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SEVERAL NON-STANDARD PROBLEMS FOR THE STATIONARY STOKES SYSTEM

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Abstract. This paper studies the Stokes system $-\Delta\mathbf{u}+\nabla\rho=\mathbf{f}$, $\nabla\cdot\mathbf{u}=\chi$ in Ω with three boundary conditions: 1) $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{g}$, $\mathbf{n}\times(\nabla\times\mathbf{u})=\mathbf{n}\times\mathbf{h}$ on $\partial\Omega$; 2) $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{g}$, $\tau\cdot[\partial\mathbf{u}/\partial\mathbf{n}-\rho\mathbf{n}+b\mathbf{u}]=\mathbf{h}\cdot\tau$ on $\partial\Omega$; 3) $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{g}$, $[T(\mathbf{u},\rho)\mathbf{n}+b\mathbf{u}]\cdot\tau=\mathbf{h}\cdot\tau$ on $\partial\Omega$. Here Ω is a bounded simply connected planar domain. We find a necessary and sufficient condition for the existence of a solution in Sobolev spaces $W^{s,q}(\Omega;\mathbb{R}^2)\times W^{s-1,q}(\Omega)$ with $1+1/q<s<\infty$, in Besov spaces $B^{q,r}_s(\Omega;\mathbb{R}^2)\times B^{q,r}_{s-1}(\Omega)$ with $1+1/q<s<\infty$, and classical solutions in $\mathcal{C}^{k,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathcal{C}^{k-1,\alpha}(\overline{\Omega})$ with $0<\alpha<1$, $k\in\mathbb{N}$.

1. Introduction

This paper studies the stationary Stokes system

$$-\Delta\mathbf{u}+\nabla\rho=\mathbf{f},\quad \nabla\cdot\mathbf{u}=\chi\quad\text{in }\Omega \tag{1}$$

with the boundary condition

$$\mathbf{u_n}=\mathbf{g_n},\quad \mathbf{n}\times(\nabla\times\mathbf{u})=\mathbf{n}\times\mathbf{h}\quad\text{on }\partial\Omega \tag{2}$$

or with the boundary condition

$$\mathbf{u_n}=\mathbf{g_n},\quad \left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}-\rho\mathbf{n}+b\mathbf{u}\right]_\tau=\mathbf{h}_\tau\text{ on }\partial\Omega \tag{3}$$

or with the boundary condition

$$\mathbf{u_n}=\mathbf{g_n},\quad [T(\mathbf{u},\rho)\mathbf{n}+b\mathbf{u}]_\tau=\mathbf{h}_\tau\text{ on }\partial\Omega \tag{4}$$

on a bounded domain $\Omega\subset\mathbb{R}^2$ with connected Lipschitz boundary. Here

$$T(\mathbf{u},\rho)\mathbf{n}=[2\hat\nabla\mathbf{u}-\rho I]\mathbf{n},\quad \hat\nabla\mathbf{u}=\frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T].$$

We denote by $\mathbf{n}=\mathbf{n}^\Omega$ the outward unit normal vector of Ω. If $\mathbf{v}$ is a vector, then $\mathbf{v_n}=(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ is the normal part of $\mathbf{v}$, and $\mathbf{v}_\tau=\mathbf{v}-\mathbf{v_n}$ is the tangential part of $\mathbf{v}$.

These problems are so called problems of Navier's type. Such problems come up in many practical applications, e.g. fluid dynamics, electromagnetic field applications, and decomposition of vector fields. For motivation of the problem (1), (2) see [32, pp. 87–98], [33, pp. 129–131], [1], [2], [18]. M. Amara, E. Chacón Vera and D. Trujillo studied this problem on a polygon $\Omega\subset\mathbb{R}^2$ for $\rho\in L^2(\Omega)$ and $\mathbf{u}\in\{\mathbf{v}\in L^2(\Omega;\mathbb{R}^2);\nabla\times\mathbf{v}\in L^2(\Omega;\mathbb{R}^2),\nabla\cdot\mathbf{v}\in L^2(\Omega)\}$ (see [4]). Under the additional condition on the domain that $\{\mathbf{v}\in L^2(\Omega;\mathbb{R}^2);\nabla\times\mathbf{v}\equiv 0,\nabla\cdot\mathbf{v}\equiv 0\}=\{0\}$, they proved the solvability of the problem for $\mathbf{f}\in L^2(\Omega,\mathbb{R}^2)$, $\chi\equiv 0$, $\mathbf{g}\equiv 0$ and $\mathbf{h}\in L^2(\partial\Omega;\mathbb{R}^2)$. It is proved that a velocity $\mathbf{u}$ is unique. J. M. Bernard treated

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this problem in $W^{1,2}(\Omega,\mathbb{R}^3)\times L^2(\Omega)$ for simply connected $\Omega\subset\mathbb{R}^3$ with boundary of class $\mathcal{C}^{1,1}$ (see [13]). He proved that there exists a solution of (1), (2) if $\mathbf{f}$, χ, $\mathbf{g}$ and h satisfy some finitely many conditions. Ch. Amrouche and A. Rejaba studied this problem for $\Omega\subset\mathbb{R}^3$ with boundary of class $\mathcal{C}^{1,1}$ (see [7]). They proved that for $\mathbf{f}\in[H_0^{p'}(div,\Omega)]'$, $\chi\in L^p(\Omega)$, $\mathbf{g}\in W^{1-1/p,p}(\partial\Omega;\mathbb{R}^3)$ and $\mathbf{h}\times\mathbf{n}\in W^{-1/p,p}(\partial\Omega;\mathbb{R}^3)$ there exists a solution in $W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\Omega)$. Here $H_0(div,\Omega)=\{\mathbf{v}\in L^p(\Omega;\mathbb{R}^3);\nabla\cdot\mathbf{v}\in L^p(\Omega),\mathbf{v}\cdot\mathbf{n}=0\text{ on }\partial\Omega\}$. A velocity $\mathbf{u}$ is unique and a pressure ρ is unique up to an additive constant. Ch. Amrouche, P. Penel and N. Seloula proved in [6] that if $\partial\Omega$ is of class $\mathcal{C}^{2,1}$, $\mathbf{f}\in L^p(\Omega;\mathbb{R}^3)$, $\chi\in W^{1,p}(\Omega)$, $\mathbf{g}\in W^{2-1/p,p}(\partial\Omega;\mathbb{R}^3)$ and $\mathbf{h}\times\mathbf{n}\in W^{1-1/p,p}(\partial\Omega;\mathbb{R}^3)$, then $(\mathbf{u},\rho)\in W^{2,p}(\Omega,\mathbb{R}^3)\times W^{1,p}(\Omega)$. Ch. Amrouche and N. Seloula studied this problem in $W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\Omega)$ and in $L^p(\Omega,\mathbb{R}^3)\times W^{-1,p}(\Omega)$ for $\Omega\subset\mathbb{R}^3$ with boundary of class $\mathcal{C}^{2,1}$ (see [8] and [9]). They found necessary and sufficient conditions for the existence of solutions satisfying $\int_C\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma=0$ for each component C of $\partial\Omega$. J. H. Bramble and P. Lee supposed in [15] that $\Omega\subset\mathbb{R}^3$ has boundary of class $\mathcal{C}^{k+2}$. They proved that for $\mathbf{f}\in H^{k-1}(\Omega;\mathbb{R}^3)$, $\chi\equiv 0$, $\mathbf{g}\in H^{k-3/2}(\partial\Omega;\mathbb{R}^3)$ and $\mathbf{h}\times\mathbf{n}\in W^{k+1/2}(\partial\Omega;\mathbb{R}^3)$, there exists a unique solution $(\mathbf{u},\rho)\in H^{k+1}(\Omega,\mathbb{R}^3)\times H^k(\Omega)$ of (1), (2) such that $\int_\Omega\rho=0$. C. Bardos treated the problem on a bounded planar domain with boundary of class $\mathcal{C}^2$ in [11]. He proved that if $\mathbf{f}\equiv 0$, $\chi\equiv 0$, $h\equiv 0$ and $\mathbf{g}\cdot\mathbf{n}\in L^{3/2}(\partial\Omega)$ with $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\,\mathrm{d}\sigma=0$, then there exists a distribution ρ and a unique $\mathbf{u}\in H^2(\Omega,\mathbb{R}^2)$ such that $(\mathbf{u},\rho)$ is a solution of the problem (1), (2). Ch. Amrouche and M. Meslameni assumed in [5] that $\Omega\subset\mathbb{R}^3$ is an exterior domain with connected boundary of class $\mathcal{C}^{1,1}$. Under some compatibility conditions they proved the unique solvability of the problem in $W_0^{1,2}(\Omega;\mathbb{R}^3)\times L^2(\Omega)$, where $W_0^{1,2}(\Omega):=\{v\in L^2_{loc}(\Omega);\partial_j v\in L^2(\Omega),v(\mathbf{x})/\sqrt{1+|\mathbf{x}|^2}\in L^2(\Omega)\}$. If the boundary of Ω is of class $\mathcal{C}^{2,1}$ then they proved that the solution is in $W_1^{2,2}(\Omega)\times W_1^{1,2}(\Omega)$ provided $f\in W_1^{0,2}(\Omega)$, $\xi\in W_1^{1,2}(\Omega)$, $g\in H^{3/2}(\partial\Omega)$, $h\in H^{1/2}(\partial\Omega)$.

Ch. Amrouche and A. Rejaba studied in [7] the problem (1), (3) for $\Omega\subset\mathbb{R}^3$ with boundary of class $\mathcal{C}^{2,1}$. They proved that for $b\equiv 0$, $\mathbf{f}\in[H_0^{p'}(div,\Omega)]'$, $\chi\in L^p(\Omega)$, $\mathbf{g}\in W^{1-1/p,p}(\partial\Omega;\mathbb{R}^3)$, $\mathbf{h}\times\mathbf{n}\in W^{-1/p,p}(\partial\Omega;\mathbb{R}^3)$ satisfying $\mathbf{h}\cdot\mathbf{n}\equiv 0$ on $\partial\Omega$,

$$\int_\Omega\chi\,\mathrm{d}\mathbf{x}=\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\,\mathrm{d}\sigma,$$

there exists a solution in $W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\Omega)$. A velocity $\mathbf{u}$ is unique, a pressure ρ is unique up to an additive constant. Y. Raudin treated in [49] the problem for $b\equiv 0$, $\mathbf{f}\equiv 0$ and $\chi\equiv 0$ in the half-space in weighted spaces $W_l^{1,p}(\mathbb{R}^m_+,\mathbb{R}^m)\times W_l^{0,p}(\mathbb{R}^m_+)$.

The problem (1), (4) is an appropriate physical model for flow problems with free boundaries and for flows past chemically reacting walls. It occurs in such liquids as lubricants, hydraulic fluids, biological fluids etc. R. Verfürth proved in [57] the solvability of the problem in the factor space $H^1(\Omega;\mathbb{R}^m)/\mathcal{S}$ for a simply connected bounded domain $\Omega\subset\mathbb{R}^m$, $m=2,3$, with boundary of class $\mathcal{C}^3$. (Here $\mathcal{S}$ is the space of rigid body rotations of Ω.) Ch. Amrouche and A. Rejaba supposed in [7] that $\Omega\subset\mathbb{R}^3$ has boundary of class $\mathcal{C}^{2,1}$. They proved that for $b\equiv 0$, $\mathbf{f}\in[H_0^{p'}(div,\Omega)]'$, $\chi\in L^p(\Omega)$, $\mathbf{g}\in W^{1-1/p,p}(\partial\Omega;\mathbb{R}^3)$, $\mathbf{h}\times\mathbf{n}\in W^{-1/p,p}(\partial\Omega;\mathbb{R}^3)$ satisfying several conditions there exists a solution in $W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\Omega)$. Here $H_0(div,\Omega)=\{\mathbf{v}\in L^p(\Omega);\mathbb{R}^3;\nabla\cdot\mathbf{v}\in L^p(\Omega),\mathbf{v}\cdot\mathbf{n}=0\text{ on }\partial\Omega\}$. A velocity $\mathbf{u}$ is unique. If $\mathbf{f}\in L^p(\Omega;\mathbb{R}^3)$, $\chi\in W^{1,p}(\Omega)$, $\mathbf{g}\cdot\mathbf{n}\in W^{2-1/p,p}(\partial\Omega)$ and $\mathbf{h}_\tau\in W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$, then $\mathbf{u} \in W^{2,p}(\Omega;\mathbb{R}^3)$, $\rho \in W^{1,p}(\Omega)$. Ch. Amrouche and A. Rejaba also studied very weak solutions of the problem in $L^p(\Omega;\mathbb{R}^3) \times W^{-1,p}(\Omega)$. T. Z. Boulmezaoud studied in [14] the problem for $b \equiv 0$ in the half-space in some weighted Sobolev spaces. A. Kozhesnikov and O. Lepsky treated in [30] the problem for $b \equiv 0$ in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary of class $\mathcal{C}^\infty$ in spaces $H^k(\Omega,\mathbb{R}^3)\times H^{k-1}(\Omega)$. N. Tanaka studied the problem for $b \equiv 0$ in the half-space $\mathbb{R}^3_+$ in homogeneous Sobolev spaces $D^{k,2}(\mathbb{R}^3_+;\mathbb{R}^3) \times D^{k-1,2}(\mathbb{R}^3_+)$. (See [52].)

Since there are almost no results concerning these problems in the plane, this article is devoted to the study of the problems for planar domains. In the whole paper we suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with connected Lipschitz boundary. First we study the auxiliary problem

$$-\Delta\mathbf{u} + \nabla\rho = 0,\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u}\cdot\mathbf{n} = g,\ \rho = h \text{ on } \partial\Omega. \tag{5}$$

(This problem is interesting by itself and it was studied from the numerical point of view in [18].) We reduce this problem to the couple of two Dirichlet problems for the Laplace equation. It enables us to utilize the whole theory of the Laplace equation. We restrict our-self to the homogeneous Stokes system because we need only the problem (5) in the sequel. The results can be extended to a non-homogeneous Stokes system by a standard way - as for the problem (1), (2).

