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BACKWARD ORBITS OF OPERATORS

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ABSTRACT. Let T be a Banach space operator with dense non-closed range. Then T has backward orbits which grow arbitrarily fast.

1. Introduction

Let X be a Banach space. As usual, we denote by $B(X)$ the set of all bounded linear operators acting on X. For $T \in B(X)$ let $R(T)$ denote the range, $R(T) = TX$, and $N(T)$ the kernel, $N(T) = \{x \in X : Tx = 0\}$. Denote by $R^\infty(T)$ the infinite range of T, $R^\infty(T) = \bigcap_{n\in\mathbb{N}} R(T^n)$.

Let $T \in B(X)$ and $x_0 \in X$. A backward orbit of x_0 is any sequence $(x_n)_{n\in\mathbb{N}}$ of vectors in X satisfying $Tx_n = x_{n-1} \quad (n \in \mathbb{N})$. The set of all vectors x_0 having a backward orbit is called the algebraic core of T and denoted by $\operatorname{co}(T)$. The notion was introduced in [11] and has applications in the local spectral theory, see e.g. [3], [4], [5], [7], [10].

Equivalently, $\operatorname{co}(T)$ is the largest linear manifold $L \subset X$ such that $TL = L$.

It is easy to see that $\operatorname{co}(T) \subset R^\infty(T)$ but the equality is not true in general.

It is well known that if $T \in B(X)$ has dense range then $R^\infty(T)$ is also dense, see [2], p. 45. In fact, in this case $\operatorname{co}(T)$ is also dense.

Proposition 1.1. *Let $T \in B(X)$ be an operator with dense range. Then* $\overline{\operatorname{co}(T)} = X$.

Proof. Since $\operatorname{co}(tT) = \operatorname{co}(T)$ for all $t \neq 0$, without loss of generality we may assume that $\|T\| = 1$.

Let $V_0 \subset X$ be a non-empty open subset. We show that $V_0 \cap \operatorname{co}(T) \neq \emptyset$.

Since $\overline{R(T)} = X$, there exists $u_1 \in X$ with $Tu_1 \in V_0$. There exists an open neighbourhood V_1 of u_1 such that $\operatorname{diam} V_1 \leq 1/2$ and $T\overline{V_1} \subset V_0$.

Similarly, there exists $u_2 \in X$ with $Tu_2 \in V_1$, and an open neighbourhood V_2 of u_2 such that $\operatorname{diam} V_2 \leq \frac{1}{4}$ and $T\overline{V_2} \subset V_1$.

By induction, we construct non-empty open subsets $V_k \subset X \quad (k \in \mathbb{N})$ such that $\operatorname{diam} V_k \leq 2^{-k}$ and $T\overline{V_k} \subset V_{k-1} \quad (k \in \mathbb{N})$.

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For each $k \in \mathbb{N}$ we have $T^k\overline{V_k} \subset T^{k-1}\overline{V_{k-1}} \subset \cdots \subset T\overline{V_1} \subset V_0$ and $\operatorname{diam} T^k\overline{V_k} \le 2^{-k}$. Hence $\bigcap_{k\in\mathbb{N}} T^k\overline{V_k}$ is non-empty and contains a single point x_0.

Similarly, for $j = 1, 2, \dots$ and $k \ge j$ we have $T^{k-j}\overline{V_k} \subset T^{k-j-1}\overline{V_{k-1}} \subset \cdots \subset T\overline{V_{j+1}} \subset \overline{V_j}$ and $\operatorname{diam} T^{k-j}\overline{V_k} \le 2^{-k}$. Let $\bigcap_{k\ge j} T^{k-j}\overline{V_k} = \{x_j\}$. Then $Tx_j \in \bigcap_{k\in\mathbb{N}} T^{k+1-j}\overline{V_k} = \{x_{j-1}\}$ for all $j \in \mathbb{N}$. Hence $x_0 \in V_0 \cap \operatorname{co}(T)$. □

Remark 1.2. It is worth noting that a similarly defined analytic core (the set of all vectors $x_0 \in X$ such that there exists a sequence $(x_j)_{j\in\mathbb{N}} \subset X$ with $Tx_j = x_{j-1} \quad (j \in \mathbb{N})$ and $\sup_j \|x_j\|^{1/j} < \infty$) is not necessarily dense if T has dense range. Example: let H be the Hilbert space with an orthonormal basis $e_n \quad (n \in \mathbb{Z})$ and let $T \in B(H)$ be the weighted bilateral shift defined by $Te_{-n} = n^{-1}e_{-n+1} \quad (n \in \mathbb{N})$ and $Te_n = e_{n+1} \quad (n \ge 0)$. Then T has dense range but the analytic core of T contains only the zero vector.

