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Abstract

In this paper we introduce a numerical scheme for fluid-structure interaction problems in two or three space dimensions: A flexible elastic plate is interacting with a viscous, compressible barotropic fluid. Hence the physical domain of definition (the domain of Eulerian coordinates) is changing in time. We introduce a fully discrete scheme that is stable, satisfies geometric conservation, mass conservation and the positivity of the density. We also prove that the scheme is consistent with the definition of continuous weak solutions.

Keywords: fluid-structure interaction, compressible Navier–Stokes, stability, consistency, finite elements, elastic plates, incremental time-stepping.

1 Introduction

In the recent decades, there is an increasing attendance of mathematicians on the subject of fluid–structure interaction (FSI) problems due to their numerous applications. This includes blood flow through a vessel, oil flows through an elastic pipe but also oscillations of suspension bridges, lifting of airplanes, bouncing of elastic balls or the rotation of wind turbines, see [2, 5, 10, 38] and the references therein.

We will consider the particular setting where the solid (or the structure) is a shell or a plate. This means that it is modeled as a thin object of one dimension less than the fluid. For related up-to-date *modeling and model reductions* on *plates* and *shells* see [15, 16, 39] and the references therein. The fluid will be considered to be governed by the compressible Navier–Stokes equation. We are interested in the development of *Galerkin schemes* which are connected to the setting of *weak solutions*. Most of the mathematical effort in this setting so far was devoted to incompressible fluids for *weak solutions* with a *fixed prescribed scalar direction of displacement of the shell*. Well posedness results commonly show that a weak solution exists *until a self-touching* of the solid is approached. For *incompressible Newtonian fluids* we name the following results [3, 9, 19, 20, 31, 32, 35, 42, 43, 44]. On the other hand, the theory for *compressible flows* is much less developed. Only recently the existence of weak solutions in the above setting was shown [4], see also [46].

The numerical results of fluid-structure interactions are rich and diverse. The numerical analysis for the incompressible flows is developed in accordance with the existence theory, see the kinematical splitting schemes developed in [9, 11, 12, 36], see also [10, 28, 34, 45] for more simulation results. Without a surprise, the numerical theory for compressible fluids interacting with shells or plates is quite sparse. We mention [1, 18] for the stability analysis with a given variable geometry and [27, 40] for some numerical simulations. It seems that a numerical strategy for compressible flows interacting with elastic structures stayed undeveloped due to the high nonlinearity of the problem originating from the fluid and its sensitive coupling to its geometry.

This paper aims to fill that gap and enrich the theory on fluid-structure interactions by introducing a (fully discrete) numerical approximation scheme which is in coherence with the known continuous existence theory.

In particular we study numerics for the interaction between a compressible barotropic fluid flow with an elastic shell in the time-space domain $I \times \Omega(t)$, where $\Omega = \Omega(t) \subset R^d$ ($d \in \{2, 3\}$, $t \in I = [0, T]$) is a time dependent domain defined by its unsteady boundary.

The boundary of Ω consists of a time dependent elastic shell $\Gamma_S(t)$ on the top surface of the fluid (whose projection in d^{th}-direction is Σ given below), and fixed solid walls $\Gamma_D = \partial\Omega \backslash \Gamma_S$ for the other parts of the boundary. Throughout the paper we reserve $\mathbf{r} = (x_1, \ldots, x_{d-1})$ as the coordinates for the plate displacement $\eta : \Sigma \to \mathbb{R}$, i.e. the distance of the shell above the horizontal plane $x_d = H$. We define $\mathbf{x} = (\mathbf{r}, x_d)$ as the Eulerian coordinates in the domain

$$\Omega(t) := \{(\mathbf{r}, x_d) \in \Sigma \times \mathbb{R} : 0 < x_d < H + \eta(\mathbf{r})\}.$$

We denote by $\widehat{\Omega} = \Omega^0 = \Sigma \times [0, H]$ the reference domain, with $\Sigma = [0, L_1] \times \cdots \times [0, L_{d-1}]$ being a $(d-1)$-dimensional time-independent domain. Accordingly we introduce the following the one-to-one invertible mapping

$$\mathcal{A} : \widehat{\Omega}(\widehat{\mathbf{x}}) \to \Omega(\mathbf{x}), \quad \mathbf{x} = \mathcal{A}(t, \widehat{\mathbf{x}}) = \mathcal{A}(t, \widehat{\mathbf{r}}, \widehat{x}_d) = \left(\widehat{\mathbf{r}}, \widehat{x}_d \frac{H + \eta(t, \mathbf{r})}{H}\right). \tag{1.1}$$

Here and hereafter, we distinguish the functions on the reference domain by the superscript "$\widehat{\ }$" except the ALE mapping. We denote by $\widehat{\nabla} \equiv \nabla_{\widehat{\mathbf{x}}}$ and $\widehat{\mathrm{div}} \equiv \mathrm{div}_{\widehat{\mathbf{x}}}$. Furhter, we denote $\mathbb{J}$ and $\mathcal{J}$ as the Jacobian of the mapping $\mathcal{A}$ and its determinant:

$$\mathbb{J} = \widehat{\nabla}\mathcal{A}, \quad \mathcal{J} = \det(\mathbb{J}).$$

We present Figure 1 for a two dimensional example of the domain and ALE mapping.

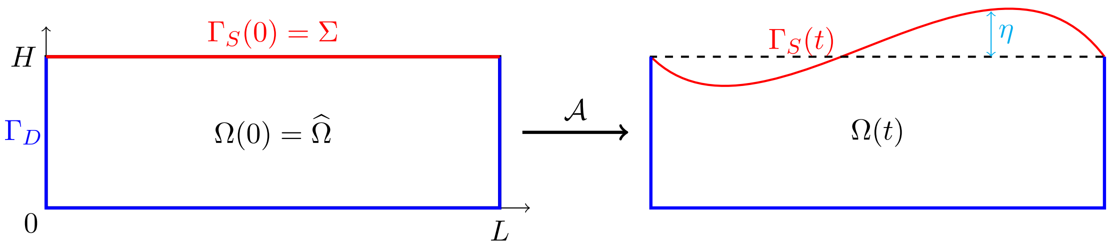

Figure 1: Time dependent domain and the ALE mapping

The evolution of the fluid flow is modeled by the Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \text{ in } I \times \Omega, \tag{1.2a}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}\boldsymbol{\tau} + \varrho \mathbf{f}, \text{ in } I \times \Omega. \tag{1.2b}$$

where $\varrho = \varrho(t, \mathbf{x})$ is the fluid density and $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the velocity field, $\boldsymbol{\tau}$ is the Cauchy stress

$$\boldsymbol{\tau} = \mathbf{S}(\nabla \mathbf{u}) - p(\varrho)\mathbf{I}, \quad \mathbf{S}(\nabla \mathbf{u}) = 2\mu \mathbf{D}(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\, \mathbf{I}, \quad \mathbf{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2}, \quad p(\varrho) = a\varrho^\gamma$$

for $a > 0$, $\gamma > 1$. The viscosity coefficients satisfy $\mu > 0$ and $\mu + \lambda \geq 0$. The motion of the shell is given by