We prove that if $(\mathbf{u},\rho)$ is a solution of the homogeneous Stokes system then $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic functions. This shows the relations between the problem (1), (2) for $\mathbf{f} \equiv 0$, $\chi \equiv 0$ and the problem (5). The problem (1), (2) leads to the Dirichlet problem for the Laplace equation with an unknown $(\partial_1 u_2 - \partial_2 u_1)$. After solving this problem we find a function ρ such that $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic functions. Now, it is enough to study the problem (5) instead of the original problem (1), (2). Then we extend the results for (1), (2) to the non-homogeneous Stokes system by a standard way.

We show that the problem (1), (3) and the problem (1), (4) are compact perturbations of the problem (1), (2). We find a necessary and sufficient condition for the existence of a solution in the Sobolev spaces $W^{s,q}(\Omega;\mathbb{R}^2)\times W^{s-1,q}(\Omega)$ for $\partial\Omega \in \mathcal{C}^{k,1}$ and $1+1/q < s \le k$, in the Besov spaces $B^{q,r}_s(\Omega;\mathbb{R}^2) \times B^{q,r}_{s-1}(\Omega)$ for $\partial\Omega \in \mathcal{C}^{k,1}$ and $1+1/q < s < k$, and in $\mathcal{C}^{k-1,\alpha}(\Omega,\mathbb{R}^2)\times \mathcal{C}^{k-2,\alpha}(\Omega)$ for $\partial\Omega \in \mathcal{C}^{k,\alpha}$. The problem (1), (2) is studied in these spaces and also in less regular spaces on domains with less regular boundaries. We show the existence of a solution $(\mathbf{u},\rho) \in W^{t,p}(\Omega;\mathbb{R}^2)\times W^{s,q}(\Omega)$ with $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$, or $(\mathbf{u},\rho) \in B^{p,\beta}_t(\Omega;\mathbb{R}^2)\times B^{q,r}_s(\Omega)$ with $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$, or $(\mathbf{u},\rho) \in \mathcal{C}^{k,\gamma}(\Omega,\mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\Omega)$ with $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\beta}(\Omega)$. If we study the problem (1), (2) in the sense of non-tangential limits, we can suppose only that $\partial\Omega$ is Lipschitz.

2. The auxiliary problem (5)

In this section we study the problem (5). **In the whole paper we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with connected Lipschitz boundary.** We denote by $\mathbf{n} = \mathbf{n}^\Omega$ the outward unit normal vector of Ω, and by $\tau = \tau^\Omega = (-n^\Omega_2, n^\Omega_1)$ the unit tangential vector of $\partial\Omega$. If we study the problem in Sobolev spaces $W^{t,p}(\Omega,\mathbb{R}^2) \times W^{s,q}(\Omega)$ or in Besov spaces $B^{p,\beta}_t(\Omega,\mathbb{R}^2) \times B^{q,r}_s(\Omega)$, then the boundary conditions are fulfilled in the sense of traces. For the homogeneous Stokes system and for boundary conditions from Lebesgue spaces we look for solutions that satisfy the boundary conditions in the sense of non-tangential limits (i.e. in weaker sense). Let us introduce necessary notations.

Let $k \in \mathbb{N}_0$, $1 < p < \infty$. We denote the Sobolev space $W^{k,p}(\Omega) = \{u \in L^p(\Omega); \partial^\beta u \in L^p(\Omega)\ \forall |\beta| \leq k\}$. If $s = k + \lambda$ with $0 < \lambda < 1$ denote by $W^{s,p}(\Omega)$ the space of all $u \in W^{k,p}(\Omega)$ such that

$$\sum_{|\beta|=k} \int_\Omega \int_\Omega \frac{|\partial^\beta u(x) - \partial^\beta u(y)|^p}{|x-y|^{2+p\lambda}}\ \mathrm{d}x\ \mathrm{d}y < \infty.$$

If $s \geq 0$ denote by $\mathring{W}^{s,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ (the space of infinitely differentiable functions with compact support in Ω) in $W^{s,p}(\Omega)$ and by $W^{-s,p}(\Omega)$ the dual space of $\mathring{W}^{s,p/(p-1)}(\Omega)$. If $t > \tau$ then $W^{t,p}(\Omega) \subset W^{\tau,p}(\Omega)$.

If $s \in \mathbb{R}^1$ and $1 < p, q \leq \infty$, denote by $B_s^{p,q}(\mathbb{R}^2)$ the Besov space. (For the definition see for example [56].) If $k \in \mathbb{N}_0$, $s = k + \lambda$ with $0 < \lambda < 1$ and $p, q < \infty$ then $B_s^{\infty,\infty}(\mathbb{R}^2) = \mathcal{C}^{k,\lambda}(\mathbb{R}^2)$ and $u \in B_s^{p,q}(\mathbb{R}^2)$ if $u \in W^{k,p}(\mathbb{R}^2)$ and

$$\sum_{|\beta|=k} \int_0^\infty \left(\int_{\mathbb{R}^2} \int_{\{y \in \mathbb{R}^3; |x-y|<t\}} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|^p}{t^2}\ \mathrm{d}y\ \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}t}{t^{\lambda q+1}} < \infty.$$

By $B_s^{p,q}(\Omega)$ we denote the space of restrictions of functions from $B_s^{p,q}(\mathbb{R}^2)$ onto Ω. The norm on $B_s^{p,q}(\Omega)$ is defined by

$$\|u\|_{B_s^{p,q}(\Omega)} = \inf\{\|f\|_{B_s^{p,q}(\mathbb{R}^2)}; f = u \text{ on } \Omega\}.$$

If $s > t$ then $B_s^{p,q}(\Omega) \subset B_t^{p,q}(\Omega)$. If s is not integer and $p < \infty$ then $B_s^{p,p}(\Omega) = W^{s,p}(\Omega)$.

If $\mathbf{x} \in \partial\Omega$, $a > 0$ denote the non-tangential approach regions of opening a at the point $\mathbf{x}$ by

$$\Gamma_a(\mathbf{x}) = \{\mathbf{y} \in \Omega; |\mathbf{x} - \mathbf{y}| < (1+a)\,\mathrm{dist}(\mathbf{y}, \partial\Omega)\}.$$

If now $\mathbf{v}$ is a vector function defined in Ω, we denote the non-tangential maximal function of $\mathbf{v}$ on $\partial\Omega$ by

$$M_a(\mathbf{v})(\mathbf{x}) = M_a^\Omega(\mathbf{v})(\mathbf{x}) := \sup\{|\mathbf{v}(\mathbf{y})|; \mathbf{y} \in \Gamma_a(\mathbf{x})\}.$$

It is well known that there exists $c > 0$ such that for $a, b > c$ and $1 \leq q < \infty$ there exist $C_1, C_2 > 0$ such that

$$\|M_a(v)\|_{L^q(\partial\Omega)} \leq C_1 \|M_b(v)\|_{L^q(\partial\Omega)} \leq C_2 \|M_a(v)\|_{L^q(\partial\Omega)}$$

for any continuous function v in Ω. (See, e.g. [27] and [51, p. 62].) **We fix $a > c$ and write $\Gamma(\mathbf{x})$ instead of $\Gamma_a(\mathbf{x})$.** Next, define the non-tangential limit of $\mathbf{v}$ at $\mathbf{x} \in \partial\Omega$

$$\mathbf{v}(\mathbf{x}) := \lim_{\Gamma(\mathbf{x}) \ni \mathbf{y} \to \mathbf{x}} \mathbf{v}(\mathbf{y})$$

whenever the limit exists.

Definition 2.1. Let $1 < p, q < \infty$, $g \in L^p(\partial\Omega)$, $h \in L^q(\partial\Omega)$. We say that $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem

$$-\Delta \mathbf{u} + \nabla \rho = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{6a}$$

$$\mathbf{u} \cdot \mathbf{n} = g, \quad \rho = h \quad \text{on } \partial\Omega, \tag{6b}$$

if $\mathbf{u} = (u_1, u_2) \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$, $\rho \in \mathcal{C}^1(\Omega)$ solve (6a), $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$, there exist non-tangential limits of $\mathbf{u}$ and ρ at almost all points of $\partial\Omega$, and these limits satisfy the boundary conditions (6b).

We need the following auxiliary lemma:

Lemma 2.2. *Let $1 < p, q < \infty$, $G \in W^{1,p}(\partial\Omega)$ be such that $\partial G/\partial\tau = \tilde{g}$. Let $\varphi \in \mathcal{C}^\infty(\Omega)$, $\Delta\varphi = 0$ in Ω, $\varphi = G$ in the sense of non-tangential limit almost everywhere on $\partial\Omega$, and $M_a(\varphi), M_a(\nabla\varphi) \in L^p(\partial\Omega)$. Suppose that there exists non-tangential limit of $\nabla\varphi$ at almost all points of $\partial\Omega$. Define $v_1 = \partial_2\varphi$, $v_2 = -\partial_1\varphi$, $\mathbf{v} = (v_1, v_2)$, $t \equiv 0$. Then $(\mathbf{v}, t)$ is an L^p-L^q-solution of the problem*

$$-\Delta\mathbf{v} + \nabla t = 0,\ \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v}\cdot\mathbf{n}^\Omega = \tilde{g},\ t = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Proof. We have $-\Delta\mathbf{v} + \nabla t = (-\partial_2\Delta\varphi, \partial_1\Delta\varphi) = 0$, $\nabla\cdot\mathbf{v} = \partial_1\partial_2\varphi - \partial_2\partial_1\varphi = 0$,

$$\tilde{g} = \frac{\partial G}{\partial\tau} = -n_2\partial_1\varphi + n_1\partial_2\varphi = \mathbf{n}\cdot(\partial_2\varphi, -\partial_1\varphi) = \mathbf{n}\cdot\mathbf{v}.$$

□

Now we are able to study uniqueness of an L^p-L^q solution.

Proposition 2.3. *Let $1 < p, q < \infty$. Suppose that one of the following conditions is satisfied:*

- *$p \le 2 \le q$,*
- *$\partial\Omega$ is of class $\mathcal{C}^1$,*
- *Ω is convex.*

Put $G(\mathbf{x}) := -\frac{1}{2}(x_1^2 + x_2^2)$. Then there exists $\varphi \in \mathcal{C}^\infty(\Omega)$ such that $\Delta\varphi = 0$ in Ω, $\varphi = G$ in the sense of non-tangential limit almost everywhere on $\partial\Omega$, $M_a(\varphi) + M_a(\nabla\varphi) \in L^p(\partial\Omega)$. Define $\mathbf{w}(\mathbf{x}) := (\partial_2\varphi(\mathbf{x}) + x_2, -\partial_1\varphi(\mathbf{x}) - x_1)$. Then $(\mathbf{w}, 0)$ is a non-trivial L^p-L^q-solution of the problem (6) with trivial data $g \equiv 0$, $h \equiv 0$. Moreover, $(\partial_1 w_2 - \partial_2 w_1) \equiv -2$. If $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6) with $g \equiv 0$, $h \equiv 0$, then there is a constant c such that $(\mathbf{u}, \rho) = c(\mathbf{w}, 0)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,\gamma}$ with $k \in \mathbb{N}$ and $0 < \gamma < 1$, then $\mathbf{w} \in \mathcal{C}^{k-1,\gamma}(\Omega, \mathbb{R}^2)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ with $k \in \mathbb{N}$, $1 < p, \beta < \infty$, $1/p < t < k$, then $\mathbf{w} \in W^{k,p}(\Omega, \mathbb{R}^2) \subset B_t^{p,\beta}(\Omega, \mathbb{R}^2) \cap W^{t,p}(\Omega, \mathbb{R}^2)$.

Proof. Clearly $\partial G(\mathbf{x})/\partial\tau = n_2(\mathbf{x})x_1 - n_1(\mathbf{x})x_2$. According to [36, Proposition 7.3] there exists $\varphi \in \mathcal{C}^\infty(\Omega)$ such that $\Delta\varphi = 0$ in Ω, $\varphi = G$ in the sense of non-tangential limit almost everywhere on $\partial\Omega$, $M_a(\varphi) + M_a(\nabla\varphi) \in L^p(\partial\Omega)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,\gamma}$ with $\gamma \in (0, 1)$ then $\varphi \in \mathcal{C}^{k,\gamma}(\overline{\Omega})$ by [36, Proposition 7.5]. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$ then $\varphi \in W^{k+1,p}(\Omega)$ by Theorem 5.6.

Lemma 2.2 gives that $(\mathbf{w}, 0)$ is an L^p-L^q-solution of the problem (6) with $g \equiv 0$, $h \equiv 0$. One has $\partial_1 w_2 - \partial_2 w_1 = -2 - \Delta\varphi = -2$. Thus $\mathbf{w}$ is non-trivial.