In general, a backward orbit of a vector x_0 is not unique. It is easy to see that backward orbits are unique if T is injective. Moreover, for injective operators there is a simple description of the algebraic core $\operatorname{co}(T)$.

Proposition 1.3. *Let $T \in B(X)$ be an injective operator. Then $\operatorname{co}(T) = R^\infty(T)$.*

Proof. The inclusion $\operatorname{co}(T) \subset R^\infty(T)$ is true in general.

Let $x_0 \in R^\infty(T) = \bigcap_{j=0}^\infty R(T^j)$. For each $j \ge 0$ let $x_j \in X$ satisfy $T^j x_j = x_0$. Then $T^j(Tx_{j+1} - x_j) = x_0 - x_0 = 0$. Since T is injective, $Tx_{j+1} = x_j$ for all $j \ge 0$ and $x_0 \in \operatorname{co}(T)$. □

2. Large backward orbits

If $T \in B(X)$ and $x_0 \in X$ then its (forward) orbit $(T^n x_0)$ may grow only exponentially, $\|T^n x_0\| \le \|T\|^n \cdot \|x_0\| \quad (n \in \mathbb{N})$. The same is true for backward orbits if T is invertible: then $\|T^{-n}x_0\| \le \|T^{-1}\|^n \cdot \|x_0\| \quad (n \in \mathbb{N})$.

However, if T is not invertible, then backward orbits may grow arbitrarily fast. The following theorem is the main result of this paper.

Theorem 2.1. *Let $T \in B(X)$ satisfy $\overline{R(T)} = X \ne R(T)$. Let $(b_j)_{j=0}^\infty$ be a sequence of positive numbers, $y \in X$ and $\varepsilon > 0$. Then there exist vectors $x_j \in X \quad (j \ge 0)$ such that $\|x_0 - y\| < \varepsilon$, $Tx_{j+1} = x_j$ and*

$$\|x_{j+1}\| \ge b_j \|x_j\|$$

for all $j \ge 0$.

Before proving Theorem 2.1 we need two simple lemmas.

Lemma 2.2. *(see [8], Lemma 1) Let X be an infinite-dimensional Banach space, let $F \subset X$ be a finite-dimensional subspace and $\varepsilon > 0$. Then there exists a subspace $M \subset X$ with $\operatorname{codim} M < \infty$ such that*

$$\|f + m\| \ge (1 - \varepsilon) \max\Big\{\|f\|, \frac{\|m\|}{2}\Big\}$$

for all $f \in F$ and $m \in M$.

If X is a Hilbert space then one can take $M = F^\perp$. So M in Lemma 2.2 may be viewed as a Banach space version of the orthogonal complement.

Lemma 2.3. *Let $T \in B(X)$ be an operator with dense range and $M \subset X$ a subspace of finite codimension. Then $\mathrm{co}\,(T) \cap M$ is dense in M.*

Proof. Let $n = \operatorname{codim} M$. If $x_1, \dots, x_{n+1} \in \mathrm{co}\,(T)$, then there exists a non-trivial linear combination $x := \sum_{i=1}^{n+1} \alpha_i x_i \in M$. So $x \in \mathrm{co}\,(T) \cap M$ and $\dim \mathrm{co}\,(T)/(\mathrm{co}\,(T) \cap M) \le n$. Let $F \subset \mathrm{co}\,(T)$ be a subspace with $\dim F \le n$ such that $\mathrm{co}\,(T) = (\mathrm{co}\,(T) \cap M) + F$. Then

$$X = \overline{\mathrm{co}\,(T)} = \overline{\mathrm{co}\,(T) \cap M} + F$$

and $\operatorname{codim} \overline{\mathrm{co}\,(T) \cap M} \le \dim F \le n$. Since $\overline{\mathrm{co}\,(T) \cap M} \subset M$ and $\operatorname{codim} M = n$, we have $\overline{\mathrm{co}\,(T) \cap M} = M$. □

Proof of Theorem 2.1.