$$\varepsilon_0 \varrho_S \partial_t z + K'(\eta) = g + \mathbf{e}_d \cdot \mathbf{F}, \quad z = \partial_t \eta, \quad \text{on} \quad I \times \Sigma, \tag{1.2c}$$

where z is the velocity of the shell, $\varrho_S > 0$ is the density of the shell, $\mathbf{e}_d = (0, 0, 1)^T$ for $d = 3$ ($\mathbf{e}_d = (0, 1)^T$ for $d = 2$), $g = g(t, \mathbf{r})$ is a given function, and $\mathbf{F}$ is given by

$$\mathbf{F} = -(\boldsymbol{\tau} \cdot \mathbf{n}) \circ \mathcal{A}\, \mathcal{J},$$

where $\boldsymbol{\tau} = \mathbf{S} - p\mathbf{I}$ is the Cauchy stress. For the sake of simplicity, we assume throughout the paper that

$$\varepsilon_0 \varrho_S = 1.$$

As elastic energy $K(\eta)$ we use the following linearized energy

$$K(\eta) = \frac{\alpha|\nabla^2\eta|^2}{2} + \frac{\beta|\nabla\eta|^2}{2}, \quad \alpha, \beta > 0,$$

which leads to the following L^2-Gradient:

$$K'(\eta) = \alpha\Delta^2\eta - \beta\Delta\eta.$$

We refer to Ciarlet and Roquefort [15] and references therein for the details of the model and also other choices of $K(\eta)$. To close the system we propose the following boundary conditions

$$\mathbf{u}|_{\Gamma_D} = 0, \tag{1.2d}$$

$$\eta|_{\partial\Sigma} = 0, \quad \nabla\eta|_{\partial\Sigma} = 0, \tag{1.2e}$$

and initial data

$$\varrho(0) = \varrho_0, \quad (\varrho\mathbf{u})(0) = \mathbf{q}_0 \text{ in } \Omega(0), \tag{1.2f}$$

$$\eta(0,\cdot) = \eta_0, \quad z(0,\cdot) = z_0 \quad \text{in} \quad \Sigma, \tag{1.2g}$$

where $\eta_0(x_1) = 0$ and $z_0(x_1)$ is a given function. We also need a compatibility condition between the shell and the fluid

$$\mathbf{u}(t,\mathbf{x})|_{\Gamma_S} \circ \mathcal{A} = z(t,\mathbf{r})\mathbf{e}_d. \tag{1.2h}$$

The purpose of the present paper is to introduce a fully discrete numerical scheme that is equipped with suitable physical and mathematical properties. By that we mean that it satisfies in particular:

(a) A fully discretized weak continuity equation that can be renormalized in the sense of DiPerna and Lions, such that the error for convex renormalizations is positive.

(b) Mass conservation and positivity of the discrete density is preserved.

(c) A fully coupled momentum equation in the spirit of Definition 2.1 on the discrete level.

(d) A discrete energy inequality for the coupled system (analogous to the continuous energy inequality (2.1)).

(e) The scheme is consistent with the continuous weak solutions introduced in [4] (See also Definition 2.1). This means in particular, that if the discrete deformation, density and velocity converge (strongly) to some limit triple, this limit triple is indeed a weak solution of the continuous problem.

(f) The scheme exists for a minimal time-interval. I.e. for every $\delta_0 \in (0, H/2)$ there is a minimal time T_0, such that a-priori $\inf_{[0,T_0]} \eta(t,\mathbf{r}) \geq \delta_0 - H$.

The existence of weak solutions for compressible viscous barotropic fluids interacting with an elastic plate is only recent [4]. It follows the seminal existence proof for weak solutions of the compressible Navier–Stokes equations [41, 26]. Note that the existence approach introduced in [4] can not be adapted to numerical approximations in a straight forward manner since it uses fixed point theorems and regularization operators on the continuous level. Indeed, the introduction of a numerical scheme that satisfies all conditions above turns out to be rather sophisticated. In particular, in order to capture the material time-derivative at the interacting interface, we have to introduce a corrector flow field (the function $\mathbf{w}$, below) that depends (linearly) on the elastic deformation η which allows to approximate the material derivative (the Eulerian time derivative) of the elastic solid.

We will consider a fully coupled implicit Euler scheme with respect to the time derivative. The spatial discretization of the deformation is done by piecewise polynomials. All three quantities are prescribed w.r.t a fixed steady reference mesh. This provides a nonlinear system for which we can prove the existence of solutions every time–step via a homotopy argument (See Theorem 4.15).

The *critical highlight* of the present paper is given in Theorem 3.7 and Theorem 4.6 where it is shown that the introduced scheme satisfies a discrete version of the energy inequality. It turns out that for compressible fluids only a fully non-linear implicit scheme does satisfy an energy inequality (See Remark 3.2). This is in contrast to incompressible fluids, which can be linearized (see e.g. [9]). While the strategy to get energy stable schemes for the compressible barotropic Navier–Stokes system is quite standard if the fluid domain Ω is fixed, see e.g. [29, 33, 37], it becomes rather difficult when a time dependent domain is considered. We would like to mention here the stability results of [1, 18] where the moving domain is a given function. As far as we know, this is the first result on energy stable numerical solutions for the FSI problem with compressible fluids even in two space dimensions. The *technical highlight* is the consistency of solutions, see Theorem 3.12 and Theorem 4.13. This is due to the fact that in case of fluid-structure interaction the space of test function is *a part of the weak solution* (see Definition 2.1). For that one has to ensure that the space of test functions of the limit weak solution (that depends on the limit geometry) can indeed be approximated. For that reason, the consistency of solutions is sensitive to the regularity of solutions–hence the consistency is the only part of the paper where there are restrictions on the barotropic exponent γ. In the fully discrete case the restriction is that $\gamma > \frac{6}{5}$ in the semi-discrete setup there is no restriction on γ.

The **main result** of the present paper is the existence of numerical solutions which satisfy (a)–(e) stated above. We first introduce the troubles related to the variable Eulerian geometry by studying the case of a discrete in time, but continuous in space model for which we prove properties (a)–(e). In the second part of the paper we study the fully discrete case for which (a)–(e) can also be shown. Finally we prove the existence of the fully discrete solutions. For all the results in the present paper we have to assume that $\alpha > 0$.

For the better readability we state here where the respective results are shown:

(a) See Lemma 3.4 (semi-discrete) and Lemma 4.4 (fully-discrete) for the renormalized equation.

(b) See (3.3) for the conservation of mass, Lemma 3.5 for the non-negativity of the discrete density and Theorem 4.15 for the existence of numerical solutions and the strict positivity of density.

(c) See Definition 3.1 (semi-discrete) and Definition 4.2 (fully discrete) for the fully coupled momentum problem.

(d) See Theorem 3.7 (semi-discrete) and Theorem 4.6 (fully discrete) for the energy inequality.

(e) See Theorem 3.12 (semi-discrete) and Theorem 4.13 (fully discrete) for the consistency of the schemes.

(f) See Lemma 3.10 (semi-discrete) and Corollary 4.7 (fully discrete) for the minimal time interval of existence.

We wish to point out that the scheme is built in such a way that one may prove that any subsequence of a numerical approximation converges weakly to a continuous solution.[1] The convergence result for the very same scheme will be the content of an independent paper.

2 Preliminaries

In this section, we introduce the necessary notations, the time discretization and time difference operators.