Let $(\mathbf{u}, \rho)$ be an L^p-L^q-solution of the problem (6) with $g \equiv 0$, $h \equiv 0$. Then $\rho \equiv 0$ by [36, Proposition 4.1]. Since $\Delta\mathbf{u} = 0$ one has $\mathbf{u} \in \mathcal{C}^\infty(\Omega; \mathbb{R}^2)$ by [38, Theorem 2.18.2]. Since Ω is simply connected there exists a sequence of simply connected domains Ω_k with Lipschitz boundary such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \cup_k\Omega_k$. Fix $\mathbf{z} \in \Omega_1$. Since $\nabla\cdot\mathbf{u} \equiv 0$ Lemma 5.1 gives

$$\int_{\partial\Omega_k} \mathbf{u}\cdot\mathbf{n}\, \mathrm{d}\sigma = 0.$$

According to [20, Theorem 1] there exists a unique $\psi \in W^{1,4}(\Omega_k) \subset \mathcal{C}(\overline{\Omega}_k)$ such that $\mathbf{u} = (\partial_2\psi, -\partial_1\psi)$ in Ω_k and $\psi(\mathbf{z}) = 0$. Since Ω_k was arbitrary we have $\mathbf{u} = (\partial_2\psi, -\partial_1\psi)$ in Ω. Moreover, $\psi \in \mathcal{C}^\infty(\Omega)$. Since $\Delta\mathbf{u} = 0$ one has

$$\partial_j \Delta\psi = \Delta\partial_j\psi = \Delta(-1)^j u_{3-j} = 0.$$

Therefore there exists a constant d such that $\Delta\psi \equiv d$. Clearly, $\Delta(\psi + dG/2) = 0$. Since $M_a(\nabla(\psi + dG/2)) \in L^p(\partial\Omega)$, [38, Theorem 5.6.1] gives that $M_a(\psi + dG/2) \in L^p(\partial\Omega)$ and there exists a non-tangential limit of $\psi + dG/2$ at almost all points of $\partial\Omega$. Moreover, $\psi + dG/2 \in W^{1,p}(\partial\Omega)$ and

$$\frac{\partial(\psi + dG/2)}{\partial\tau} = \mathbf{u}\cdot\mathbf{n} + \frac{d}{2}\frac{\partial G}{\partial\tau} = \frac{d}{2}\frac{\partial G}{\partial\tau}.$$

by [37, Lemma 11.2]. Therefore there exists a constant $\tilde d$ such that $\psi + dG/2 = dG/2 + \tilde d$ on $\partial\Omega$. So, $\Delta(\psi + dG/2 - d\varphi/2) = 0$ in Ω, $\psi + dG/2 - d\varphi/2 = \tilde d$ on $\partial\Omega$. From the uniqueness of the regular L^p-solution of the Dirichlet problem for the Laplace equation ([36, Proposition 7.3]) we infer that $\psi + dG/2 - d\varphi/2 \equiv \tilde d$. Thus $\mathbf{u} = (\partial_2\psi, -\partial_1\psi) = \frac{d}{2}\mathbf{w}$. □

Theorem 2.4. *Let $\mathbf{w}$ be the vector function from Proposition 2.3.*

(1) Let $1 < p, q < \infty$. Suppose that one of the following conditions is satisfied:
  - *$p \le 2 \le q$,*
  - *$\partial\Omega$ is of class $\mathcal{C}^1$,*
  - *Ω is convex.*

  If $g \in L^p(\partial\Omega)$, $h \in L^q(\partial\Omega)$, then there exists an L^p-L^q-solution $(\mathbf{u}, \rho)$ of the problem (6) if and only if the condition

$$\int_{\partial\Omega} g \, \mathrm{d}\sigma = 0 \tag{8}$$

  holds true. If $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6) then $\mathbf{u} \in B^{p,\max(p,2)}_{1/p}(\Omega, \mathbb{R}^2)$, $\rho \in B^{q,\max(q,2)}_{1/q}(\Omega)$, and the general form of an L^p-L^q-solution of the problem (6) is $(\mathbf{u} + c\mathbf{w}, \rho)$ where $c \in \mathbb{R}^1$.

(2) Let $k \in \mathbb{N}$, $1 < p, q < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s \le k+1$, $s - 1/q \notin \mathbb{N}_0$, $1/p < t \le k$, $t - 1/p \notin \mathbb{N}_0$, and $t \le s+1$. Suppose that $s + 1 - 2/q \ge t - 2/p$. (That is true if $p \le q$ or $t \le s-1$.) If $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$, then there exists a solution $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, R^2) \times W^{s,q}(\Omega)$ of the problem (6) if and only if the condition (8) holds. If $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, R^2) \times W^{s,q}(\Omega)$ is a solution of (6) then $(\mathbf{u}, \rho)$ is an L^p-L^q solution of (6), and the general form of a solution of the problem (6) in $W^{t,p}(\Omega, R^2) \times W^{s,q}(\Omega)$ is $(\mathbf{u} + c\mathbf{w}, \rho)$ where $c \in \mathbb{R}^1$.

(3) Let $k \in \mathbb{N}$, $1 < p, q, r, \beta < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s < k+1$, $1/p < t < k$, and $t \le s+1$, $(s+1) - 2/q \ge t - 2/p$. If $t = s+1$ or $(s+1) - 2/q = t - 2/p$ suppose moreover that $p \le q$ and $r \le \beta$. If $g \in B^{p,\beta}_{t-1/p}(\partial\Omega)$, $h \in B^{q,r}_{s-1/q}(\partial\Omega)$, then there exists a solution $(\mathbf{u}, \rho) \in B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ of the problem (6) if and only if the condition (8) holds. If $(\mathbf{u}, \rho) \in B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ is a solution of (6) then $(\mathbf{u}, \rho)$ is an L^p-L^q solution of (6), and the general form of a solution of the problem (6) in $B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ is $(\mathbf{u} + c\mathbf{w}, \rho)$ where $c \in \mathbb{R}^1$.

(4) Let $k, m, n \in N_0$, $0 < \alpha, \beta, \gamma < 1$, and $m + \beta \le k + \alpha$, $n + \gamma \le m + 1 + \beta$, $n + 1 + \gamma \le k + \alpha$. Suppose that $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$. If $h \in \mathcal{C}^{m,\beta}(\partial\Omega)$, $g \in \mathcal{C}^{n,\gamma}(\partial\Omega)$, then there exists a solution $(\mathbf{u}, \rho) \in \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega})$ of the problem (6) if and only if the condition (8) holds. The general form of a solution of the problem (6) in $\mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega})$ is $(\mathbf{u} + c\mathbf{w}, \rho)$ where $c \in \mathbb{R}^1$.

Proof. Suppose first that $(\mathbf{u}, \rho)$ is a solution of the problem (6) in the cases (2), (3). Then $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$, and there exist non-tangential limits of $\mathbf{u}$ and ρ at almost all points of $\partial\Omega$ by Proposition 5.8. The trace of $\mathbf{u}$ is equal to the non-tangential limit of $\mathbf{u}$, and the trace of ρ is equal to the non-tangential limit of ρ by [43, Proposition 3.37]. So, $(\mathbf{u}, \rho)$ is an L^p-L^q solution of (6).

$\mathbf{u} \in B^{p,\max(p,2)}_{1/p}(\Omega, \mathbb{R}^2)$, $\rho \in B^{q,\max(q,2)}_{1/q}(\Omega)$ whenever $(\mathbf{u}, \rho)$ is an L^p-L^q solution of (6) by [36, Theorem 4.2].

If $(\mathbf{u}, \rho)$ is a solution of (6) (in the case (1), (2), (3) or (4)) then the general form of a solution of the problem (6) in the same sense is $(\mathbf{u} + c\mathbf{w}, \rho)$ where $c \in \mathbb{R}^1$. (See Proposition 2.3.)

If $(\mathbf{u}, \rho)$ is a solution of (6) (in the case (1), (2), (3) or (4)) then the condition (8) holds by Lemma 5.1.

We now prove the existence of a solution. Suppose that the condition (8) is fulfilled. If $(\mathbf{u}, \rho)$ is an L^p-L^q solution of the problem (6), then $\rho \in \mathcal{C}^\infty(\Omega)$, $\Delta\rho = 0$ in Ω by [29, p. 10], $\rho = h$ on $\partial\Omega$ in the sense of the non-tangential limit and $M_a(\rho) \in L^q(\partial\Omega)$, i.e. ρ is an L^q-solution of the Dirichlet problem for the Laplace equation

$$\Delta\rho = 0 \quad \text{in } \Omega, \qquad \rho = h \quad \text{on } \partial\Omega. \tag{9}$$

We solve this problem. In the case (1) there exists an L^q-solution of the problem (9) by [36, Proposition 7.2]. Moreover, $\rho \in B^{q,\max(q,2)}_{1/q}(\Omega)$. In the case (2) there exists a solution $\rho \in W^{s,q}(\Omega)$ of (9) by Theorem 5.6. In the case (3) there exists a solution $\rho \in B^{q,r}_s(\Omega)$ of (9) by [36, Proposition 7.8]. In the case (4) there exists a solution $\rho \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ of (9) by [36, Proposition7.4 and Proposition 7.5].

According to [36, Lemma 3.1] there exists $\mathbf{\Phi} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2) \cap \mathcal{C}^\infty(\Omega; \mathbb{R}^2)$ such that $(\mathbf{\Phi}, \rho)$ is a solution of the Stokes system (6a) in Ω, and $\mathbf{\Phi} \in B^{q,\max(2,q)}_{1/q+1}(\Omega, \mathbb{R}^2)$ in the case (1), $\mathbf{\Phi} \in W^{s+1,q}(\Omega, \mathbb{R}^2) \hookrightarrow W^{t,p}(\Omega, \mathbb{R}^2)$ in the case (2), $\mathbf{\Phi} \in B^{q,r}_{s+1}(\Omega, \mathbb{R}^2) \hookrightarrow B^{p,\beta}_t(\Omega, \mathbb{R}^2)$ in the case (3), $\mathbf{\Phi} \in \mathcal{C}^{m+1,\beta}(\overline{\Omega}, \mathbb{R}^2)$ in the case (4). (For the inclusions of the spaces see [12, Theorem 3.8] and Proposition 5.7.)

$\int_{\partial\Omega} \mathbf{\Phi} \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma = 0$ by Lemma 5.1. Note that $(\mathbf{u}, \rho)$ is a solution of the problem (6) if and only if for $\mathbf{v} = \mathbf{u} - \mathbf{\Phi}$, $t \equiv 0$ the couple $(\mathbf{v}, t)$ is a solution of the problem (7) with $\tilde{g} := g - \mathbf{\Phi} \cdot \mathbf{n}^\Omega$. Notice $\int_{\partial\Omega} \tilde{g} \, \mathrm{d}\sigma = 0$.

We now construct a function G such that $\partial G/\partial\tau = \tilde{g}$ on $\partial\Omega$. Fix $\mathbf{z} \in \partial\Omega$. For $\mathbf{x} \in \partial\Omega$ we denote by $L(x)$ the part of $\partial\Omega$ between points $\mathbf{z}$ and $\mathbf{x}$. (From the beginning $\mathbf{z}$ we go along $\partial\Omega$ in the direction τ.) Define

$$G(\mathbf{x}) := \int_{L(\mathbf{x})} \tilde{g} \, \mathrm{d}\sigma.$$

Since $\int_{\partial\Omega} \tilde{g} \, \mathrm{d}\sigma = 0$, the function G is well defined on $\partial\Omega$. Clearly, $\partial G/\partial\tau = \tilde{g}$. Since $\tilde{g} \in L^p(\partial\Omega)$, the function G is bounded (and even continuous). Since $\partial G/\partial\tau = \tilde{g} \in L^p(\partial\Omega)$, we infer that $G \in W^{1,p}(\partial\Omega)$. In the case (2) the function

$\tilde{g} \in W^{t-1/p,p}(\partial\Omega)$ by [22, Theorem 1.5.1.2], and therefore $G \in W^{t+1-1/p,p}(\partial\Omega)$. In the case (3) the function $\tilde{g} \in B^{p,\beta}_{t-1/p}(\partial\Omega)$ by [25, Chapter VI, Theorem 1], and thus $G \in B^{p,\beta}_{t+1-1/p}(\partial\Omega)$. Clearly, $G \in \mathcal{C}^{n+1,\gamma}(\partial\Omega)$ in the case (4).

According to [36, Proposition 7.3] there exists $\varphi \in \mathcal{C}^\infty(\Omega)$ such that $\Delta\varphi = 0$ in Ω, $\varphi = G$ in the sense of non-tangential limit almost everywhere on $\partial\Omega$, $M_a(\varphi)+M_a(\nabla\varphi) \in L^p(\partial\Omega)$. Moreover, $\varphi \in W^{t+1,p}(\Omega)$ in the case (2), $\varphi \in B^{p,\beta}_{t+1}(\Omega)$ in the case (3), $\varphi \in \mathcal{C}^{n+1,\gamma}(\overline{\Omega})$ in the case (4), and $\varphi = G$ on $\partial\Omega$ in the sense of traces. (See [36, Proposition 7.3, Proposition 7.5, Proposition 7.7 and Proposition 7.8].) [38, Theorem 5.6.1] gives that there exists a non-tangential limit of $\nabla\varphi$ at almost all points of $\partial\Omega$. The non-tangential limit of $\nabla\varphi$ is equal to the trace of $\nabla\varphi$ by [43, Proposition 3.37].

Define $v_1 = \partial_2\varphi$, $v_2 = -\partial_1\varphi$, $\mathbf{v} = (v_1, v_2)$, $t \equiv 0$. Then $(\mathbf{v}, t)$ is a solution of the problem (7) by Lemma 2.2. Put $\mathbf{u} := \mathbf{v} + \mathbf{\Phi}$. Then $(\mathbf{u}, \rho)$ is a solution of the problem (6). □

3. The problem (1), (2)

In this section we study the problem (1), (2) (the problem with so called Hodge conditions) for planar domains. If $\mathbf{u}$ and ρ depend only on x_1, x_2 in $G = \Omega\times(c_1, c_2)$ with $\Omega \subset \mathbb{R}^2$, and $u_3 \equiv 0$ then $\nabla \times \mathbf{u} = (0, 0, \partial_1 u_2 - \partial_2 u_1)$. On $\Omega \times \{c_j\}$ we have $\mathbf{n}^G \times (\nabla \times \mathbf{u}) = (0, 0, 0)$. On $\partial\Omega \times (c_1, c_2)$ we have $\mathbf{n}^G = (n_1^\Omega, n_2^\Omega, 0)$ and $\mathbf{n}^G \times (\nabla\times\mathbf{u}) = (\partial_1 u_2 - \partial_2 u_1)(n_2^\Omega, -n_1^\Omega, 0)$. So, the problem (1), (2) can be reduced to the problem

$$-\Delta\mathbf{u} + \nabla\rho = \mathbf{f},\ \nabla\cdot\mathbf{u} = \chi \text{ in } \Omega, \tag{10a}$$

$$\mathbf{u}\cdot\mathbf{n} = g,\ \partial_1 u_2 - \partial_2 u_1 = b \text{ on } \partial\Omega. \tag{10b}$$

(Compare [46].) If $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2) \cup B^{p,\beta}_t(\Omega, \mathbb{R}^2)$ and $\partial_1 u_2 - \partial_2 u_1 \in W^{s,q}(\Omega) \cup B^{q,r}_s(\Omega)$ with $1/p < t$ and $1/q < s$ then the boundary conditions will be satisfied in the sense of traces. For the homogeneous system (6a) and $g \in L^p(\partial\Omega)$, $b \in L^q(\partial\Omega)$ we shall again study solutions satisfying the boundary conditions in the sense of non-tangential limits.