Without loss of generality we may assume that $0 < \varepsilon < 1$.

Find $x_{0,0} \in \mathrm{co}\,(T)$ with $\|x_{0,0} - y\| < \varepsilon/2$. Find vectors $x_{0,j} \in X$ such that $Tx_{0,j} = x_{0,j-1} \quad (j \in \mathbb{N})$.

We construct inductively vectors $x_{k,j} \in \mathrm{co}\,(T), k, j \ge 0$ such that

$$Tx_{k,j+1} = x_{k,j} \qquad (k, j \ge 0), \tag{2.1}$$

$$\|x_{k+1,j} - x_{k,j}\| < \frac{\varepsilon}{2^{k+2}} \qquad (0 \le j \le k) \tag{2.2}$$

and

$$\|x_{k,j+1}\| \ge b_j(1 + 2^{-k})\|x_{k,j}\| \qquad (0 \le j \le k-1). \tag{2.3}$$

Let $k \ge 0$ and suppose that the vectors $x_{0,j}, x_{1,j}, \dots, x_{k,j} \quad (j \ge 0)$ satisfying (2.1), (2.2) and (2.3) have already been constructed.

Choose $\varepsilon' > 0$ such that

$$\varepsilon' < \frac{\varepsilon \min\{1, \|x_{k,j}\| : 0 \le j \le k\}}{2^{k+5} b_k \cdot \max\{1, \|T\|^k\} \cdot \|x_{k,k}\|}.$$

Let $F = \bigvee_{j=0}^{k+1} x_{k,j}$. Then $\dim F < \infty$. Let $M' \subset X$ be a subspace of finite codimension satisfying

$$\|f + m\| \ge (1 - 2^{-k-4}) \max\{\|f\|, \|m\|/2\} \qquad (f \in F, m \in M'), \tag{1}$$

which exists by Lemma 2.2. Then $\operatorname{codim} T^{-j}M' < \infty$ for all j. Let $M = \bigcap_{j=0}^{k} T^{-j}M'$. Then $\operatorname{codim} M < \infty$.

Since TM is not closed, the restriction $T|M$ is not bounded below. Moreover, $\mathrm{co}\,(T) \cap M$ is dense in M by Lemma 2.3. So there exists $u_{k+1} \in \mathrm{co}\,(T) \cap M$ such that $\|u_{k+1}\| = 1$ and

$$\|Tu_{k+1}\| < \varepsilon'.$$

For $j = 0, \dots, k+1$ set

$$x_{k+1,j} = x_{k,j} + 4b_k\|x_{k,k}\|T^{k+1-j}u_{k+1}. \tag{2}$$

Clearly $x_{k+1,j} \in \mathrm{co}\,(T) \quad (j = 0, \dots, k+1)$.

For $j > k+1$ choose vectors $x_{k+1,j} \in \mathrm{co}\,(T)$ satisfying $Tx_{k+1,j} = x_{k+1,j-1}$. Clearly vectors $x_{k+1,j}$ satisfy (2.1).

For $j = 0, 1, \dots, k$ we have by (2),

$$\begin{aligned} \big\|x_{k+1,j} - x_{k,j}\big\| &= 4b_k\|x_{k,k}\| \cdot \|T^{k-j+1}u_{k+1}\| \\ \le& 4b_k\|x_{k,k}\| \cdot \|T^{k-j}\| \cdot \|Tu_{k+1}\| \le 4b_k\|x_{k,k}\| \cdot \|T^{k-j}\| \cdot \varepsilon' < \frac{\varepsilon}{2^{k+2}}. \end{aligned}$$

Hence the vectors $x_{k+1,j}$ satisfy (2.2).