Weak solutions

We begin by introducing the following concept of weak solutions developed in [4, 46] where the existence of weak solutions (until a self-contact of the boundary) under appropriate initial conditions was shown. Indeed, existence could be shown in the following *continuous* spaces

- The *deformation* is usually assumed to be in the following Bochener space[2] $\eta \in W^I$, where $W^I := L^2(0,T;W_0^{2,2}(\Sigma)) \cap W^{1,2}(0,T;L^2(\Sigma))$.
- The *density* $\varrho \in Q^I$, were $Q^I := L^\infty(0,T;L^\gamma(\Omega(t))$. This means that $\varrho(t) \in L^\gamma(\Omega(t))$ for almost every t and that the essential supremum over the respective norms is bounded.
- The *velocity* $\mathbf{u} \in V^I$, where

$$V^I := \{\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega(t)) \,:\, \mathbf{u}(\mathbf{r}, H + \eta(\mathbf{r})) = \partial_t \eta(\mathbf{r})\mathbf{e}_d \text{ for all } \mathbf{r} \in \Sigma \text{ and } \mathbf{u} \equiv 0 \text{ on } \Gamma_D\}$$

[1] Please observe, that the lower bound on γ is the very same as was requested in [4].

[2] Throughout the paper we make use of the standard notation of Bochner spaces, Sobolev spaces and Lebesgue spaces, see for instance [25] for more details.

Definition 2.1 (Weak solution).
A weak solution to (1.2)–(1.2g) is a triple $(\eta, \varrho, \mathbf{u}) \in W^I \times Q^I \times V^I$ that satisfies the following

$$\int_0^T \frac{d}{dt} \int_{\Omega_\tau} \varrho\varphi \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_{\Omega_\tau} (\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla\varphi) \,\mathrm{d}x \,\mathrm{d}t = 0$$

for all $\psi \in C^\infty\left(\bar{I} \times \mathbb{R}^d\right)$;

$$\begin{aligned}
&\int_0^T \frac{d}{dt} \int_{\Omega_\tau} \varrho\mathbf{u} \cdot \mathbf{\Psi} \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_{\Omega_\tau} (\varrho\mathbf{u} \cdot \partial_t\mathbf{\Psi} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla\mathbf{\Psi}) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_\tau} (\mathbf{S}(\nabla\mathbf{u}) : \nabla\mathbf{\Psi} - a\varrho^\gamma \mathrm{div}\mathbf{\Psi}) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \left(\frac{d}{dt} \int_\Sigma \partial_t\eta\psi \,d\mathbf{r} - \int_\Sigma \Big(\partial_t\eta\partial_t\psi + K'(\eta)\psi\Big) \,d\mathbf{r} \right) \mathrm{d}t \\
&= \int_0^T \int_{\Omega_\tau} \varrho\mathbf{f} \cdot \mathbf{\Psi} \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Sigma g\psi \,d\mathbf{r} \,\mathrm{d}t
\end{aligned}$$

for all $(\mathbf{\Psi}, \psi) \in C_0^\infty([0, T] \times \mathbb{R}^d) \times C_0^\infty(\Sigma)$ with $\mathbf{\Psi}(\mathbf{r}, H + \eta) = \psi(\mathbf{r})\mathbf{e}_d$ on Σ and $\mathbf{\Psi} \equiv 0$ on Γ_D. Moreover, the solution satisfies the energy estimates

$$\begin{aligned}
&\sup_{t\in[0,T]} \left(\int_{\Omega_\tau} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{H}(\varrho) \right) \mathrm{d}x + \int_\Sigma \left(z^2 + K(\eta)\right) d\mathbf{r} \right) + \int_0^T \int_{\Omega_\tau} \mathbf{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \int_{\Omega_\tau} \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \mathcal{H}(\varrho_0) \right) \mathrm{d}x + \int_\Sigma \left(|\eta_0|^2 + |z_0|^2 + K(\eta(0))\right) d\mathbf{r} + \int_0^T \left(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Sigma)} \right) \mathrm{d}t
\end{aligned} \tag{2.1}$$

where $\mathcal{H}(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma$ represents the internal energy of the fluid.

Time discretization

We divide the time interval by N_t subintervals and set $\tau = T/N_t$ as the size of the time step. For simplicity, we write $t^k = k\tau$ and $I^k = [t^k, t^{k+1})$ for all $k = 0, 1, \ldots, N_t$. Further, we denote v_τ^k as the approximation of v at the time t^k. Next, we start the time discretization with the piecewise constant in time approximations of the domain (mesh)

$$\Omega_\tau(t) = \widehat{\Omega} \text{ for } t < \tau; \quad \Omega_\tau(t) = \Omega_\tau^k \text{ for all } t \in I^k,\ k = 1, 2, \ldots, N_t.$$

Note that the deformation of the domain is related to the ALE mapping, that is also approximated as a piecewise constant in time function

$$\mathcal{A}_\tau(t) = \mathcal{A}_\tau^0 \text{ for } t < \tau; \quad \mathcal{A}_\tau(t) = \mathcal{A}_\tau^k \text{ for all } t \in I^k,\ k = 1, 2, \ldots, N_t.$$

Further, we continue the time discretization of all unknowns, including the test functions, by piecewise constant in time functions on the fixed reference domain $\widehat{\Omega}$

$$\widehat{v}_\tau(t, \widehat{\mathbf{x}}) = \widehat{v}_\tau^0(\widehat{\mathbf{x}}) \text{ for } t < \tau; \quad \widehat{v}_\tau(t, \cdot) = \widehat{v}_\tau^k(\widehat{\mathbf{x}}) \text{ for all } t \in [k\tau, (k+1)\tau), \quad k = 1, 2, \ldots, N_t, \quad \widehat{\mathbf{x}} \in \widehat{\Omega} \tag{2.2}$$

where $\widehat{v} \in \{\widehat{\varrho}, \widehat{\mathbf{u}}, \widehat{p}, \widehat{\eta}, \widehat{z}, \widehat{\mathbf{w}}, \widehat{\varphi}, \widehat{\mathbf{\Psi}}, \widehat{\psi}\}$. To recover the functions from the reference domain back to the current domain, we take

$$v_\tau^k = \widehat{v}_\tau^k \circ (\mathcal{A}_\tau^k)^{-1} \quad \text{and} \quad v_\tau = \widehat{v}_\tau \circ \mathcal{A}_\tau^{-1} \tag{2.3}$$

for $v \in \{\varrho, \mathbf{u}, p, \eta, z, \mathbf{w}, \varphi, \mathbf{\Psi}, \psi\}$. Finally, we define a projection operator

$$\Pi_t[v] = \sum_{k=0}^{N_t} \int_{I^k} \frac{1_{I^k}(t)}{\tau} v \,\mathrm{d}t,\ \forall\, k \in \{0, 1, \ldots, N_t\}, \text{ and } 1_{I^k}(t) = \begin{cases} 1 & \text{if } t \in I^k \\ 0 & \text{otherwise .} \end{cases} \tag{2.4}$$

where $1_{I^k}(t)$ is the characteristic function

$$1_{I^k}(t) = \begin{cases} 1 & \text{if } t \in I^k \\ 0 & \text{otherwise .} \end{cases} \tag{2.5}$$

ALE mapping

In consistent with (2.3), we define the deformation rate of the fluid domain at time t^k $(k = 1, 2, \dots, N_t)$ as

$$\widehat{\mathbf{w}}_\tau^k = \frac{\mathcal{A}_\tau^k - \mathcal{A}_\tau^{k-1}}{\tau} = \left(\mathbf{0}_{d-1}, \frac{\eta_\tau^k - \eta_\tau^{k-1}}{\tau}\frac{\widehat{x}_d}{H}\right)^T, \quad \mathbf{w}_\tau^k = \widehat{\mathbf{w}}_\tau^k \circ (\mathcal{A}_\tau^k)^{-1} = \left(\mathbf{0}_{d-1}, \frac{\eta_\tau^k - \eta_\tau^{k-1}}{\tau}\frac{x_d}{\eta_\tau^k + H}\right)^T, \tag{2.6}$$

where $\mathbf{0}_{d-1}$ is $(d-1)$-dimensional zero vector.