Definition 3.1. Let $1 < p, q < \infty$, $g \in L^p(\partial\Omega)$, $b \in L^q(\partial\Omega)$. We say that $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6a), (10b) if $\mathbf{u} = (u_1, u_2) \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$, $\rho \in \mathcal{C}^1(\Omega)$ solve (6a), $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\partial_1 u_2 - \partial_2 u_1) \in L^q(\partial\Omega)$, there exist non-tangential limits of $\mathbf{u}$ and $\partial_1 u_2 - \partial_2 u_1$ at almost all points of $\partial\Omega$, and these limits satisfy the boundary conditions (10b).

First we find relations between the problems (6a), (10b) and (6).

Lemma 3.2. *Let $(\mathbf{u}, \rho)$ be a solution of the Stokes system (6a) in an open set $G \subset \mathbb{R}^2$. Then $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic function in G.*

Proof. $\mathbf{u} \in \mathcal{C}^\infty(G, \mathbb{R}^2)$, $p \in \mathcal{C}^\infty(G)$ by [48, §1.2]. Since $\partial_1 u_1 = -\partial_2 u_2$, we have

$$\partial_1(\partial_1 u_2 - \partial_2 u_1) = \partial_1^2 u_2 - \partial_2\partial_1 u_1 = \partial_1^2 u_2 + \partial_2^2 u_2 = \Delta u_2 = \partial_2\rho,$$

$$\partial_2(\partial_1 u_2 - \partial_2 u_1) = \partial_1\partial_2 u_2 - \partial_2^2 u_1 = -\partial_1^2 u_1 - \partial_2^2 u_1 = -\Delta u_1 = -\partial_1\rho.$$

[10, Proposition 3.2] gives that $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is holomorphic in G. □

Theorem 3.3. *Let $1 < p, q < \infty$. Suppose that one of the following conditions is satisfied:*

- *$p \leq 2 \leq q$,*
- *$\partial\Omega$ is of class $\mathcal{C}^1$,*
- *Ω is convex.*

If $g \in L^p(\partial\Omega)$, $b \in L^q(\partial\Omega)$, then there exists an L^p-L^q-solution $(\mathbf{u}, \rho)$ of the problem (6a), (10b) if and only if the condition (8) holds true. If $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6a), (10b), then the general form of an L^p-L^q-solution of the problem is $(\mathbf{u}, \rho + c)$, where c is a constant. Moreover, $\mathbf{u} \in B^{p,\max(p,2)}_{1/p}(\Omega, \mathbb{R}^2)$, $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,\max(q,2)}_{1/q}(\Omega)$, $\rho \in B^{q,\max(q,2)}_{1/q}(\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$ and there exists a non-tangential limit of ρ at almost all points of $\partial\Omega$. Let $0 < \alpha < 1$. If $\alpha \leq 1/2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $b \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ then $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $\rho \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. If $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$ and $b \in \mathcal{C}^{k,\alpha}(\partial\Omega)$ with $k \in \mathbb{N}$ then $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$ and $\rho \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$. If $q = 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $1/q < s < 1 + 1/q$, $b \in W^{s-1/q,q}(\partial\Omega)$ then $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$, $\rho \in W^{s,q}(\Omega)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$, $1/q < s \leq k + 1$ with $k \in \mathbb{N}$, $s - 1/q \notin \mathbb{N}$ and $b \in W^{s-1/q,q}(\partial\Omega)$, then $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$, $\rho \in W^{s,q}(\Omega)$. If $q = 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $1/q < s < 1 + 1/q$, $1 < r < \infty$, $b \in B^{q,r}_{s-1/q}(\partial\Omega)$ then $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$, $\rho \in B^{q,r}_s(\Omega)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$, $1/q < s < k + 1$ with $k \in \mathbb{N}$, $1 < r < \infty$ and $b \in B^{q,r}_{s-1/q}(\partial\Omega)$, then $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$, $\rho \in B^{q,r}_s(\Omega)$.

Proof. Suppose first that $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6a), (10b). Then $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic function by Lemma 3.2. So, $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,\max(q,2)}_{1/q}(\Omega)$ by [42, Corollary 4.4]. Lemma 5.5 gives that $M_a(\rho) \in L^q(\partial\Omega)$ and there exists a non-tangential limit of ρ at almost all points of $\partial\Omega$. So, $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6) for some h. Theorem 2.4 gives that (8) holds, $\mathbf{u} \in B^{p,\max(p,2)}_{1/p}(\Omega, \mathbb{R}^2)$ and $\rho \in B^{q,\max(q,2)}_{1/q}(\Omega)$.

Let now $g \equiv 0$, $b \equiv 0$. If c is a constant then $(0, c)$ is an L^p-L^q-solution of the problem (6a), (10b). Let now $(\mathbf{u}, \rho)$ be an L^p-L^q-solution of the problem (6a), (10b). Then $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic function by Lemma 3.2. So, $\omega := (\partial_1 u_2 - \partial_2 u_1)$ is an L^q-solution of the Dirichlet problem

$$\Delta\omega = 0 \quad \text{in } \Omega, \quad \omega = b \quad \text{on } \partial\Omega. \tag{11}$$

Since $b \equiv 0$, [36, Proposition 7.2] gives that $\omega \equiv 0$. Since $i\rho$ is a holomorphic function, there exists a constant c such that $\rho \equiv c$. Therefore, $(\mathbf{u}, \rho - c)$ is an L^p-L^q-solution of the problem (6) with $h \equiv 0$. According to Proposition 2.3 there exists a constant β such that $\mathbf{u} = \beta\mathbf{w}$. So, $0 = (\partial_1 u_2 - \partial_2 u_1) = \beta(\partial_1 w_2 - \partial_2 w_1) = -2\beta$. Thus $\beta = 0$ and $\mathbf{u} \equiv 0$.

Let now (8) holds. According to [36, Proposition 7.2] there is a unique L^q-solution ω of the Dirichlet problem for the Laplace equation (11). Moreover, $\omega \in B^{q,\max(q,2)}_{1/q}(\Omega)$. According to [10, Theorem 16.3] there exists a real function $\tilde{\rho}$ such that $\omega + i\tilde{\rho}$ is holomorphic. Lemma 5.5 gives that $M_a(\tilde{\rho}) \in L^q(\partial\Omega)$ and there exists a non-tangential limit h of $\tilde{\rho}$ at almost all points of $\partial\Omega$. According to Theorem 2.4 there exists an L^p-L^q-solution $(\mathbf{v}, \rho)$ of the problem (6). The function $(\partial_1 v_2 - \partial_2 v_1) + i\rho$ is holomorphic by Lemma 3.2. Since ρ and $\tilde{\rho}$ are L^q-solutions of the Dirichlet problem $\Delta\rho = 0$ in Ω, $\rho = h$ on $\partial\Omega$, [36, Proposition 7.2] gives that $\tilde{\rho} = \rho$. Since $[(\partial_1 v_2 - \partial_2 v_1) + i\rho] - (\omega + i\tilde{\rho}) = (\partial_1 v_2 - \partial_2 v_1) - \omega$ is a holomorphic function, there exists a constant c such that $(\partial_1 v_2 - \partial_2 v_1) - \omega = c$. (See [10, Proposition 3.6].) Thus $(\mathbf{v},\rho)$ is an L^p-L^q-solution of the Hodge problem (6a),

$$\mathbf{v}\cdot\mathbf{n}=\mathbf{g},\quad \partial_1 v_2-\partial_2 v_1=b+c\quad \text{on } \partial\Omega. \tag{12}$$

Let $\mathbf{w}$ be the vector function from Proposition 2.3. Then $(\mathbf{w},0)$ is an L^p-L^q-solution of the Hodge problem (6a),

$$\mathbf{w}\cdot\mathbf{n}=0,\quad \partial_1 w_2-\partial_2 w_1=-2\quad \text{on } \partial\Omega. \tag{13}$$

Define $\mathbf{u}:=\mathbf{v}+(c/2)\mathbf{w}$. Then $(\mathbf{u},\rho)$ is an L^p-L^q-solution of the problem (6a), (10b).

Suppose that $\alpha\le 1/2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $b\in\mathcal{C}^{0,\alpha}(\partial\Omega)$. Then $(\partial_1 u_2-\partial_2 u_1)=\omega\in\mathcal{C}^{0,\alpha}(\overline{\Omega})$ by [36, Proposition 7.4]. Since $(\partial_1 u_2-\partial_2 u_1)+i\rho$ is holomorphic, Lemma 5.2 gives that $\rho\in\mathcal{C}^{0,\alpha}(\overline{\Omega})$. If $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$ and $b\in\mathcal{C}^{k,\alpha}(\partial\Omega)$ then $(\partial_1 u_2-\partial_2 u_1)\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$ by [36, Proposition 7.5]. Lemma 5.2 gives that $\rho\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$.

Suppose that $q=2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $1/q<s<1+1/q$, $b\in W^{s-1/q,q}(\partial\Omega)$. Then $(\partial_1 u_2-\partial_2 u_1)=\omega\in W^{s,q}(\Omega)$ by [36, Proposition 7.6]. Since $(\partial_1 u_2-\partial_2 u_1)+i\rho$ is holomorphic, Lemma 5.3 gives that $\rho\in W^{s,q}(\Omega)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$, $1/q<s\le k+1$, $s-1/q\notin\mathbb{N}$ and $b\in W^{s-1/q,q}(\partial\Omega)$ with $k\in\mathbb{N}$, then $(\partial_1 u_2-\partial_2 u_1)\in W^{s,q}(\Omega)$ by Theorem 5.6. Lemma 5.3 gives that $\rho\in W^{s,q}(\Omega)$.

Suppose that $q=2$ or $\partial\Omega$ is of class $\mathcal{C}^1$ or Ω is convex, and $1/q<s<1+1/q$, $1<r<\infty$, $b\in B^{q,r}_{s-1/q}(\partial\Omega)$. Then $(\partial_1 u_2-\partial_2 u_1)=\omega\in B^{q,r}_s(\Omega)$ by [36, Proposition 7.6]. Since $(\partial_1 u_2-\partial_2 u_1)+i\rho$ is holomorphic, Lemma 5.4 gives that $\rho\in B^{q,r}_s(\Omega)$. If $\partial\Omega$ is of class $\mathcal{C}^{k,1}$, $1/q<s<k+1$, and $b\in B^{q,r}_{s-1/q}(\partial\Omega)$ with $k\in\mathbb{N}$, then $(\partial_1 u_2-\partial_2 u_1)\in B^{q,r}_s(\Omega)$ by Theorem 5.6. Lemma 5.4 gives $\rho\in B^{q,r}_s(\Omega)$. □

Theorem 3.4. *Let $1<p,q<\infty$. Suppose that one of the following conditions is satisfied:*

- $p,q\le 2$,
- $\partial\Omega$ *is of class* $\mathcal{C}^1$,
- Ω *is convex.*

If $g\in L^p(\partial\Omega)$, $b\in W^{1,q}(\partial\Omega)$, then there exists an L^p-L^q-solution $(\mathbf{u},\rho)$ of the problem (6a), (10b),

$$M_a(\nabla(\partial_1 u_2-\partial_2 u_1))\in L^q(\partial\Omega) \tag{14}$$

if and only if the condition (8) holds true. If $(\mathbf{u},\rho)$ is a solution of the problem (6a), (10b) in this sense, then the general form of a solution of the problem in this sense is $(\mathbf{u},\rho+c)$, where c is a constant. Moreover, $\mathbf{u}\in B^{p,\max(p,2)}_{1/p}(\Omega,\mathbb{R}^2)$, $(\partial_1 u_2-\partial_2 u_1)\in B^{q,\max(q,2)}_{1+1/q}(\Omega)\hookrightarrow\mathcal{C}(\overline{\Omega})$, $\rho\in B^{q,\max(q,2)}_{1+1/q}(\Omega)\hookrightarrow\mathcal{C}(\overline{\Omega})$, $M_a(\nabla\rho)\in L^q(\partial\Omega)$.

Proof. Let $(\mathbf{u},\rho)$ be an L^p-L^q-solution of the problem (6a), (10b), (14). Put $\omega=(\partial_1 u_2-\partial_2 u_1)$. Then $\omega+i\rho$ is a holomorphic function by Lemma 3.2. Lemma 5.5 gives that $M_a(\rho),M_a(\nabla\rho)\in L^q(\partial\Omega)$ and there exists a non-tangential limit of ρ at almost all points of $\partial\Omega$. Since ω and ρ are harmonic functions, [42, Corollary 4.4] gives that $\omega,\rho\in B^{q,\max(q,2)}_{1+1/q}(\Omega)$. One has $B^{q,\max(q,2)}_{1+1/q}(\Omega)\hookrightarrow\mathcal{C}(\overline{\Omega})$ by [56, Proposition 4.6]. So, $(\mathbf{u},\rho+c)$, where c is a constant, is the general form of an L^p-L^q-solution of the problem (6a), (10b), (14). (See Theorem 3.3.) Moreover, (8) holds.

Suppose that (8) holds true. Sobolev's embedding theorem gives that $b\in\mathcal{C}(\partial\Omega)$. Put $r=\max(2,q)$. According to Theorem 3.3 there exists an L^p-L^r solution $(\mathbf{u},\rho)$

of the problem (6a), (10b). Since $q \le r$, $(\mathbf{u}, \rho)$ is an L^p-L^q solution of the problem (6a), (10b). Put $\omega = (\partial_1 u_2 - \partial_2 u_1)$. Then $\omega + i\rho$ is a holomorphic function by Lemma 3.2. In particular, ω is an L^r-solution of the Dirichlet problem for the Laplace equation (11). Since $b \in W^{1,q}(\partial\Omega)$, [36, Proposition 7.3] gives that $M_a(\omega), M_a(\nabla\omega) \in L^q(\partial\Omega)$. $\square$

Before we shall study the non-homogeneous problem (10), we prove the following auxiliary lemma. We denote $B(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^2; |\mathbf{x} - \mathbf{y}| < r\}$.

