the cone spectral radius, *Proc. Amer. Math. Soc.* 141 (2013), 2741–2754.
[25] B. Lemmens and R.D. Nussbaum, *Nonlinear Perron-Frobenius Theory*, Cambridge University Press, 2012.
[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, A reprint of the 1977 and 1979 editions, Springer, 1996.
[27] B. Lins and R.D. Nussbaum, Denjoy-Wolff theorems, Hilbert metric nonexpansive maps on reproduction-decimation operators, *J. Funct. Anal.* 254 (2008), 203–245.
[28] G.L. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: A brief introduction, *J. Math. Sci.* (N. Y.) 140, no.3 (2007), 426–444.
[29] G.L. Litvinov, V.P. Maslov and G.B. Shpiz, Idempotent functional analysis: An algebraic approach, *Math Notes* 69, no. 5-6 (2001) 696–729.
[30] G.L. Litvinov and V.P. Maslov (eds.), Idempotent mathematics and mathematical physics, *Contemp. Math.* Vol. 377, Amer.Math. Soc., Providence, RI, 2005.
[31] J. Mallet-Paret and R.D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, *Discrete and Continuous Dynamical Systems*, vol 8, num 3 (2002), 519–562.
[32] J. Mallet-Paret and R. D. Nussbaum, Generalizing the Krein-Rutman theorem, measures of noncompactness and the fixed point index, *J. Fixed Point Theory and Applications* 7 (2010), 103–143.
[33] J. Mallet-Paret and R. D. Nussbaum, Inequivalent measures of noncompactness, Inequivalent measures of noncompactness, *Ann. Mat. Pura Appl.* (4) 190 (2011), 453–488.
[34] J. Mallet-Paret and R. D. Nussbaum, Inequivalent measures of noncompactness, Inequivalent measures of noncompactness and the radius of the essential radius, *Proc. Amer. Math. Soc.* 139 (2011), 917–930.
[35] V. Müller and A. Peperko, Generalized spectral radius and its max algebra version, *Linear Algebra Appl.* 439 (2013), 1006–1016.
[36] V. Müller and A. Peperko, On the spectrum in max-algebra, *Linear Algebra Appl.* 485 (2015), 250–266.

[37] R.D. Nussbaum, Eigenvalues of nonlinear operators and the linear Krein-Rutman, in: Fixed Point Theory (Sherbrooke, Quebec, 1980), E. Fadell and G. Fournier, editors, Lecture notes in Mathematics 886, *Springer-Verlag*, Berlin (1981), 309–331.
[38] L. Pachter and B. Sturmfels (eds.), *Algebraic statistics for computational biology*, Cambridge Univ. Press, New York, 2005.
[39] P. Santucci and M. Väth, On the definition of eigenvalues of nonlinear operators, *Nonlinear Anal.* 40 (2000), 565-576.
[40] G. B. Shpiz, An eigenvector existence theorem in idempotent analysis, *Mathematical Notes* 82, 3-4 (2007), 410–417.
[41] W. Wnuk, *Banach lattices with order continuous norms*, Polish Scientific Publ., PWN, Warszawa, 1999.

Vladimir Müller
Institute of Mathematics, Czech Academy of Sciences
Žitna 25
115 67 Prague, Czech Republic
email: muller@math.cas.cz

Aljoša Peperko
Faculty of Mechanical Engineering
University of Ljubljana
Aškerčeva 6
SI-1000 Ljubljana, Slovenia
and
Institute of Mathematics, Physics and Mechanics
Jadranska 19
SI-1000 Ljubljana, Slovenia
e-mails : aljosa.peperko@fmf.uni-lj.si , aljosa.peperko@fs.uni-lj.si