For convenience, we introduce $\mathbf{X}_i^j$ as the mapping from $\Omega_\tau(t^i)$ to $\Omega_\tau(t^j)$, i.e.,

$$\mathbf{X}_i^j: \quad \Omega_\tau(t^i) \mapsto \Omega_\tau(t^j), \qquad \mathbf{X}_i^j(\mathbf{x}^i) = \mathcal{A}_\tau^j \circ (\mathcal{A}_\tau^i)^{-1}(\mathbf{x}^i) \quad \text{for all } \mathbf{x}^i \in \Omega_\tau(t^i). \tag{2.7}$$

Recalling the definition of the ALE mapping (1.1), the Jacobian of $\mathbf{X}_i^j$ and its determinant read

$$\mathbb{J}_i^j = \frac{\partial \mathbf{X}_i^j(\mathbf{x}^i)}{\partial \mathbf{x}^i}, \quad \text{and} \quad \mathcal{J}_i^j = \det\left(\mathbb{J}_i^j\right) = \frac{\eta^j + H}{\eta^i + H}, \tag{2.8}$$

respectively. From the above notations it is easy to check

$$\tau \operatorname{div}\mathbf{w}_\tau^k = 1 - \mathcal{J}_k^{k-1}. \tag{2.9}$$

Further we observe, that if $\eta_\tau^k(\mathbf{r}) \in (\delta_0 - H, H_{\max} - H)$ for all $k \in \{1, ..., N_t\}$ and all $\mathbf{r} \in \Sigma$, then

$$0 < \frac{\delta_0}{H_{\max}} \le \mathcal{J}_i^j \le \frac{H_{\max}}{\delta_0}, \quad i, j \in \{1, ..., N_t\}. \tag{2.10}$$

In order to transfer between the current domain and the reference domain, we recall the chain-rule and properties of the Piola transformation from [13]

$$\mathrm{d}x = \mathcal{J}\,\mathrm{d}\widehat{x}, \quad \mathrm{d}S(x) = |\mathcal{J}\mathbb{J}^{-T}\widehat{\mathbf{n}}|\mathrm{d}S(\widehat{x}), \quad \mathbf{n} = \frac{\mathcal{J}\mathbb{J}^{-T}\widehat{\mathbf{n}}}{|\mathcal{J}\mathbb{J}^{-T}\widehat{\mathbf{n}}|}, \quad \mathcal{J}\operatorname{div}\mathbf{q} = \widehat{\operatorname{div}}\left(\mathcal{J}\mathbb{J}^{-T}\widehat{\mathbf{q}}\right), \quad \mathbb{J}^T\nabla_{\mathbf{x}} r = \widehat{\nabla}\widehat{r}, \tag{2.11}$$

for a scalar function r and a vector filed $\mathbf{q}$. Finally we denote for simplicity

$$\widehat{\operatorname{div}\mathbf{q}} := \operatorname{div}\mathbf{q} \circ \mathcal{A} = \frac{1}{\mathcal{J}}\widehat{\operatorname{div}}\left(\mathcal{J}\mathbb{J}^{-T}\widehat{\mathbf{q}}\right), \quad \widehat{\nabla r} := \nabla r \circ \mathcal{A} = \mathbb{J}^{-T}\widehat{\nabla}\widehat{r}. \tag{2.12}$$

Time difference operators

First, let us introduce the discrete derivative in time for the displacement of the shell. As the shell function $\eta(t, \mathbf{r})$ is defined on the time independent reference domain, we apply the standard backward Euler discretization for the family of functions $r^k : \Sigma \to \mathbb{R}$, $k \in \{0, ..., N_t\}$:

$$\delta_t r^k(\mathbf{r}) = \frac{r^k(\mathbf{r}) - r^{k-1}(\mathbf{r})}{\tau}, \; \delta_t^2 r^k(\mathbf{r}) = \delta_t(\delta_t r^k(\mathbf{r})) = \frac{\delta_t r^k(\mathbf{r}) - \delta_t r^{k-1}(\mathbf{r})}{\tau}. \tag{2.13}$$

Next, for the fluid part, it is necessary to use the material time derivative in order to discretize our scheme properly. Since the domain Ω_τ is changing in time discretely we divide the material derivative in the bulk part (inside the domain where the deformation of the elastic shell is of minor importance) and the boundary part, where we approximate it by z^k. We define in the following the discrete material derivative that reflects the change of the domain as

$$D_t^{\mathcal{A}} r_\tau^k = \frac{r_\tau^k - r_\tau^{k-1} \circ \mathbf{X}_k^{k-1}}{\tau}, \tag{2.14}$$

where $\mathbf{X}_k^{k-1} = \mathcal{A}_\tau^{k-1} \circ (\mathcal{A}_\tau^k)^{-1}$ is the mapping from $\Omega_\tau(t^k)$ to $\Omega_\tau(t^{k-1})$, see (2.7). In the following we deviate the material derivative in a fluid part and a shell part by the following heuristics. Let q be some quantity defined on the current domain and $\mathbf{u}$ be the fluid velocity on the same domain. We deviate

$$\lim_{\tau \to 0} \partial_t q(t + \tau, \mathbf{x} + \tau\mathbf{u}(t, \mathbf{x})) = \partial_t q(t, \mathbf{x}) + \mathbf{u} \cdot \nabla q(t, \mathbf{x}) = \partial_t q(t, \mathbf{x}) + \mathbf{w} \cdot \nabla q(t, \mathbf{x}) + (\mathbf{u} - \mathbf{w}) \cdot \nabla q(t, \mathbf{x}),$$

where $\mathbf{w}$ denotes the velocity of the moving domain, and $(\mathbf{u} - \mathbf{w})$ is the relative velocity of the fluid with respect to the moving domain. Note that the first two terms on the right hand side capture the material time-derivative. Indeed $D_t^{\mathcal{A}} r_\tau^k$ is approximating that part. We give an insight by the following heuristics at the time $t = t^k$:

$$
\begin{aligned}
D_t^{\mathcal{A}} r^k &= \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1}}{\tau} = \frac{r(t,\mathbf{x}) - r(t-\tau, \mathbf{X}_k^{k-1}(\mathbf{x}))}{\tau} \\
&= \frac{r(t,\mathbf{x}) - r(t-\tau,\mathbf{x})}{\tau} + \frac{r(t-\tau,\mathbf{x}) - r(t-\tau, \mathbf{X}_k^{k-1})}{\tau} \\
&= \frac{r(t,\mathbf{x}) - r(t-\tau,\mathbf{x})}{\tau} + \frac{r(t-\tau,\mathbf{x}) - r(t-\tau, \mathbf{X}_k^{k-1})}{\mathbf{x} - \mathbf{X}_k^{k-1}} \frac{\mathbf{x} - (\mathbf{x} - \mathbf{w}^k \tau)}{\tau} \\
&\sim \partial_t r + \mathbf{w} \cdot \nabla r
\end{aligned}
$$