Lemma 3.5. *Let $1 < q, r < \infty$, $1/q < s < \infty$, $m \in \mathbb{N}_0$, $0 < \alpha < 1$.*

- *(a) If $\mathbf{f} \in B^{q,r}_{s-1}(\Omega, \mathbb{R}^2)$, $\chi \in B^{q,r}_s(\Omega)$ then there exists a solution $(\mathbf{u}, \rho) \in B^{q,r}_{s+1}(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ of (10a).*
- *(b) If $\mathbf{f} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s,q}(\Omega)$ then there exists a solution $(\mathbf{u}, \rho) \in W^{s+1,q}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of (10a).*
- *(c) Let $\chi \in \mathcal{C}^{m,\alpha}(\overline{\Omega})$, $\mathbf{f} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$. If $m \ge 1$ suppose that $\mathbf{f} \in \mathcal{C}^{m-1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$. Then there exists a solution $(\mathbf{u}, \rho) \in \mathcal{C}^{m+1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\alpha}(\overline{\Omega})$ of (10a).*
- *(d) If $\mathbf{f} \in B^{q,r}_{s-1}(\Omega, \mathbb{R}^2)$, $\chi \in B^{q,r}_s(\Omega)$ and $(\mathbf{u}, \rho)$ is a solution of (10a) in the sense of distributions, then $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$ if and only if $\rho \in B^{q,r}_s(\Omega)$.*
- *(e) If $\mathbf{f} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s,q}(\Omega)$ and $(\mathbf{u}, \rho)$ is a solution of (10a) in the sense of distributions, then $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$ if and only if $\rho \in W^{s,q}(\Omega)$.*
- *(f) Let $\chi \in \mathcal{C}^{m,\alpha}(\overline{\Omega})$, $\mathbf{f} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$. If $m \ge 1$ suppose that $\mathbf{f} \in \mathcal{C}^{m-1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$. If $(\mathbf{u}, \rho)$ is a solution of (10a) in the sense of distributions, then $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\alpha}(\overline{\Omega})$ if and only if $\rho \in \mathcal{C}^{m,\alpha}(\overline{\Omega})$.*

Proof. (a) Choose $r \in (0, \infty)$ such that $\overline{\Omega} \subset B(0; r)$. We can suppose that $\mathbf{f} \in B^{q,r}_{s-1}(B(0; r), \mathbb{R}^2)$, $\chi \in B^{q,r}_s(B(0; r))$. According to [36, Theorem 5.4] there exists a solution $(\mathbf{u}, \rho) \in B^{q,r}_{s+1}(B(0; r), \mathbb{R}^2) \times B^{q,r}_s(B(0; r))$ of (10a) in $B(0; r)$.

(b) If $s \notin \mathbb{N}$ then (b) is a consequence of (a). Let now $s \in \mathbb{N}$. We can suppose that $\mathbf{f} \in W^{s-1,q}(B(0; r), \mathbb{R}^2)$, $\chi \in W^{s,q}(B(0; r))$ by [3, Theorem 5.24]. According to [36, Theorem 5.5] there exists a solution $(\mathbf{u}, \rho) \in W^{s+1,q}(B(0; r), \mathbb{R}^2) \times W^{s,q}(B(0; r))$ of (10a) in $B(0; r)$.

(c) Choose $r \in (0, \infty)$ such that $\overline{\Omega} \subset B(0; r)$. Since $\mathcal{C}^{m,\alpha}(\overline{\Omega}) = B^{\infty,\infty}_\alpha(\Omega)$ by [56, Theorem 1.122], we can suppose that $\chi \in B^{\infty,\infty}_\alpha(\overline{B(0; r)}) = \mathcal{C}^{m,\alpha}(\overline{B(0; r)})$. If $m \ge 1$ we can suppose that $\mathbf{f} \in \mathcal{C}^{m-1,\alpha}(\overline{B(0; r)}, \mathbb{R}^2)$. If $m = 0$ we can suppose that $\mathbf{f} \in L^\infty(B(0; r), \mathbb{R}^2)$. According to [21, Theorem 8.34, Theorem 6.14 and Theorem 6.19] there exists $\mathbf{v} \in \mathcal{C}^{m+1,\alpha}(\overline{B(0; r)}; \mathbb{R}^2)$ such that $-\Delta\mathbf{v} = \mathbf{f}$ in $B(0; r)$. Put $\rho = (\chi - \nabla \cdot \mathbf{v})$. Then $\rho \in \mathcal{C}^{m,\alpha}(\overline{B(0; r)})$. According to [21, Theorem 6.14 and Theorem 6.19] there exists $\varphi \in \mathcal{C}^{m+2,\alpha}(\overline{B(0; r)})$ such that $\Delta\varphi = \rho$. Put $\mathbf{u} := \mathbf{v} + \nabla\varphi$. Then

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} + \Delta\varphi = \chi,$$
$$-\Delta\mathbf{u} + \nabla\rho = -\Delta\mathbf{v} - \nabla\Delta\varphi + \nabla\rho = \mathbf{f}.$$

So $(\mathbf{u}, \rho) \in \mathcal{C}^{m+1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\alpha}(\overline{\Omega})$ is a solution of (10a).

(d), (e), (f): Put $X = B^{q,r}_{s+1}(\Omega, \mathbb{R}^2)$ and $Y = B^{q,r}_s(\Omega)$ in the case (d), $X = W^{s+1,q}(\Omega, \mathbb{R}^2)$ and $Y = W^{s,q}(\Omega)$ in the case (e), $X = \mathcal{C}^{m+1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ and $Y = \mathcal{C}^{m,\alpha}(\overline{\Omega})$ in the case (f). We have proved that there exists a solution $(\mathbf{v}, \tau) \in X \times Y$ of (10a). Put $\mathbf{U} := \mathbf{u} - \mathbf{v}$, $P := \rho - \tau$. Clearly, $(\partial_1 u_2 - \partial_2 u_1) \in Y$ if and only if $(\partial_1 U_2 - \partial_2 U_1) \in Y$. Similarly, $\rho \in Y$ if and only if $P \in Y$. Since $(\mathbf{U}, P)$ is a solution of (6a), Lemma 3.2, Lemma 5.4, Lemma 5.3 and Lemma 5.2 give that $(\partial_1 U_2 - \partial_2 U_1) \in Y$ if and only if $P \in Y$. $\square$

Theorem 3.6. *Let $k \in \mathbb{N}_0$, $1 < p, q, r, \beta < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,\alpha}$, $0 < \alpha \leq 1$.*

(1) Let $k \in \mathbb{N}$, $\alpha = 1$, $1/q < s \leq k+1$, $s - 1/q \notin \mathbb{N}_0$, $1/p < t \leq k$, $t - 1/p \notin \mathbb{N}_0$, and $t \leq s+1$. Suppose that $s - 2/q \geq t - 2/p$. (That is true if $p \leq q$ or $t \leq s-1$.) Let $g \in W^{t-1/p,p}(\partial\Omega)$, $b \in W^{s-1/q,q}(\partial\Omega)$, $\mathbf{f} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s,q}(\Omega)$.

(a) Let $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2)$, $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$ and ρ be a distribution in Ω. Suppose that $(\mathbf{u}, \rho)$ is a solution of the problem (10). Then

$$\int_{\partial\Omega} g \,\mathrm{d}\sigma = \int_{\Omega} \chi \,\mathrm{d}\mathbf{x} \tag{15}$$

and $\rho \in W^{s,q}(\Omega)$. If $\mathbf{f} \equiv 0$, $\chi \equiv 0$, then $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (10). If $b \equiv 0$, $g \equiv 0$, $\chi \equiv 0$, $\mathbf{f} \equiv 0$ then $\mathbf{u} \equiv 0$, ρ is constant.

(b) Suppose (15). Then there exists $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ with $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$ such that $(\mathbf{u}, \rho)$ is a solution of the problem (10). In particular, if $1/q < s \leq k-1$, $s - 1/q \notin \mathbb{N}_0$, $g \in W^{s+1-1/q,q}(\partial\Omega)$, $b \in W^{s-1/q,q}(\partial\Omega)$, $\mathbf{f} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s,q}(\Omega)$, then there exists a solution $(\mathbf{u}, \rho) \in W^{s+1,q}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of the problem (10).

(2) Let $k \in \mathbb{N}$, $\alpha = 1$, $1/q < s < k+1$, $1/p < t < k$, and $t \leq s+1$, $(s+1) - 2/q \geq t - 2/p$. If $t = s+1$ or $(s+1) - 2/q = t - 2/p$ suppose moreover that $p \leq q$ and $r \leq \beta$. Let $g \in B^{p,\beta}_{t-1/p}(\partial\Omega)$, $b \in B^{q,r}_{s-1/q}(\partial\Omega)$, $\mathbf{f} \in B^{q,r}_{s-1}(\Omega, \mathbb{R}^2)$, $\chi \in B^{q,r}_s(\Omega)$.

(a) Let $\mathbf{u} \in B^{p,\beta}_t(\Omega, \mathbb{R}^2)$ with $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$ and ρ be a distribution in Ω. Suppose that $(\mathbf{u}, \rho)$ is a solution of the problem (10). Then (15) holds and $\rho \in B^{q,r}_s(\Omega)$. If $\mathbf{f} \equiv 0$, $\chi \equiv 0$, then $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (10). If $b \equiv 0$, $g \equiv 0$, $\chi \equiv 0$, $\mathbf{f} \equiv 0$ then $\mathbf{u} \equiv 0$, ρ is constant.

(b) Suppose (15). Then there exists $(\mathbf{u}, \rho) \in B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ with $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$ such that $(\mathbf{u}, \rho)$ is a solution of the problem (10). In particular, if $1/q < s < k-1$, $g \in B^{q,r}_{s+1-1/q}(\partial\Omega)$, $b \in B^{q,r}_{s-1/q}(\partial\Omega)$, $\mathbf{f} \in B^{q,r}_{s-1}(\Omega, \mathbb{R}^2)$, $\chi \in B^{q,r}_s(\Omega)$, then there is a solution $(\mathbf{u}, \rho) \in B^{q,r}_{s+1}(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ of the problem (10).

(3) Let $m, n \in \mathbb{N}_0$, $0 < \alpha, \beta, \gamma < 1$, and $m + \beta \leq k + \alpha$, $n + \gamma \leq m + 1 + \beta$, $n + 1 + \gamma \leq k + \alpha$. Let $b \in \mathcal{C}^{m,\beta}(\partial\Omega)$, $g \in \mathcal{C}^{n,\gamma}(\partial\Omega)$, $\chi \in \mathcal{C}^{m,\beta}(\overline{\Omega})$, $\mathbf{f} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$. If $m \geq 1$ suppose $\mathbf{f} \in \mathcal{C}^{m-1,\beta}(\overline{\Omega}, \mathbb{R}^2)$.

(a) Let $\mathbf{u} \in \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2)$, $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ and ρ be a distribution in Ω. Let $(\mathbf{u}, \rho)$ be a solution of the problem (10). Then $\rho \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ and (15) holds. If $b \equiv 0$, $g \equiv 0$, $\chi \equiv 0$, $\mathbf{f} \equiv 0$ then $\mathbf{u} \equiv 0$, ρ is constant.

(b) If (15) holds, then there exists $(\mathbf{u}, \rho) \in \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega})$ with $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ such that $(\mathbf{u}, \rho)$ is a solution of the problem (10). In particular, if $\partial\Omega \in \mathcal{C}^{m+2,\alpha}$, $g \in \mathcal{C}^{m+1,\alpha}(\partial\Omega)$, $b \in \mathcal{C}^{m,\alpha}(\partial\Omega)$, $\chi \in \mathcal{C}^{m,\alpha}(\overline{\Omega})$, $\mathbf{f} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$ for $m = 0$ and $\mathbf{f} \in \mathcal{C}^{m-1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ for $m \in \mathbb{N}$, then there exists a solution $(\mathbf{u}, \rho) \in \mathcal{C}^{m+1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\alpha}(\overline{\Omega})$ of the problem (10).

Proof. Let $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2)$ with the vorticity $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$ and a distribution ρ in Ω be such that $(\mathbf{u}, \rho)$ is a solution of the problem (10). Then $\rho \in W^{s,q}(\Omega)$ by Lemma 3.5. According to Lemma 3.5 there exists $(\mathbf{v}, \tau) \in W^{s+1,q}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ such that $-\Delta \mathbf{v} + \nabla \tau = \mathbf{f}$, $\nabla \cdot \mathbf{v} = \chi$. The Divergence theorem gives

$$\int_{\partial\Omega} \mathbf{n}^{\Omega} \cdot \mathbf{v} \, \mathrm{d}\sigma = \int_{\Omega} \chi \, \mathrm{d}\mathbf{x}. \tag{16}$$

According to [12, Theorem 3.8] one has $W^{s+1,q}(\Omega) \hookrightarrow W^{t,p}(\Omega)$. Thus $(\mathbf{u}-\mathbf{v}, \rho-\tau) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$, $\Delta(\mathbf{u}-\mathbf{v}) = \nabla(\rho - \tau)$, $\nabla \cdot (\mathbf{u}-\mathbf{v}) = 0$ in Ω. Theorem 2.4 gives

$$\int_{\partial\Omega} \mathbf{n}^{\Omega} \cdot (\mathbf{u}-\mathbf{v}) \, \mathrm{d}\sigma = 0.$$

From this and (16) we obtain (15). Let now $\mathbf{f} \equiv 0$, $\chi \equiv 0$. Put $h := \rho$ on $\partial\Omega$. Theorem 2.4 gives that $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6). In particular, $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$ and there exist non-tangential limits of $\mathbf{u}$ and ρ at almost all points of $\partial\Omega$. Lemma 3.2 gives that $(\partial_1 u_2 - \partial_2 u_1) + i\rho$ is a holomorphic function in Ω. According to Lemma 5.5 there exists a non-tangential limit of $(\partial_1 u_2 - \partial_2 u_1)$ at almost all points of $\partial\Omega$ and $M_a(\partial_1 u_2 - \partial_2 u_1) \in L^q(\partial\Omega)$. So, $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (10). If $b \equiv 0$, $g \equiv 0$, $\chi \equiv 0$, $\mathbf{f} \equiv 0$ then $\mathbf{u} \equiv 0$, ρ is constant by Theorem 3.3.

Let $\mathbf{u} \in B^{p,\beta}_t(\Omega, \mathbb{R}^2)$ with the vorticity $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$ and a distribution ρ in Ω be such that $(\mathbf{u}, \rho)$ is a solution of the problem (10). Then $\rho \in B^{q,r}_s(\Omega)$ by Lemma 3.5. Since $B^{p,\beta}_t(\Omega, \mathbb{R}^2) \hookrightarrow W^{t-\epsilon,p}(\Omega, \mathbb{R}^2)$, $B^{q,r}_s(\Omega) \hookrightarrow W^{s-\epsilon,q}(\Omega)$ for $\epsilon > 0$ by [54, §4.6.1, Theorem], (1a) forces (2a).