For $j = 0, \dots, k-1$ we have

$$\|x_{k+1,j+1}\| = \Big\|x_{k,j+1} + 4b_k\|x_{k,k}\| \cdot T^{k-j}u_{k+1}\Big\|,$$

where $x_{k,j+1} \in F$ and $T^{k-j}u_{k+1} \in M'$. So

$$\|x_{k+1,j+1}\| \ge (1 - 2^{-k-4})\|x_{k,j+1}\| \ge (1 - 2^{-k-4})(1 + 2^{-k})b_j\|x_{k,j}\|$$

by (1) and the induction assumption. On the other hand,

$$\begin{aligned} \|x_{k+1,j}\| &\le \|x_{k,j}\| + 4b_k\|x_{k,k}\| \cdot \|T^{k-j+1}u_{k+1}\| \\ \le& \|x_{k,j}\| + 4b_k\|x_{k,k}\| \cdot \|T^{k-j}\|\varepsilon' \le \|x_{k,j}\|(1 + 2^{-k-3}). \end{aligned}$$

So

$$\|x_{k+1,j+1}\| \ge \frac{b_j(1 - 2^{-k-4})(1 + 2^{-k})\|x_{k+1,j}\|}{1 + 2^{-k-3}} \ge b_j(1 + 2^{-k-1})\|x_{k+1,j}\|$$

since $(1 - 2^{-k-4})(1 + 2^{-k}) \ge (1 + 2^{-k-3})(1 + 2^{-k-1})$.

For $j = k$ we have by (1),

$$\begin{aligned} \|x_{k+1,k+1}\| &= \Big\|x_{k,k+1} + 4b_k\|x_{k,k}\| \cdot u_{k+1}\Big\| \\ \ge& \frac{1 - 2^{-k-4}}{2} \cdot 4b_k\|x_{k,k}\| = 2(1 - 2^{-k-4})b_k\|x_{k,k}\| \end{aligned}$$

and

$$\begin{aligned} \|x_{k+1,k}\| &\le \|x_{k,k}\| + 4b_k\|x_{k,k}\| \cdot \|Tu_{k+1}\| \\ \le& \|x_{k,k}\|(1 + 4b_k\varepsilon') \le \|x_{k,k}\|(1 + 2^{-k-3}). \end{aligned}$$

Thus

$$\|x_{k+1,k+1}\| \ge \frac{2b_k(1 - 2^{-k-4})\|x_{k+1,k}\|}{1 + 2^{-k-3}} \ge b_k(1 + 2^{-k-1})\|x_{k+1,k}\|.$$

Hence $x_{k+1,j}$ satisfy (2.3).

Suppose that the vectors $x_{k,j} \in \mathrm{co}\,(T), k, j = 0, 1, \dots$ satisfying (2.1), (2.2) and (2.3) have been constructed. Clearly $(x_{k,j})_k$ is a Cauchy sequence for each j. Let $x_j = \lim_{k\to\infty} x_{k,j}$. We have

$$\|y - x_0\| \le \|y - x_{0,0}\| + \|x_{0,0} - x_{1,0}\| + \|x_{2,0} - x_{1,0}\| + \cdots < \frac{\varepsilon}{2} + \sum_{j=0}^{\infty} \frac{\varepsilon}{2^{j+2}} = \varepsilon.$$

Moreover, for each $j \geq 1$ we have

$$Tx_j = \lim_{k\to\infty} Tx_{k,j} = \lim_{k\to\infty} x_{k,j-1} = x_{j-1}.$$

Finally, for each $j \geq 0$ we have

$$\|x_{j+1}\| = \lim_{k\to\infty} \|x_{k,j+1}\| \geq \lim_{k\to\infty} b_j(1+2^{-k})\|x_{k,j}\| = b_j\|x_j\|.$$

So $\|x_{j+1}\| \geq b_j\|x_j\|$ for all $j \geq 0$. □

Corollary 2.4. *(cf. [9], Theorem 3) Let $T \in B(X)$ be an operator with $\overline{R(T)} = X \neq R(T)$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Then there exists a unit vector $u \in X$ such that*

$$\|T^{j+1}u\| \leq \varepsilon\|T^j u\| \qquad (j = 0, 1, \dots, n).$$

Corollary 2.5. *Let $T \in B(X)$ be an operator such that $\overline{R(T)} = X \neq R(T)$. Let $(a_j)_{j=0}^\infty$ be a sequence of positive numbers, $y \in X$, $\|y\| > a_0$ and $\varepsilon > 0$. Then there exist vectors $x_j \in \operatorname{co}(T)$ $(j \geq 0)$ such that $\|x_0 - y\| < \varepsilon$, $Tx_{j+1} = x_j$ and*

$$\|x_j\| \geq a_j$$

for all $j \geq 0$.