Further, we observe

$$
\begin{aligned}
D_t^{\mathcal{A}} r^k &= \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1}}{\tau} = \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} + \frac{\mathcal{J}_k^{k-1} - 1}{\tau} r^{k-1} \circ \mathbf{X}_k^{k-1} \\
&= \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} - \mathrm{div} \mathbf{w}_\tau^k \, r^{k-1} \circ \mathbf{X}_k^{k-1},
\end{aligned}
$$

which, as can be seen below turns out to be the suitable deviation in order to get *a-priori* estimates. In addition, the above calculation motivates the use of the following non-standard time difference operator approximating the Eulerian time derivative

$$
D_t r^k = \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau}. \tag{2.15}
$$

We summarize the heuristics by the following discrete version of the Reynolds transport theorem.

Lemma 2.2 (Discrete Reynolds transport)**.**
For the time difference operator defined in (2.13) *and* (2.15)*, we have the following discrete analogy of the Reynolds transport theorem.*

$$
\delta_t \int_{\Omega_\tau^k} r^k \, \mathrm{d}x = \int_{\Omega_\tau^k} D_t r^k \, \mathrm{d}x = \int_{\Omega_\tau^k} \left(D_t^{\mathcal{A}} r^k + \mathrm{div} \mathbf{w}_\tau^k \, r^{k-1} \circ \mathbf{X}_k^{k-1} \right) \mathrm{d}x. \tag{2.16}
$$

Proof. From the definition of time difference operators and the determinant of the Jacobian given in (2.8), we easily get

$$
\begin{aligned}
\delta_t \int_{\Omega_\tau^k} r^k \, \mathrm{d}x &= \frac{1}{\tau} \left(\int_{\Omega_\tau^k} r^k \, \mathrm{d}x - \int_{\Omega_\tau^{k-1}} r^{k-1} \, \mathrm{d}x \right) \\
&= \frac{1}{\tau} \left(\int_{\Omega_\tau^k} r^k \, \mathrm{d}x - \int_{\Omega_\tau^k} r^{k-1} \circ \mathbf{X}_k^{k-1} \det \left(\frac{\partial \mathbf{x}^{k-1}}{\partial \mathbf{x}^k} \right) \mathrm{d}\mathbf{x}^k \right) = \int_{\Omega_\tau^k} \frac{r^k - r^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \, \mathrm{d}x \\
&= \int_{\Omega_\tau^k} D_t r^k \, \mathrm{d}x = \int_{\Omega_\tau^k} \left(D_t^{\mathcal{A}} r^k + \mathrm{div} \mathbf{w}_\tau^k \, r^{k-1} \circ \mathbf{X}_k^{k-1} \right) \mathrm{d}x.
\end{aligned}
$$

□

Note that the discrete Reynolds transport holds also for any $C \subset \Omega_\tau^k$. Thus we obtain the geometric conservation law by taking $r = 1$

$$
\frac{1}{\tau} \left(|C^k| - |C^{k-1}| \right) = \int_{C^k} \mathrm{div} \mathbf{w}_\tau^k \, \mathrm{d}x = \int_{\partial C^k} \mathbf{w}_\tau^k \cdot \mathbf{n} \, \mathrm{d}\mathbf{r}. \tag{2.17}
$$

3 Semi-discrete scheme

This section introduces the necessary tools and observations with respect to the time discretization. Due to the overwhelming technical notation in the fully discrete case we decided to include this semi-discrete section. We wish to emphasize that the main objective of this section is to explain the methodology. Hence we will assume within this section that the discrete in time but continuous solutions in space introduced below exist and are bounded in spaces in such a way that the discrete energy is well defined. We assume further (for this section) that the all needed test functions are admissible without further justification.

3.1 The scheme

The analysis is best understood when considering the scheme in the current domain (which is changing in each time step). However, for applications the scheme defined on the reference domain seems more handable to be implemented (see also the next section). Hence we first introduce the semi-discrete ALE scheme on the current domain followed by its equivalent formulation on the fixed reference geometry.For the spaces of existence we simply assume, that $W(\Sigma) \subset W_0^{2,2}(\Sigma)$, $Q(\Omega_\tau) \subset L^\gamma(\Omega_\tau)$ and $V(\Omega_\tau) \subset W^{1,s}(\Omega_\tau)$ for all $s < 2$.

Definition 3.1 (Semi-discrete scheme on the current domain)**.**
We seek the solution $(\eta_\tau^k, \varrho_\tau^k, \mathbf{u}_\tau^k) \in (W(\Sigma), Q(\Omega_\tau^k), V(\Omega_\tau^k))$ for all $k \in \{1, \dots, N_t\}$ and for all (admissible) $(\psi_\tau, \varphi_\tau, \mathbf{\Psi}_\tau) \in (W(\Sigma), Q(\Omega_\tau^k), V(\Omega_\tau^k))$ with $\mathbf{\Psi}_\tau|_{\Gamma_S} \circ \mathcal{A}_\tau = \psi_\tau \mathbf{e}_d$, such that the following hold:

$$\int_{\Omega_\tau^k} D_t \varrho_\tau^k \varphi_\tau \, dx + \int_{\Omega_\tau^k} \operatorname{div}(\varrho_\tau^k \mathbf{v}_\tau^k) \varphi_\tau \, dx = 0; \tag{3.1a}$$

$$\begin{aligned}\int_{\Omega_\tau^k} D_t \left(\varrho_\tau^k \mathbf{u}_\tau^k\right) \cdot \mathbf{\Psi}_\tau + \operatorname{div}(\varrho_\tau^k \mathbf{u}_\tau^k \otimes \mathbf{v}_\tau^k) \cdot \mathbf{\Psi}_\tau \, dx + \int_{\Omega_\tau^k} \mathbf{S}(\nabla \mathbf{u}_\tau^k) : \nabla \mathbf{\Psi}_\tau \, dx - \int_{\Omega_\tau^k} p(\varrho_\tau^k) \operatorname{div} \mathbf{\Psi}_\tau \, dx \\ + \int_\Sigma \delta_t z_\tau^k \psi_\tau \, d\mathbf{r} + \alpha \int_\Sigma \Delta \eta_\tau^k \Delta \psi_\tau \, d\mathbf{r} + \beta \int_\Sigma \nabla \eta_\tau^k \cdot \nabla \psi_\tau \, d\mathbf{r} = \int_{\Omega_\tau^k} \varrho_\tau^k \mathbf{f}_\tau^k \cdot \mathbf{\Psi}_\tau \, dx + \int_\Sigma g_\tau^k \psi_\tau \, d\mathbf{r};\end{aligned} \tag{3.1b}$$

where

$$z_\tau^k = \delta_t \eta_\tau^k, \quad \mathbf{v}_\tau^k = \mathbf{u}_\tau^k - \mathbf{w}_\tau^k.$$

and

$$g_\tau^k := \frac{1}{\tau} \int_{I^k} g \, dt \text{ and } \mathbf{f}_\tau^k := \frac{1}{\tau} \int_{I^k} \mathbf{f} \, dt$$