Let $\mathbf{u} \in \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2)$, $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ and ρ be a distribution in Ω. Let $(\mathbf{u}, \rho)$ be a solution of the problem (10). Then $\rho \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ by Lemma 3.5. Choose $p, q \in (1, \infty)$ such that $1/p < \gamma$, $1/q < \beta$. Choose $t \in (1/p, \gamma)$, $s \in (1/q, \beta)$. Then $\mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \hookrightarrow W^{t,p}(\Omega, \mathbb{R}^2)$, $\mathcal{C}^{m,\beta}(\overline{\Omega}) \hookrightarrow W^{s,q}(\Omega)$ by [31, Remark 6.8.3]. So, (3a) is a consequence of (1a).

We now construct a solution of the problem (10). Assume (15). Suppose first that $\mathbf{f} \equiv 0$, $\chi \equiv 0$. According to Theorem 3.3 there exists an L^p-L^q-solution $(\mathbf{u}, \rho)$ of the problem (10). Moreover $M_a(\rho) \in L^q(\partial\Omega)$ and there exists a non-tangential limit of ρ at almost all points of $\partial\Omega$. Further, $(\partial_1 u_2 - \partial_2 u_1) \in W^{s,q}(\Omega)$ and $\rho \in W^{s,q}(\Omega)$ in the case (1), $(\partial_1 u_2 - \partial_2 u_1) \in B^{q,r}_s(\Omega)$ and $\rho \in B^{q,r}_s(\Omega)$ in the case (2), $(\partial_1 u_2 - \partial_2 u_1) \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ and $\rho \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ in the case (3). Denote by h the trace of ρ. Then $(\mathbf{u}, \rho)$ is an L^p-L^q-solution of the problem (6). Theorem 2.4 gives that $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2)$ in the case (1), $\mathbf{u} \in B^{p,\beta}_t(\Omega, \mathbb{R}^2)$ in the case (2), $\mathbf{u} \in \mathcal{C}^{n,\gamma}(\Omega, \mathbb{R}^2)$ in the case (3).

Let now (15) hold and $\mathbf{f}$ and χ be arbitrary. According to Lemma 3.5 there exists a solution $(\mathbf{v}, \tau)$ of $-\Delta \mathbf{v} + \nabla \tau = \mathbf{f}$, $\nabla \cdot \mathbf{v} = \chi$ in Ω. Moreover, $(\mathbf{v}, \tau) \in W^{s+1,q}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega) \hookrightarrow W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ in the case (1), $(\mathbf{v}, \tau) \in B^{q,r}_{s+1}(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega) \hookrightarrow B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ in the case (2), and $(\mathbf{v}, \tau) \in \mathcal{C}^{m+1,\beta}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega}) \hookrightarrow \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega})$ in the case (3). Put $G = g - \mathbf{n}^{\Omega} \cdot \mathbf{v}$, $B = b - (\partial_1 v_2 - \partial_2 v_1)$ on $\partial\Omega$. Then $G \in W^{t-1/p,p}(\partial\Omega)$ and $B \in W^{s-1/q,q}(\partial\Omega)$ in the case (1) by [22, Theorem 1.5.1.2], $G \in B^{p,\beta}_{t-1/p}(\partial\Omega)$ and $B \in B^{q,r}_{s-1/q}(\partial\Omega)$ in the case (2) by [25, Chapter VIII, Theorem 2]. Since

$$\int_{\partial\Omega} \mathbf{n}^\Omega \cdot \mathbf{v} \, \mathrm{d}\sigma = \int_\Omega \chi \, \mathrm{d}\mathbf{x}$$

we deduce that

$$\int_{\partial\Omega} G \, \mathrm{d}\sigma = 0.$$

We have proved that there exists a solution $(\mathbf{V}, T)$ of the problem

$$\Delta \mathbf{V} = \nabla T, \quad \nabla \cdot \mathbf{V} = 0 \quad \text{in } \Omega,$$

$$\mathbf{V} \cdot \mathbf{n}^\Omega = G, \quad (\partial_1 V_2 - \partial_2 V_1) = B \quad \text{on } \partial\Omega$$

such that $(\mathbf{V}, T) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ with $(\partial_1 V_2 - \partial_2 V_1) \in W^{s,q}(\Omega)$ in the case (1), $(\mathbf{V}, T) \in B^{p,\beta}_t(\Omega, \mathbb{R}^2) \times B^{q,r}_s(\Omega)$ with $(\partial_1 V_2 - \partial_2 V_1) \in B^{q,r}_s(\Omega)$ in the case (2), $(\mathbf{V}, T) \in \mathcal{C}^{n,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{m,\beta}(\overline{\Omega})$ with $(\partial_1 V_2 - \partial_2 V_1) \in \mathcal{C}^{m,\beta}(\overline{\Omega})$ in the case (3). Put $\mathbf{u} = \mathbf{v} + \mathbf{V}$, $\rho = \tau + T$. Then $(\mathbf{u}, \rho)$ is a solution of (10). □

4. The other problems

Theorem 4.1. *Let $k \in \mathbb{N}$, $k \geq 2$, $1 < q, r < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,\alpha}$, $0 < \alpha \leq 1$, $b \in \mathcal{C}^{k-2,\alpha}(\partial\Omega)$, $b \geq 0$.*

(1) Let $\alpha = 1$, $1/q + 1 < s \leq k$, $s - 1/q \notin \mathbb{N}$, $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{h} \in W^{s-1-1/q,q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{f} \in W^{s-2,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s-1,q}(\Omega)$.

(a) Then there exists a solution $(\mathbf{u}, \rho) \in W^{s,q}(\Omega, \mathbb{R}^2) \times W^{s-1,q}(\Omega)$ of the problem (1), (3) if and only if (15) holds. The general form of a solution of the problem (1), (3) in $W^{s,q}(\Omega, \mathbb{R}^2) \times W^{s-1,q}(\Omega)$ is $(\mathbf{u}, \rho + c)$ where c is a constant.

(b) Suppose

$$\int_{\partial\Omega} b \, \mathrm{d}\sigma > 0. \tag{17}$$

Then there exists a solution $(\mathbf{u}, \rho) \in W^{s,q}(\Omega, \mathbb{R}^2) \times W^{s-1,q}(\Omega)$ of the problem (1), (4) if and only if (15) holds. The general form of a solution of the problem (1), (4) in $W^{s,q}(\Omega, \mathbb{R}^2) \times W^{s-1,q}(\Omega)$ is $(\mathbf{u}, \rho + c)$ where c is a constant.

(2) Let $\alpha = 1$, $1/q + 1 < s < k$, $\mathbf{g} \in B^{q,r}_{s-1/q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{h} \in B^{q,r}_{s-1-1/q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{f} \in B^{q,r}_{s-2}(\Omega, \mathbb{R}^2)$, $\chi \in B^{q,r}_{s-1}(\Omega)$.

(a) Then there exists a solution $(\mathbf{u}, \rho) \in B^{q,r}_s(\Omega, \mathbb{R}^2) \times B^{q,r}_{s-1}(\Omega)$ of the problem (1), (3) if and only if (15) holds. The general form of a solution of the problem (1), (3) in $B^{q,r}_s(\Omega, \mathbb{R}^2) \times B^{q,r}_{s-1}(\Omega)$ is $(\mathbf{u}, \rho + c)$ where c is a constant.

(b) Suppose (17). Then there exists a solution $(\mathbf{u}, \rho) \in B^{q,r}_s(\Omega, \mathbb{R}^2) \times B^{q,r}_{s-1}(\Omega)$ of the problem (1), (4) if and only if (15) holds. The general form of a solution of the problem (1), (4) in $B^{q,r}_s(\Omega, \mathbb{R}^2) \times B^{q,r}_{s-1}(\Omega)$ is $(\mathbf{u}, \rho + c)$ where c is a constant.

(3) Let $k > 2$, $0 < \alpha < 1$, $\mathbf{g} \in \mathcal{C}^{k-1,\alpha}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{h} \in \mathcal{C}^{k-2,\alpha}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{f} \in \mathcal{C}^{k-3,\alpha}(\overline{\Omega}, \mathbb{R}^2)$, $\chi \in \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$.

(a) Then there exists a solution $(\mathbf{u}, \rho) \in \mathcal{C}^{k-1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ of the problem (1), (3) if and only if (15) holds. The general form of a

solution of the problem (1), (3) *in* $\mathcal{C}^{k-1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ *is* $(\mathbf{u}, \rho + c)$ *where* c *is a constant.*

(b) *Suppose* (17). *Then there exists a solution* $(\mathbf{u}, \rho) \in \mathcal{C}^{k-1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ *of the problem* (1), (4) *if and only if* (15) *holds. The general form of a solution of the problem* (1), (4) *in* $\mathcal{C}^{k-1,\alpha}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ *is* $(\mathbf{u}, \rho + c)$ *where* c *is a constant.*

Proof. Clearly, if $\mathbf{u} \equiv 0$ and ρ is constant then $(\mathbf{u}, \rho)$ is a solution of the problem (1), (3) and the problem (1), (4) with $\mathbf{f} \equiv 0$, $\chi \equiv 0$, $\mathbf{g} \equiv 0$, $\mathbf{h} \equiv 0$.

If $(\mathbf{u}, \rho)$ is a solution of the problem (1), (3) or the problem (1), (4), then $(\mathbf{u}, \rho)$ is a solution of some problem of the type (10). So, (15) holds by Theorem 3.6.

Let us denote $X = W^{s,q}(\Omega, \mathbb{R}^2)$, $Y = W^{s-1,q}(\Omega)$, $A = W^{s-1/q,q}(\partial\Omega)$, $B = W^{s-1-1/q,q}(\partial\Omega)$, $D = W^{s-2,q}(\Omega, \mathbb{R}^2)$ in the case (1); $X = B_s^{q,r}(\Omega, \mathbb{R}^2)$, $Y = B_{s-1}^{q,r}(\Omega)$, $A = B_{s-1/q}^{q,r}(\partial\Omega)$, $B = B_{s-1-1/q}^{q,r}(\partial\Omega)$, $D = B_{s-2}^{q,r}(\Omega, \mathbb{R}^2)$ in the case (2); $X = \mathcal{C}^{k-1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$, $Y = \mathcal{C}^{k-2,\alpha}(\overline{\Omega})$, $A = \mathcal{C}^{k-1,\alpha}(\partial\Omega)$, $B = \mathcal{C}^{k-2,\alpha}(\partial\Omega)$, $D = \mathcal{C}^{k-3,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ in the case (3); $X_\tau := \{\mathbf{u} \in X; \mathbf{u}_\mathbf{n} = 0\}$, $Y_0 = \{\rho \in Y; \int_\Omega \rho \, \mathrm{d}\mathbf{x} = 0\}$, $\widehat{X}_\tau := \{\mathbf{u} \in W^{2,2}(\Omega, \mathbb{R}^2); \mathbf{u}_\mathbf{n} = 0\}$, $\widehat{Y}_0 = \{\rho \in W^{1,2}(\Omega); \int_\Omega \rho \, \mathrm{d}\mathbf{x} = 0\}$, $\hat{B} = W^{1/2,2}(\partial\Omega)$, $\hat{D} = L^2(\Omega, \mathbb{R}^2)$. For $\lambda > 0$ define

$$V_\lambda(\mathbf{u}, \rho) := [-\Delta\mathbf{u} + \nabla\rho, \nabla \cdot \mathbf{u}, \lambda(\partial_1 u_2 - \partial_2 u_1)].$$

Then $V_\lambda : X_\tau \times Y_0 \to D \times Y_0 \times B$, $V_\lambda : \widehat{X}_\tau \times \widehat{Y}_0 \to \widehat{D} \times \widehat{Y}_0 \times \widehat{B}$ are isomorphisms by Theorem 3.6.

Denote by $\kappa := \nabla \cdot \mathbf{n}^\Omega$ the curvature of $\partial\Omega$. Remember that $\kappa := \nabla \cdot \mathbf{n}$ where $\mathbf{n}$ is the unit exterior normal of Ω. Since $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$, one has $\kappa \in \mathcal{C}^{k-2,\alpha}$. If $\mathbf{u} \in \widehat{X}_\tau$ then

$$\tau \cdot [\partial\mathbf{u}/\partial n] = (\partial_1 u_2 - \partial_2 u_1) - \kappa\tau \cdot \mathbf{u} \tag{18}$$

by [26, Lemma 4.1], and

$$\tau \cdot [(\hat{\nabla}\mathbf{u})\mathbf{n}^\Omega] = \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1) - \kappa\tau \cdot \mathbf{u} \tag{19}$$

by [16, Lemma 2.1]. We now show (18) and (19) for $\mathbf{u} \in X_\tau$. Denote $[\mathbf{f}, \chi, h] = V_1(\mathbf{u}, 0) \in D \times Y_0 \times B$. We can choose $\mathbf{f}_k \in D \cap \widehat{D}$, $\chi_k \in Y_0 \cap \widehat{Y}_0$ and $h_k \in B \cap \widehat{B}$ such that $[\mathbf{f}_k, \chi_k, h_k] \to [\mathbf{f}, \chi, h]$ as $k \to \infty$ in $D \times Y_0 \times B$. Denote $[\mathbf{u}_k, \rho_k] = V_1^{-1}[\mathbf{f}_k, \chi_k, h_k]$. Then $[\mathbf{u}_k, \rho_k] \in [X_\tau \times Y_0] \cap [\widehat{X}_\tau \times \widehat{Y}_0]$ and $[\mathbf{u}_k, \rho_k] \to [\mathbf{u}, \rho]$ as $k \to \infty$ in $X_\tau \times Y_0$. Using (18) and (19) for $\mathbf{u}_k$ and continuity of the trace of $\mathbf{v}$ and $\nabla\mathbf{v}$ for $\mathbf{v} \in X$, we deduce (18) and (19).