Proof. Set $b_j = \frac{a_{j+1}}{a_j}$ $(j \geq 0)$. By Theorem 2.1, there exist vectors $x_j \in \operatorname{co}(T)$ $(j = 0, 1, \dots)$ such that $\|x_0 - y\| < \min\{\varepsilon, \|y\| - a_0\}$, $Tx_{j+1} = x_j$ and $\|x_{j+1}\| \geq b_j\|x_j\|$ for all $j \geq 0$. We have $\|x_0\| \geq \|y\| - (\|y\| - a_0) = a_0$ and, by induction,

$$\|x_j\| \geq b_{j-1}\|x_{j-1}\| \geq b_{j-1}a_{j-1} = a_j$$

for all $j \geq 0$. □

If T is injective then the backward orbit is unique and exists for each vector $x_0 \in R^\infty(T) = \operatorname{co}(T)$. So Theorem 2.1 and Corollary 2.5 become simpler.

Theorem 2.6. *Let $T \in B(X)$ be an injective operator such that $\overline{R(T)} = X \neq R(T)$. Let $(b_j)_{j\geq 0}$ be a sequence of positive numbers, $y \in X$ and $\varepsilon > 0$. Then there exists $x \in R^\infty(T)$ such that $\|x - y\| < \varepsilon$ and*

$$\|T^{-j-1}x\| \geq b_j\|T^{-j}x\|$$

for all $j \geq 0$.

Corollary 2.7. *Let $T \in B(X)$ be an injective operator such that $\overline{R(T)} = X \neq R(T)$. Let $(a_j)_{j=0}^\infty$ be a sequence of positive numbers. Let $y \in X$ satisfy $\|y\| > a_0$ and let $\varepsilon > 0$. Then there exists $x \in R^\infty(T)$ such that $\|x - y\| < \varepsilon$ and*

$$\|T^{-j}x\| \geq a_j$$

for all $j \geq 0$.

An analogous result can be formulated also for strongly continuous semigroups of operators.

Corollary 2.8. *Let $T(t)_{t\geq 0}$ be a strongly continuous semigroup of operators acting on X. Suppose that $T(1)$ is injective and $\overline{R(T(1))} = X \neq R(T(1))$ (and hence $T(t)$ is injective, non-surjective with dense range for each $t > 0$). Let $f : [0,\infty) \to [0,\infty)$ be a continuous function. Then there exists $x \in \bigcap_{t\geq 0} R(T(t))$ such that $\|T(t)^{-1}x\| > f(t)$ for all $t \geq 0$.*

Proof. Let $K = \max\{\|T(t)\| : 0 \leq t \leq 1\}$. By Corollary 2.5, there exists $x \in \bigcap_{t\geq 0} R(T(t)) = \bigcap_{n\in\mathbb{N}} R(T(n))$ such that

$$\|T(n)^{-1}x\| \geq K \max\{f(t) : n \leq t \leq n+1\}$$

for all integers $n \geq 0$.

For $n \leq t \leq n+1$ we have $T(t-n)T(t)^{-1}x = T(n)^{-1}x$. So

$$\|T(t)^{-1}x\| \geq K^{-1}\|T(n)^{-1}x\| \geq \max\{f(t) : n \leq t \leq n+1\} \geq f(t).$$

□

3. Backward orbits and hypercyclicity

Let $T \in B(X)$. A vector $x \in X$ is called hypercyclic for T if its (forward) orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. An operator $T \in B(X)$ is called hypercyclic if there exists a vector that is hypercyclic for T. It is well known that any hypercyclic operator has a dense residual set of hypercyclic vectors.