The scheme is supplemented with the initial data

$$\varrho_\tau^0 = \varrho_0, \quad \mathbf{u}_\tau^0 = \mathbf{u}_0, \quad \eta_\tau^0 = 0, \quad z_\tau^0 = 0,$$

and the boundary conditions

$$\mathbf{u}_\tau^k|_{\partial\Omega_\tau} = \mathbf{w}_\tau^k|_{\partial\Omega_\tau}.$$

Remark 3.2. Please observe that the above scheme is fully implicit and nonlinear. This means that both velocity $\mathbf{u}_\tau^k$ and density ϱ_τ^k are coupled to *their domain of definition* Ω_τ^k, which is determined by the unknown η_τ^k for each time step $k = 1, 2, \dots, N_T$. This is in striking contrast to the approaches from incompressible flows [9] where the velocity and pressure can be solved for each time step in the domain of the previous step. Here a common problem for compressible fluids reveals itself: Due to the fact that the *renormalized* density equation is necessary to derive an energy inequality out of the (discrete) scheme seems to enforce an implicit and non-linear scheme. Indeed, until today there is no time discrete scheme for compressible fluids that provides energy estimates which is not both *nonlinear* and *fully implicit* even for fixed domains. Unfortunately, in our investigations it turned out that also for fluid-structure interactions there is no space to allow explicit in time parts of the solutions. Nevertheless, we can solve the scheme (3.1) by rewriting its equivalent formulation on the reference domain $\widehat{\Omega}$ to avoid the problem of solving unknowns on an unknown domain, see scheme (3.2) given below in Definition 3.3. Though the scheme (3.2) is also a fully implicit and nonlinear scheme, we can solve the nonlinear system iteratively on the given reference domain. Furthermore, we will show that a full discretization in time and space actually possesses a solution, see Theorem 4.15. In addition, we can assure that for a positive time interval that the fully discrete scheme is well-defined.

Equivalently, we may consider the scheme in the reference domain in accordance with the notation (2.3).

Definition 3.3 (Semi-discrete scheme on reference domain)**.**
We seek the solution $\eta_\tau^k \in W$, such that $\min_{k,\mathbf{r}} \eta_\tau^k(\mathbf{r}) \geq \delta_0$ and $(\widehat{\varrho}_\tau^k, \widehat{\mathbf{u}}_\tau^k) \in Q(\widehat{\Omega}) \times V(\widehat{\Omega})$, such that $\widehat{\varrho}_\tau^k \widehat{\mathbf{u}}_\tau^k \in L^1(\widehat{\Omega})$ and that the following hold:

$$\int_{\widehat{\Omega}} \frac{\widehat{\varrho}_\tau^k \mathcal{J}_0^k - \widehat{\varrho}_\tau^{k-1} \mathcal{J}_0^{k-1}}{\tau} \widehat{\varphi}_\tau \, d\widehat{x} + \int_{\widehat{\Omega}} \widehat{\operatorname{div}(\varrho_\tau^k \mathbf{v}_\tau^k)} \, \widehat{\varphi}_\tau \mathcal{J}_0^k \, d\widehat{x} = 0; \tag{3.2a}$$

$$\int_{\widehat{\Omega}} \frac{\widehat{\varrho}_\tau^k \widehat{\mathbf{u}}_\tau^k \mathcal{J}_0^k - \widehat{\varrho}_\tau^{k-1} \widehat{\mathbf{u}}_\tau^{k-1} \mathcal{J}_0^{k-1}}{\tau} \cdot \widehat{\mathbf{\Psi}}_\tau \,\mathrm{d}\widehat{x} + \int_{\widehat{\Omega}} \widehat{\mathrm{div}(\varrho_\tau^k \mathbf{u}_\tau^k \otimes \mathbf{v}_\tau^k)} \cdot \widehat{\mathbf{\Psi}}_\tau \mathcal{J}_0^k \,\mathrm{d}\widehat{x}$$
$$+ 2\mu \int_{\widehat{\Omega}} \widehat{\mathbf{D}(\mathbf{u}_\tau^k)} : \widehat{\nabla \mathbf{\Psi}_\tau} \mathcal{J}_0^k \,\mathrm{d}\widehat{x} + \lambda \int_{\widehat{\Omega}} \widehat{\mathrm{div}\mathbf{u}_\tau^k} \widehat{\mathrm{div}\mathbf{\Psi}_\tau} \mathcal{J}_0^k \,\mathrm{d}\widehat{x} - \int_{\widehat{\Omega}} p(\widehat{\varrho}_\tau^k) \widehat{\mathrm{div}\mathbf{\Psi}_\tau} \mathcal{J}_0^k \,\mathrm{d}\widehat{x}$$
$$+ \int_\Sigma \delta_t z_\tau^k \psi_\tau \,d\mathbf{r} + \alpha \int_\Sigma \Delta \eta_\tau^k \Delta \psi_\tau \,d\mathbf{r} + \beta \int_\Sigma \nabla \eta_\tau^k \cdot \nabla \psi_\tau \,d\mathbf{r} = \int_{\widehat{\Omega}} \widehat{\varrho}_\tau^k \widehat{\mathbf{f}}_\tau^k \cdot \widehat{\mathbf{\Psi}}_\tau \mathcal{J}_0^k \,\mathrm{d}\widehat{x} + \int_\Sigma g_\tau^k \psi_\tau \,d\mathbf{r}; \quad (3.2b)$$

for all $k \in \{1, \ldots, N_t\}$ and for all admissible $(\widehat{\varphi}_\tau, \widehat{\mathbf{\Psi}}_\tau, \psi_\tau) \in (Q(\widehat{\Omega}) \times V(\widehat{\Omega}) \times W(\Sigma))$, with $\widehat{\mathbf{\Psi}}_\tau|_\Sigma = \psi_\tau \mathbf{e}_d$.

Here $\mathcal{J}_0^k$ and $\mathbb{J}_k^0$ are given in (2.8), and

$$z_\tau^k = \delta_t \eta_\tau^k, \quad \widehat{\mathbf{v}}_\tau^k = \widehat{\mathbf{u}}_\tau^k - \widehat{\mathbf{w}}_\tau^k.$$

The scheme is supplemented with the initial data

$$\varrho_\tau^0 = \varrho_0, \quad \mathbf{u}_\tau^0 = \mathbf{u}_0, \quad \eta_\tau^0 = 0, \quad z_\tau^0 = 0,$$

and the boundary conditions

$$\widehat{\mathbf{u}}_\tau^k|_{\partial\widehat{\Omega}} = \widehat{\mathbf{w}}_\tau^k|_{\partial\widehat{\Omega}}.$$

Note that the divergence and gradient operators are given in (2.11).