Define

$$W_a(\mathbf{u}, \rho) := [-\Delta\mathbf{u} + \nabla\rho, \nabla \cdot \mathbf{u}, \tau \cdot (\partial\mathbf{u}/\partial n + b\mathbf{u})],$$
$$W_b(\mathbf{u}, \rho) := [-\Delta\mathbf{u} + \nabla\rho, \nabla \cdot \mathbf{u}, \tau \cdot [T(\mathbf{u}, \rho)\mathbf{n}^\Omega + b\mathbf{u}]].$$

Then $W_a : X_\tau \times Y_0 \to D \times Y_0 \times B$, $W_b : X_\tau \times Y_0 \to D \times Y_0 \times B$ by (15). If $(\mathbf{u}, \rho) \in X_\tau \times Y_0$ then

$$W_a(\mathbf{u}, \rho) - V_1(\mathbf{u}, \rho) = W_b(\mathbf{u}, \rho) - V_{1/2}(\mathbf{u}, \rho) = [0, 0, (b - \kappa)\tau \cdot \mathbf{u}] \tag{20}$$

by (18) and (19). Note that $(b - \kappa)\tau \in \mathcal{C}^{k-2}(\partial\Omega; \mathbb{R}^2)$. (See for example [38, Lemma 1.16.8].) The trace is a compact operator from X to $B \times B$. If $\psi \in \mathcal{C}^{k-2}(\partial\Omega)$ then $v \mapsto \psi v$ is a bounded operator on B. (For the case (1) see [45, Chap. 2, §5.4, Lemma 5.5]. For the case (3) see [38, Lemma 1.16.8]. For the case (2) we use [55, §3.3.2, Theorem] and the fact that for small $\epsilon > 0$ one has $\mathcal{C}^{k-2,1}(\partial\Omega) \hookrightarrow$

$\mathcal{C}^{k-2,1-\epsilon}(\partial\Omega) = B^{\infty,\infty}_{k-1-\epsilon}(\partial\Omega)$ by [50, Chapter V, §5, Proposition 8'].) This and (20) give that $W_a - V_1$ and $W_b - V_{1/2}$ are compact operators from $X_\tau \times Y_0$ to $D \times Y_0 \times B$. Since $V_\lambda : X_\tau \times Y_0 \to D \times Y_0 \times B$ is an isomorphism, we infer that

$$W_a : X_\tau \times Y_0 \to D \times Y_0 \times B, \quad W_b : X_\tau \times Y_0 \to D \times Y_0 \times B$$

are Fredholm operators with index 0. In particular, for $s = q = r = 2$ in the case (1),

$$W_a : \widehat{X}_\tau \times \widehat{Y}_0 \to \widehat{D} \times \widehat{Y}_0 \times \widehat{B}, \quad W_b : \widehat{X}_\tau \times \widehat{Y}_0 \to \widehat{D} \times \widehat{Y}_0 \times \widehat{B}$$

are Fredholm operators with index 0.

Let now $(\mathbf{u}, \rho) \in X_\tau \times Y_0$, $W_a(\mathbf{u}, \rho) = 0$. Since $W_a : X_\tau \times Y_0 \to D \times Y_0 \times B$, $W_a : \widehat{X}_\tau \times \widehat{Y}_0 \to \widehat{D} \times \widehat{Y}_0 \times \widehat{B}$ are Fredholm operators with index 0, $(\mathbf{u}, \rho) \in \widehat{X}_\tau \times \widehat{Y}_0$ by [44, Lemma 11.9.21]. According to Green's formula

$$0 = \int_{\partial\Omega} \mathbf{u} \cdot [\partial \mathbf{u}/\partial \mathbf{n} - \rho \mathbf{n} + b\mathbf{u}] \, \mathrm{d}\sigma = \int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} b|\mathbf{u}|^2 \, \mathrm{d}\sigma.$$

Since $b \geq 0$ we infer that $\nabla \mathbf{u} \equiv 0$. Therefore there exists a vector $\mathbf{c} \in \mathbb{R}^2$ such that $\mathbf{u} \equiv \mathbf{c}$. Since $\mathbf{c} \cdot \mathbf{n} = 0$ on $\partial\Omega$ we deduce that $\mathbf{c} = 0$. Thus $\nabla \rho = \Delta \mathbf{u} = 0$ and ρ is constant. Since

$$\int_\Omega \rho \, \mathrm{d}\mathbf{x} = 0, \tag{21}$$

we infer $\rho \equiv 0$. Since $W_a : X_\tau \times Y_0 \to D \times Y_0 \times B$ is a Fredholm operator with index 0 and trivial kernel, it is an isomorphism.

Suppose now that $(\mathbf{u}, \rho) \in X_\tau \times Y_0$, $W_b(\mathbf{u}, \rho) = 0$. Since $W_b : X_\tau \times Y_0 \to D \times Y_0 \times B$, $W_b : \widehat{X}_\tau \times \widehat{Y}_0 \to \widehat{D} \times \widehat{Y}_0 \times \widehat{B}$ are Fredholm operators with index 0, [44, Lemma 11.9.21] gives $(\mathbf{u}, \rho) \in \widehat{X}_\tau \times \widehat{Y}_0$. According to Green's formula

$$0 = \int_{\partial\Omega} \mathbf{u} \cdot [T(\mathbf{u}, \rho)\mathbf{n} + b\mathbf{u}] \, \mathrm{d}\sigma = \int_\Omega 2|\hat{\nabla} \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} b|\mathbf{u}|^2 \, \mathrm{d}\sigma.$$

$b \geq 0$ forces that $\hat{\nabla} \mathbf{u} = 0$ in Ω, $b\mathbf{u} = 0$ on $\partial\Omega$. Since $\hat{\nabla} \mathbf{u} \equiv 0$, [34, Lemma 3.1] gives that $\mathbf{u} = A\mathbf{x} + \mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^2$ and $A = (a_{ij})$ is a skew symmetric matrix, i.e. $a_{ij} = -a_{ji}$. Since the surface measure of $\{\mathbf{x} \in \partial\Omega; \mathbf{u}(\mathbf{x}) = 0\}$ is positive by (17), [35, Lemma 5.1] gives $\mathbf{u} \equiv 0$. Thus $\nabla \rho = \Delta \mathbf{u} = 0$ and ρ is constant. (21) forces that $\rho \equiv 0$. Since $W_b : X_\tau \times Y_0 \to D \times Y_0 \times B$ is a Fredholm operator with index 0 and trivial kernel, it is an isomorphism.

Let $(\mathbf{u}, \rho) \in X \times Y$ be a solution of (1), (3) (or (1), (4)) with $\mathbf{f} \equiv 0$, $\chi \equiv 0$, $\mathbf{g} \equiv 0$, $\mathbf{h} \equiv 0$. Then $\mathbf{u} \in X_\tau$ and there exists $c \in \mathbb{R}^1$ such that $\rho - c \in Y_0$. Clearly, $(\mathbf{u}, \rho - c)$ solves again (1), (3) (or (1), (4)), respectively. We have proved that $\mathbf{u} \equiv 0$, $\rho - c \equiv 0$.

We now show the existence of a solution. Assume (15). According to Theorem 3.6 there exists $(\mathbf{v}, p) \in X \times Y$ such that

$$-\Delta \mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = \chi \quad \text{in } \Omega, \quad \mathbf{u_n} = \mathbf{g_n} \quad \text{on } \partial\Omega.$$

Clearly, $(\mathbf{v}, p) + W_a^{-1}(0, 0, \tau \cdot (\mathbf{h} - \partial \mathbf{v}/\partial \mathbf{n} - b\mathbf{v})) \in X \times Y$ is a solution of (1), (3), and $(\mathbf{v}, p) + W_b^{-1}(0, 0, \tau \cdot (\mathbf{h} - T(\mathbf{v}, p)\mathbf{n} - b\mathbf{v})) \in X \times Y$ is a solution of (1), (4). $\square$

Theorem 4.2. *Let $k \in \mathbb{N}$, $k \geq 2$, $1 < q, r < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,\alpha}$, $0 < \alpha \leq 1$, $b \equiv 0$.*

(1) Let $\alpha = 1$, $1/q + 1 < s \leq k$, $s - 1/q \notin \mathbb{N}$, $\mathbf{g} \in W^{s-1/q,q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{h} \in W^{s-1-1/q,q}(\partial\Omega; \mathbb{R}^2)$, $\mathbf{f} \in W^{s-2,q}(\Omega, \mathbb{R}^2)$, $\chi \in W^{s-1,q}(\Omega)$.

(a) *If Ω is not a circle then there exists a solution $(\mathbf{u},\rho)\in W^{s,q}(\Omega,\mathbb{R}^2)\times W^{s-1,q}(\Omega)$ of the problem (1), (4) if and only if (15) holds. The general form of a solution of the problem (1), (4) in $W^{s,q}(\Omega,\mathbb{R}^2)\times W^{s-1,q}(\Omega)$ is $(\mathbf{u},\rho+c)$ where c is a constant.*

(b) *Suppose that $\Omega=B(\mathbf{z};t)$ for some $\mathbf{z}\in\mathbb{R}^2$ and $t\in(0,\infty)$. Let $s\geq 2$. Define $\mathbf{v}(\mathbf{x}):=(x_2-z_2,z_1-x_1)$. Then there exists a solution $(\mathbf{u},\rho)\in W^{s,q}(\Omega,\mathbb{R}^2)\times W^{s-1,q}(\Omega)$ of the problem (1), (4) if and only if (15) and*

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\ \mathrm{d}\mathbf{x}+\int_{\partial\Omega}\mathbf{h}\cdot\mathbf{v}\ \mathrm{d}\sigma+\int_\Omega \mathbf{v}\cdot\nabla\chi\ \mathrm{d}\mathbf{x}=0. \tag{22}$$

hold, and $(\mathbf{u}+c_1\mathbf{v},\rho+c_2)$ with $c_1,c_2\in\mathbb{R}^1$ is a general form of a solution of the problem (1), (4) in $W^{s,q}(\Omega,\mathbb{R}^2)\times W^{s-1,q}(\Omega)$.

(2) *Let $\alpha=1$, $1/q+1<s<k$, $\mathbf{g}\in B^{q,r}_{s-1/q}(\partial\Omega;\mathbb{R}^2)$, $\mathbf{h}\in B^{q,r}_{s-1-1/q}(\partial\Omega;\mathbb{R}^2)$, $\mathbf{f}\in B^{q,r}_{s-2}(\Omega,\mathbb{R}^2)$, $\chi\in B^{q,r}_{s-1}(\Omega)$.*

(a) *If Ω is not a circle then there exists a solution $(\mathbf{u},\rho)\in B^{q,r}_s(\Omega,\mathbb{R}^2)\times B^{q,r}_{s-1}(\Omega)$ of the problem (1), (4) if and only if (15) holds. The general form of a solution of the problem (1), (4) in $B^{q,r}_s(\Omega,\mathbb{R}^2)\times B^{q,r}_{s-1}(\Omega)$ is $(\mathbf{u},\rho+c)$ where c is a constant.*

(b) *Suppose that $\Omega=B(\mathbf{z};t)$ for some $\mathbf{z}\in\mathbb{R}^2$ and $t\in(0,\infty)$. Let $s\geq 2$. Define $\mathbf{v}(\mathbf{x}):=(x_2-z_2,z_1-x_1)$. Then there exists a solution $(\mathbf{u},\rho)\in B^{q,r}_s(\Omega,\mathbb{R}^2)\times B^{q,r}_{s-1}(\Omega)$ of the problem (1), (4) if and only if (15) and (22) hold. The general form of a solution of the problem (1), (4) in $B^{q,r}_s(\Omega,\mathbb{R}^2)\times B^{q,r}_{s-1}(\Omega)$ is $(\mathbf{u}+c_1\mathbf{v},\rho+c_2)$ where c_1, c_2 are constants.*

(3) *Let $k>2$, $0<\alpha<1$, $\mathbf{g}\in\mathcal{C}^{k-1,\alpha}(\partial\Omega;\mathbb{R}^2)$, $\mathbf{h}\in\mathcal{C}^{k-2,\alpha}(\partial\Omega;\mathbb{R}^2)$, $\mathbf{f}\in\mathcal{C}^{k-3,\alpha}(\overline{\Omega},\mathbb{R}^2)$, $\chi\in\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$.*

(a) *If Ω is not a circle then there exists a solution $(\mathbf{u},\rho)\in\mathcal{C}^{k-1,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ of the problem (1), (4) if and only if (15) holds. The general form of a solution of the problem (1), (4) in $\mathcal{C}^{k-1,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ is $(\mathbf{u},\rho+c)$ where c is a constant.*

(b) *Suppose that $\Omega=B(\mathbf{z};t)$ for some $\mathbf{z}\in\mathbb{R}^2$ and $t\in(0,\infty)$. Define $\mathbf{v}(\mathbf{x}):=(x_2-z_2,z_1-x_1)$. Then there exists a solution $(\mathbf{u},\rho)\in\mathcal{C}^{k-1,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ of the problem (1), (4) if and only if (15) and (22) hold. The general form of a solution of the problem (1), (4) in $\mathcal{C}^{k-1,\alpha}(\overline{\Omega},\mathbb{R}^2)\times\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$ is $(\mathbf{u}+c_1\mathbf{v},\rho+c_2)$ with $c_1,c_2\in\mathbb{R}^1$.*

Proof. Clearly, if $\mathbf{u}\equiv 0$ and ρ is constant then $(\mathbf{u},\rho)$ is a solution of the problem (1), (4) with $\mathbf{f}\equiv 0$, $\chi\equiv 0$, $\mathbf{g}\equiv 0$, $\mathbf{h}\equiv 0$.

If $(\mathbf{u},\rho)$ is a solution of (1), (4), then $(\mathbf{u},\rho)$ is a solution of (1),

$$\mathbf{u}_\mathbf{n}=\mathbf{g}_\mathbf{n},\quad [T(\mathbf{u},\rho)\mathbf{n}+\mathbf{u}]_\tau=[\mathbf{h}+\mathbf{u}]_\tau \text{ on } \partial\Omega.$$

Therefore (15) holds by Theorem 4.1.