The following classical result gives a characterization of hypercyclic operators, see e.g. [1], p. 2.

Theorem 3.1. *(Birkhoff) Let X be a separable Banach space and $T \in B(X)$. The following statements are equivalent:*

(i) *T is hypercyclic;*
(ii) *for each pair of non-empty open subsets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.*

An easy consequence of the Birkhoff theorem is that an invertible operator $T \in B(X)$ is hypercyclic if and only if its inverse $T^{-1} \in B(X)$ is hypercyclic, see [1], p. 3.

It is interesting to note that this equivalence remains true even if T is only injective (not necessarily invertible). As the Birkhoff theorem, the next result is true in a more general setting, see [6]. We include the proof for the sake of convenience.

Theorem 3.2. *Let $T \in B(X)$ be an injective operator. The following conditions are equivalent:*

(i) *T is hypercyclic;*
(ii) *the set of all vectors $x \in R^\infty(T)$ with the property that $\{T^{-j}x : j = 0, 1, \dots\}^- = X$ is dense in X.*

Proof. Note that each of the conditions implies that X is separable and T has dense range. So $R^\infty(T)$ is dense.

(ii)⇒(i):

Let $U, V \subset X$ be non-empty open subsets. By (ii), there exists $x \in U \cap R^\infty(T)$ such that $\{T^{-j}x : j = 0, 1, \dots\}^- = X$. In particular, there exists $k \geq 0$ such that $T^{-k}x \in V$. So $x \in T^kV$ and $T^kV \cap U \neq \emptyset$. By the Birkhoff theorem, T is hypercyclic.

(i)⇒(ii):

Let $V \subset X$ be a nonempty open subset. We show that there exists a vector $x \in V$ whose backward orbit $\{T^{-n}x : n \in \mathbb{N}\}$ is dense in X.

Let $(U_n)_{n\in\mathbb{N}}$ be a countable base of open sets in X.

By the Birkhoff theorem, there exist $u \in U_1$ and $n_1 \in \mathbb{N}$ such that $T^{n_1}u \in V$. There exists an open neighbourhood V_1 of u such that $\operatorname{diam} V_1 \leq \frac{1}{2\max\{1,\|T\|^{n_1}\}}$, $\overline{V_1} \subset U_1$ and $T^{n_1}\overline{V_1} \subset V$.

Similarly, there exists an non-empty open set V_2 and $n_2 \in \mathbb{N}$ such that $\overline{V_2} \subset U_2$, $\operatorname{diam} V_2 \leq \frac{1}{4\max\{1,\|T\|^{n_2+n_1}\}}$ and $T^{n_2}\overline{V_2} \subset V_1$. Inductively, there exist non-empty open sets $V_3, V_4, \dots$ and positive integers $n_3, n_4, \dots$ such that $\overline{V_k} \subset U_k$, $\operatorname{diam} V_k \leq \frac{1}{2^k\max\{1,\|T\|^{n_k+\cdots+n_1}\}}$ and $T^{n_k}\overline{V_k} \subset V_{k-1}$ $(k = 2, 3, \dots)$.

Then

$$T^{n_1+n_2+\cdots+n_k}\overline{V_k} \subset T^{n_1+n_2+\cdots+n_{k-1}}\overline{V_{k-1}} \subset \cdots \subset T^{n_1}\overline{V_1} \subset V.$$

Moreover, $\operatorname{diam} T^{n_1+\cdots+n_k}\overline{V_k} \leq 2^{-k}$. Hence $\bigcap_{k\in\mathbb{N}} T^{n_1+n_2+\cdots+n_k}\overline{V_k} \neq \emptyset$. Let $x \in \bigcap_{k\in\mathbb{N}} T^{n_1+n_2+\cdots+n_k}\overline{V_k}$. Clearly $x \in R^\infty(T)$. We have $x \in V$ and

$$T^{-n_1-\cdots-n_k}x \in \overline{V_k} \subset U_k$$

for all $k \in \mathbb{N}$. Hence the backward orbit $\{T^{-n}x : n \in \mathbb{N}\}$ is dense in X.

□

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