3.2 Stability

In this section, we aim to show some stability properties for the scheme (3.1) (or equivalently (3.2)). First, we remark that the scheme (3.1) preserves the total mass. Indeed, by setting $\varphi_\tau \equiv 1$ in (3.1a) and applying the discrete Reynolds transport Lemma 2.2, we derive $\delta_t \left(\int_{\Omega_\tau^k} \varrho_\tau^k \,\mathrm{d}x \right) = \int_{\Omega_\tau^k} D_t \varrho_\tau^k \,\mathrm{d}x = 0$ for all $k = 1, \ldots, N_t$, which implies

$$\int_{\Omega_\tau^k} \varrho_\tau^k \,\mathrm{d}x = \int_{\Omega_\tau^{k-1}} \varrho_\tau^{k-1} \,\mathrm{d}x = \cdots = \int_{\widehat{\Omega}} \varrho_\tau^0 \,\mathrm{d}\widehat{x} =: M_0, \quad \text{for all } k = 1, \ldots, N_t. \quad (3.3)$$

Next, we show the renormalization of the discrete density problem.

Lemma 3.4 (Renormalized continuity equation)**.**
Let $(\varrho_\tau, \mathbf{u}_\tau) \in Q \times V$ satisfy the discrete continuity equation (3.1a) with the boundary condition $\mathbf{u}_\tau|_{\partial\Omega_\tau} = \mathbf{w}_\tau|_{\partial\Omega_\tau}$. Then for any $B \in C^1(R)$ it holds

$$\frac{1}{\tau} \left(\int_{\Omega_\tau^k} B(\varrho_\tau^k) \,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} B(\varrho_\tau^{k-1}) \,\mathrm{d}x \right) + \int_{\Omega_\tau^k} \left(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \right) \mathrm{div}\mathbf{u}_\tau^k \,\mathrm{d}x + D_0 = 0.$$

where

$$D_0 = \frac{1}{\tau} \int_{\Omega_\tau^k} \mathcal{J}_k^{k-1} \left(B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) - B(\varrho_\tau^k) - B'(\varrho_\tau^k)(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} - \varrho_\tau^k) \right) \mathrm{d}x.$$

Moreover, $D_0 \geq 0$ if B is convex.

Proof. We set $\varphi_\tau = B'(\varrho_\tau^k)$ in the discrete density equation (3.1a) and obtain

$$\int_{\Omega_\tau^k} D_t \varrho_\tau^k B'(\varrho_\tau^k) \,\mathrm{d}x + \int_{\Omega_\tau^k} \mathrm{div} \left(\varrho_\tau^k \mathbf{v}_\tau^k \right) B'(\varrho_\tau^k) \,\mathrm{d}x = 0.$$

First, by applying the Taylor expansion, we know there exist $\xi \in \mathrm{co}\{\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}, \varrho_\tau^k\}$ that

$$
\begin{aligned}
&\int_{\Omega_\tau^k} D_t \varrho_\tau^k B'(\varrho_\tau^k)\,\mathrm{d}x = \int_{\Omega_\tau^k} \frac{\varrho_\tau^k - \varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} B'(\varrho_\tau^k)\,\mathrm{d}x \\
&= \frac{1}{\tau} \int_{\Omega_\tau^k} \Big(B(\varrho_\tau^k) - B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) \mathcal{J}_k^{k-1} + \Big(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \Big) \\
&\qquad\qquad + \mathcal{J}_k^{k-1} \Big(B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) - \varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} B'(\varrho_\tau^k) \Big) \Big)\,\mathrm{d}x \\
&= \frac{1}{\tau} \int_{\Omega_\tau^k} \Big(B(\varrho_\tau^k) - B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) \mathcal{J}_k^{k-1} \Big)\,\mathrm{d}x + \frac{1}{\tau} \int_{\Omega_\tau^k} \Big(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \Big) (1 - \mathcal{J}_k^{k-1})\,\mathrm{d}x \\
&\qquad + \frac{1}{\tau} \int_{\Omega_\tau^k} \mathcal{J}_k^{k-1} \Big(B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) - B(\varrho_\tau^k) - B'(\varrho_\tau^k) (\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} - \varrho_\tau^k) \Big)\,\mathrm{d}x \\
&= \frac{1}{\tau} \left(\int_{\Omega_\tau^k} B(\varrho_\tau^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} B(\varrho_\tau^{k-1})\,\mathrm{d}x \right) + \int_{\Omega_\tau^k} \Big(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \Big) \mathrm{div} \mathbf{w}_\tau^k\,\mathrm{d}x + D_0
\end{aligned}
\tag{3.4}
$$

where we have used relation between the Jacobian and the deformation rate of the domain given in (2.9). Next, by applying integration by parts twice, we reformulate the convective term as

$$
\begin{aligned}
&\int_{\Omega_\tau^k} \mathrm{div}(\varrho_\tau^k \mathbf{v}_\tau^k) B'(\varrho_\tau^k)\,\mathrm{d}x = -\int_{\Omega_\tau^k} \varrho_\tau^k \mathbf{v}_\tau^k \cdot \nabla B'(\varrho_\tau^k)\,\mathrm{d}x = -\int_{\Omega_\tau^k} \mathbf{v}_\tau^k \cdot \nabla \Big(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \Big)\,\mathrm{d}x \\
&= \int_{\Omega_\tau^k} \mathrm{div} \mathbf{v}_\tau^k \Big(\varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) \Big)\,\mathrm{d}x,
\end{aligned}
$$

where we have used the equality $\nabla\Big(\varrho B'(\varrho) - B\Big) = \varrho \nabla B'(\varrho)$. Consequently, summing up the above equations and seeing $\mathbf{v}_\tau = \mathbf{u}_\tau - \mathbf{w}_\tau$, we complete the proof of the identity. Now, if $B \in C^2(\mathbb{R})$ is convex, we use the fact that by Taylor expansion there exists a $\xi(\mathbf{x}) \in \mathrm{co}\{\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}(\mathbf{x}), \varrho_\tau^k(\mathbf{x})\}$ for all $\mathbf{x} \in \Omega_\tau^k$, such that

$$
D_0 = \int_{\Omega_\tau^k} \tau \mathcal{J}_k^{k-1} \frac{B''(\xi)}{2} \left| D_t^{\mathcal{A}} \varrho_\tau^k \right|^2 \,\mathrm{d}x \geq 0.
$$

The general convex case follows by approximation. □

With the renormalized continuity equation in hand, we are ready to show non-negativity of the discrete density and the internal energy balance.

Lemma 3.5 (Non-negativity of density)**.**
Under the assumption of Lemma 3.4 we have $\varrho_\tau^k \geq 0$ for all $k = 1, \ldots, N_t$ provided $\varrho_\tau^0 \geq 0$.

Proof. By setting $B(\varrho) = \max\{0, -\varrho\} \geq 0$ in Lemma 3.4 and assuming $\varrho_\tau^{k-1} \geq 0$, we observe

$$
B(\varrho_\tau^{k-1}) = 0, \quad \varrho_\tau^k B'(\varrho_\tau^k) - B(\varrho_\tau^k) = 0, \ \Big(B(\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}) - B(\varrho_\tau^k) - B'(\varrho_\tau^k) (\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} - \varrho_\tau^k) \Big) \geq 0.
$$

Thus we find

$$
\int_{\Omega_\tau^k} B(\varrho_\tau^k)\,\mathrm{d}x = 0.
$$

Realizing B is a non-negative function we know that $B(\varrho_\tau^k) = 0$ holds for all $\mathbf{x} \in \Omega_\tau^k$ which implies $\varrho_\tau^k \geq 0$. As $\varrho_\tau^0 \geq 0$ we finish the proof by mathematical induction. □

Further discussion on the strictly positivity of the discrete density will be shown for the fully discrete scheme in the next section.