Let us denote $X=W^{s,q}(\Omega,\mathbb{R}^2)$, $Y=W^{s-1,q}(\Omega)$, $A=W^{s-1/q,q}(\partial\Omega)$, $B=W^{s-1-1/q,q}(\partial\Omega)$, $D=W^{s-2,q}(\Omega,\mathbb{R}^2)$ in the case (1); $X=B^{q,r}_s(\Omega,\mathbb{R}^2)$, $Y=B^{q,r}_{s-1}(\Omega)$, $A=B^{q,r}_{s-1/q}(\partial\Omega)$, $B=B^{q,r}_{s-1-1/q}(\partial\Omega)$, $D=B^{q,r}_{s-2}(\Omega,\mathbb{R}^2)$ in the case (2); $X=\mathcal{C}^{k-1,\alpha}(\overline{\Omega},\mathbb{R}^2)$, $Y=\mathcal{C}^{k-2,\alpha}(\overline{\Omega})$, $A=\mathcal{C}^{k-1,\alpha}(\partial\Omega)$, $B=\mathcal{C}^{k-2,\alpha}(\partial\Omega)$, $D=\mathcal{C}^{k-3,\alpha}(\overline{\Omega},\mathbb{R}^2)$ in the case (3); $\widehat{X}=W^{2,2}(\Omega,\mathbb{R}^2)$, $\widehat{Y}=W^{1,2}(\Omega)$, $\widehat{A}=W^{3/2,2}(\partial\Omega)$,

$\hat{B} = W^{1/2,2}(\partial\Omega)$, $\hat{D} = L^2(\Omega, \mathbb{R}^2)$. For $\lambda \geq 0$ define

$$W_\lambda(\mathbf{u}, \rho) := [-\Delta\mathbf{u} + \nabla\rho, \nabla\cdot\mathbf{u}, \mathbf{n}\cdot\mathbf{u}, \tau\cdot[T(\mathbf{u},\rho)\mathbf{n}^\Omega + \lambda\mathbf{u}]].$$

Then $W_1 : X\times Y \to D\times Y\times A\times B$ is a Fredholm operator with index 0 by Theorem 4.1. If $(\mathbf{u}, \rho) \in X \times Y$ then $W_1(\mathbf{u}, \rho) - W_0(\mathbf{u}, \rho) = [0, 0, 0, \tau\cdot\mathbf{u}]$. Since $W_1 - W_0 : X \times Y \to D \times Y \times A \times B$ is compact, the operator $W_0 : X \times Y \to D \times Y \times A \times B$ is a Fredholm operator with index 0. In particular, for $s = q = 2$ in the case (1) we obtain that $W_0 : \widehat{X} \times \widehat{Y} \to \widehat{D} \times \widehat{Y} \times \widehat{A} \times \widehat{B}$ is a Fredholm operator with index 0.

Let now $(\mathbf{u}, \rho) \in X \times Y$, $W_0(\mathbf{u}, \rho) = 0$. Since $W_0 : X \times Y \to D \times Y \times A \times B$, $W_0 : \widehat{X}_\tau \times \widehat{Y} \to \widehat{D} \times \widehat{Y} \times \widehat{A} \times \widehat{B}$ are Fredholm operators with index 0, [44, Lemma 11.9.21] gives $(\mathbf{u}, \rho) \in \widehat{X} \times \widehat{Y}$. According to Green's formula

$$0 = \int_{\partial\Omega} \mathbf{u}\cdot T(\mathbf{u}, \rho)\mathbf{n}\, \mathrm{d}\sigma = \int_\Omega 2|\hat{\nabla}\mathbf{u}|^2\, \mathrm{d}\mathbf{x}.$$

Since $\hat{\nabla}\mathbf{u} \equiv 0$, [34, Lemma 3.1] gives that $\mathbf{u}(\mathbf{x}) = (c_1 + ax_2, c_2 - ax_1)$ where $a, c_1, c_2 \in \mathbb{R}^1$. Thus $\nabla\rho = \Delta\mathbf{u} = 0$ and ρ is constant. Suppose first that $a = 0$. Then $0 = \mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot(c_1, c_2)$ on $\partial\Omega$. This forces $c_1 = c_2 = 0$ and thus $\mathbf{u} \equiv 0$. Let now $a \neq 0$. Denote $\mathbf{z} := [c_2/a, -c_1/a]$. Then $\mathbf{u}(\mathbf{x}) = a(x_2 - z_2, -(x_1 - z_1))$. Define $\varphi(\mathbf{x}) := (x_1 - z_1)^2 + (x_2 - z_2)^2$. Then we have on $\partial\Omega$

$$\frac{\partial\varphi(\mathbf{x})}{\partial\tau} = 2(x_1 - z_1, x_2 - z_2)\cdot\tau = \frac{2}{a}\,(\mathbf{n}\cdot\mathbf{u}) = 0.$$

So, φ is locally constant on $\partial\Omega$. Since $\partial\Omega$ is connected, there exists $t > 0$ such that $\partial\Omega = \partial B(\mathbf{z}; t)$ and therefore $\Omega = B(\mathbf{z}; t)$.

So, if Ω is not a circle then $W_0(\mathbf{u}, \rho) = 0$ if and only if $\mathbf{u} \equiv 0$ and ρ is constant. Since $W_0 : X \times Y \to D \times Y \times A \times B$ is a Fredholm operator with index 0, (15) is a necessary and sufficient condition for the existence of a solution $(\mathbf{u}, \rho) \in X \times Y$ of (1), (4).

Suppose $\Omega = B(\mathbf{z}; t)$. If $c_1, c_2 \in \mathbb{R}^1$ then easy calculation yields $W_0(c_1\mathbf{v}, c_2) = 0$. So, $W_0(\mathbf{u}, \rho) = 0$ if and only if $(\mathbf{u}, \rho) = (c_1\mathbf{v}, c_2)$ for some $c_1, c_2 \in \mathbb{R}^1$. If $(\mathbf{u}, \rho) \in X \times Y$ is a solution of (1), (4), then the Green formula yields

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{h}\cdot\mathbf{v}\, \mathrm{d}\sigma = \int_\Omega \sum_{i=1}^2 \Big[-\rho\partial_i v_i + v_i\partial_i\chi + \sum_{j=1}^2 \partial_i u_j(\partial_i v_j + \partial_j v_i)\Big]\, \mathrm{d}\sigma$$
$$= -\int_\Omega \mathbf{v}\cdot\nabla\chi\, \mathrm{d}\mathbf{x}.$$

Since $W_0 : X \times Y \to D \times Y \times A \times B$ is a Fredholm operator with index 0, there exists a solution $(\mathbf{u}, \rho) \in X \times Y$ of (1), (4) if and only if (15) and (22) hold. □

5. Appendix

5.1. Non-tangential limit.

Lemma 5.1. *Let $G \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary. Let $\mathbf{u} \in L^1_{loc}(G, \mathbb{R}^m)$ with $\nabla\cdot\mathbf{u} \in L^1(G)$. Suppose that $M_a(\mathbf{u}) \in L^1(\partial G)$ and there exists a non-tangential limit of $\mathbf{u}$ at almost all points of ∂G. Then*

$$\int_{\partial G} \mathbf{u}\cdot\mathbf{n}^G\, \mathrm{d}\sigma = \int_G \nabla\cdot\mathbf{u}\, \mathrm{d}\mathbf{x}.$$

(See [43, Proposition 2.4].)

5.2. **Holomorphic functions.**

Lemma 5.2. *Let $\xi+i\eta$ be a holomorphic function in Ω. If $\eta\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$ with $k\in\mathbb{N}_0$ and $0<\alpha<1$, then $\xi\in\mathcal{C}^{k,\alpha}(\overline{\Omega})$.*

Proof. Suppose first that $k=0$. The domain with Lipschitz boundary satisfies the property P_α by [40, p. 394]. Therefore $\xi\in\mathcal{C}^{0,\alpha}(\overline{\Omega})$ by [40, Corollary 3.7].

If $k\neq 0$ we use that fact that $\partial^\beta\xi+i\partial^\beta\eta$ is a holomorphic function for arbitrary multiindex β. □

Lemma 5.3. *Let $\xi+i\eta$ be a holomorphic function in Ω, $1<q<\infty$, $0<s<\infty$. If $\eta\in W^{s,q}(\Omega)$ then $\xi\in W^{s,q}(\Omega)$.*

Proof. For $s\notin\mathbb{N}$ see [40, Proposition 9.2]. Let now $s\in\mathbb{N}$. Since $W^{s,q}(\Omega)\subset W^{1/2,q}(\Omega)$ one has $\xi\in W^{1/2,q}(\Omega)\subset L^q(\Omega)$. The rest is a consequence of the fact that $\partial_1\xi=\partial_2\eta$, $\partial_2\xi=-\partial_1\eta$. □

Lemma 5.4. *Let $\xi+i\eta$ be a holomorphic function in Ω. If $\eta\in B^{q,r}_s(\Omega)$ with $1<q,r<\infty$, $0<s<\infty$, then $\xi\in B^{q,r}_s(\Omega)$.*

Proof. We can suppose that $\eta\in B^{q,r}_s(\mathbb{R}^2)$. Then $\partial_j\eta\in B^{q,r}_{s-1}(\mathbb{R}^2)$ by [47, Chapter 3, Theorem 9]. Thus $\partial_1\xi=\partial_2\eta\in B^{q,r}_{s-1}(\mathbb{R}^2)$, $\partial_2\xi=-\partial_1\eta\in B^{q,r}_{s-1}(\mathbb{R}^2)$. [41, Proposition 7.6] gives $\xi\in B^{q,r}_s(\Omega)$. □

Lemma 5.5. *Let $\xi+i\eta$ be a holomorphic function in Ω and $1<q<\infty$. If $M_a(\eta)\in L^q(\partial\Omega)$ then $M_a(\xi)\in L^q(\partial\Omega)$ and there exist non-tangential limits of ξ and η at almost all points of $\partial\Omega$.*

Proof. If u is a harmonic function in Ω then $M_a(u)\in L^q(\partial\Omega)$ if and only if $A_a(u)\in L^q(\partial\Omega)$, where

$$A_a(u)(\mathbf{x})=\left[\int_{\Gamma_a(\mathbf{x})}|\nabla u(\mathbf{y})|^2\,\mathrm{d}\mathbf{x}\right]^{1/2}.$$

(See [17, Theorem] or [28, Theorem 1.5.10].) Since $\partial_1\xi=\partial_2\eta$, $\partial_2\xi=-\partial_1\eta$ and therefore $A_a(\xi)=A_a(\eta)$, we infer $M_a(\xi)\in L^q(\partial\Omega)$. Since $M_a(\xi),M_a(\eta)\in L^q(\partial\Omega)$, there exist non-tangential limits of ξ and η at almost all points of $\partial\Omega$ by [23] and [24, Theorem 1]. □

5.3. **The Dirichlet problem for the Laplace equation.**

Theorem 5.6. *Let $\partial\Omega$ be of class $\mathcal{C}^{k,1}$ with $k\in\mathbb{N}$, and $1<p,q<\infty$, $1/p<s\le k+1$.*

(a) If $f\in W^{s-2,p}(\Omega)$, $g\in W^{s-1/p,p}(\partial\Omega)$ and $s-1/p\notin\mathbb{N}$, then there exists a unique solution $u\in W^{s,p}(\Omega)$ of the Dirichlet problem

$$\Delta u=f\quad\text{in }\Omega,\qquad u=g\quad\text{on }\partial\Omega. \tag{23}$$

(b) If $f\in B^{p,q}_{s-2}(\Omega)$, $g\in B^{p,q}_{s-1/p}(\partial\Omega)$ and $s<k+1$, then there exists a unique solution $u\in B^{p,q}_s(\Omega)$ of the Dirichlet problem (23).

Proof. If $s<1+1/p$ then (a) is a consequence of [39, Corollary 4.2]. If $s\in\mathbb{N}$, $s>1$, then (a) follows from [22, Theorem 2.4.2.5 and Theorem 2.5.1.1]. The rest we obtain by the real interpolation. (See [53, Lemma 22.3], [56, Corollary 1.111, Theorem 1.122] and [19], Theorem 6.7.) □

5.4. **Besov spaces.**

Proposition 5.7. *Let $-\infty < t \leq \tau < \infty$ and $1 < p, q, r, \beta < \infty$. Suppose that one from the following conditions holds:*

(a) $\tau > t$, $\tau - 2/q > t - 2/p$.
(b) $\tau > t$, $\tau - 2/q = t - 2/p$, $r \leq \beta$.
(c) $\tau = t$, $p \leq q$, $r \leq \beta$.

Then $B^{q,r}_\tau(\Omega) \hookrightarrow B^{p,\beta}_t(\Omega)$.

Proof. If the condition (a) holds then $B^{q,r}_\tau(\Omega) \hookrightarrow B^{p,\beta}_t(\Omega)$ by [56, Theorem 1.97]. If the condition (b) holds then $B^{q,r}_\tau(\Omega) \hookrightarrow B^{p,\beta}_t(\Omega)$ by [56, pp. 78–79]. If the condition (c) holds then $B^{q,r}_\tau(\Omega) \hookrightarrow B^{p,r}_t(\Omega) \hookrightarrow B^{p,\beta}_t(\Omega)$ by [55, §3.3.1, Theorem] and [54, §4.6.1, Theorem]. □

5.5. **Stokes system with prescribed pressure.**

Proposition 5.8. *Let $k \in \mathbb{N}$, $1 < p, q < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s \leq k+1$, $s - 1/q \notin \mathbb{N}_0$, $1/p < t \leq k$, $t - 1/p \notin \mathbb{N}_0$, and $t \leq s+1$. Suppose that $s + 1 - 2/q \geq t - 2/p$. Let $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$. Then there exists a unique solution $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of the problem*

$$-\Delta\mathbf{u} + \nabla\rho = 0, \ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u}\cdot\tau^\Omega = g, \ \rho = h \text{ on } \partial\Omega, \tag{24}$$

where $\tau^\Omega = (n^\Omega_2, -n^\Omega_1)$ is the tangential vector on $\partial\Omega$. Moreover, $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$, and there exist non-tangential limits of $\mathbf{u}$ and ρ at almost all points of $\partial\Omega$.

Proof. The proof is the same like the proof of [36, Theorem 5.2] but we use the new embedding result $W^{s+1,q}(\Omega) \hookrightarrow W^{t,p}(\Omega)$ (see [12, Theorem 3.8]). □

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