Next, by setting $B = \mathcal{H}(\varrho)$ in Lemma 3.4 and realizing $p = \varrho \mathcal{H}'(\varrho) - \mathcal{H}$, we immediately derive the following relation on the internal energy.

Corollary 3.6 (Internal energy balance)**.**
Let $(\varrho_\tau, \mathbf{u}_\tau, \eta_\tau) \in Q \times V \times W$ be the solution of the discrete problem (3.1). Then there exists $\xi(\mathbf{x}) \in \mathrm{co}\{\varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1}(\mathbf{x}), \varrho_\tau^k(\mathbf{x})\}$ such that

$$\frac{1}{\tau}\left(\int_{\Omega_\tau^k} \mathcal{H}(\varrho_\tau^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} \mathcal{H}(\varrho_\tau^{k-1})\,\mathrm{d}x\right) + \int_{\Omega_\tau^k} p(\varrho_\tau^k)\mathrm{div}\mathbf{u}_\tau^k\,\mathrm{d}x + D_1 = 0, \tag{3.5}$$

where

$$\begin{aligned} D_1 &= \frac{1}{\tau}\int_{\Omega_\tau^k} \mathcal{J}_k^{k-1}\left(\mathcal{H}(\varrho_\tau^{k-1}\circ \mathbf{X}_k^{k-1}) - \mathcal{H}(\varrho_\tau^k) - \mathcal{H}'(\varrho_\tau^k)(\varrho_\tau^{k-1}\circ \mathbf{X}_k^{k-1} - \varrho_\tau^k)\right)\mathrm{d}x \\ &= \int_{\Omega_\tau^k} \tau \mathcal{J}_k^{k-1}\frac{\mathcal{H}''(\xi)}{2}\left|D_t^{\mathcal{A}}\varrho_\tau^k\right|^2\mathrm{d}x \geq 0. \end{aligned}$$

Finally, we proceed to show the energy stability of the scheme (3.1).

Theorem 3.7 (Energy estimates)**.**
Let $\left(\varrho_\tau^k, \mathbf{u}_\tau^k, \eta_\tau^k\right)_{k=1}^{N_T}$ be a family of numerical solutions obtained by scheme (3.1). Then the following energy estimate holds

$$\begin{aligned} &\delta_t\left(\int_{\Omega_\tau^k} E_f^k\,\mathrm{d}x + \int_\Sigma E_s^k\,d\mathbf{r}\right) + \frac{\tau}{2}\int_\Sigma\left(|\delta_t z_\tau^k|^2 + \alpha\left|\Delta z_\tau^k\right|^2 + \beta\left|\nabla z_\tau^k\right|^2\right)d\mathbf{r} \\ &+ \int_{\Omega_\tau^k}\left(2\mu|\mathbf{D}(\mathbf{u}_\tau^k)|^2 + \lambda|\mathrm{div}\mathbf{u}_\tau^k|^2\right)\mathrm{d}x + \int_{\Omega_\tau^k}\tau\mathcal{J}_k^{k-1}\frac{\mathcal{H}''(\xi)}{2}\left|D_t^{\mathcal{A}}\varrho_\tau^k\right|^2\mathrm{d}x + \int_{\Omega_\tau^k}\frac{\tau}{2}\varrho_\tau^{k-1}\circ\mathbf{X}_k^{k-1}\left|D_t^{\mathcal{A}}\mathbf{u}_\tau^k\right|^2\mathrm{d}x \\ &= \int_{\Omega_\tau^k}\varrho_\tau^k\mathbf{f}_\tau^k\cdot\mathbf{u}_\tau^k\,\mathrm{d}x + \int_\Sigma g_\tau^k z_\tau^k\,d\mathbf{r}. \end{aligned} \tag{3.6}$$

where

$$E_f^k = \frac{1}{2}\varrho_\tau^k\left|\mathbf{u}_\tau^k\right|^2 + \mathcal{H}(\varrho_\tau^k), \quad E_s^k = \frac{1}{2}(|z_\tau^k|^2 + \alpha|\Delta\eta_\tau^k|^2 + \beta|\nabla\eta_\tau^k|^2).$$

Proof. Setting $\varphi_\tau = -\frac{|\mathbf{u}_\tau^k|^2}{2}$ in (3.1a), and $(\mathbf{\Psi}_\tau, \psi_\tau) = (\mathbf{u}_\tau^k, z_\tau^k)$ in (3.1b), we have

$$\sum_{i=1}^{2} I_i = 0, \quad \text{and} \sum_{i=3}^{9} I_i = 0,$$

respectively, where

$$\begin{aligned} &I_1 = -\int_{\Omega_\tau^k} D_t\varrho_\tau^k\frac{|\mathbf{u}_\tau^k|^2}{2}\mathrm{d}x, \quad I_2 = -\int_{\Omega_\tau^k}\mathrm{div}\left(\varrho_\tau^k\mathbf{v}_\tau^k\right)\frac{|\mathbf{u}_\tau^k|^2}{2}\mathrm{d}x, \\ &I_3 = \int_{\Omega_\tau^k} D_t\left(\varrho_\tau^k\mathbf{u}_\tau^k\right)\cdot\mathbf{u}_\tau^k\,\mathrm{d}x, \quad I_4 = \int_{\Omega_\tau^k}\mathrm{div}\left(\varrho_\tau^k\mathbf{u}_\tau^k\mathbf{v}_\tau^k\right)\cdot\mathbf{u}_\tau^k\,\mathrm{d}x, \quad I_5 = -\int_{\Omega_\tau^k} p(\varrho_\tau^k)\mathrm{div}\mathbf{u}_\tau^k\,\mathrm{d}x, \\ &I_6 = 2\mu\int_{\Omega_\tau^k}|\mathbf{D}(\mathbf{u}_\tau^k)|^2\,\mathrm{d}x + \lambda\int_{\Omega_\tau^k}|\mathrm{div}\mathbf{u}_\tau^k|^2\,\mathrm{d}x, \quad I_7 = \int_{\Omega_\tau^k}\varrho_\tau^k\mathbf{f}_\tau^k\cdot\mathbf{u}_\tau^k\,\mathrm{d}x + \int_\Sigma g_\tau^k z_\tau^k\,d\mathbf{r}, \\ &I_8 = \int_\Sigma\frac{z_\tau^k - z_\tau^{k-1}}{\tau}z_\tau^k\,d\mathbf{r}, \quad I_9 = \alpha\int_\Sigma\Delta\eta_\tau^k\Delta z_\tau^k\,d\mathbf{r} + \beta\int_\Sigma\nabla\eta_\tau^k\cdot\nabla z_\tau^k\,d\mathbf{r}. \end{aligned}$$

Now we proceed with the summation of all the I_i terms for $i = 1, \ldots, 12$.

Term $I_1 + I_3 + I_8$. Applying the equality $a(a-b) = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2}$ we get

$$\begin{aligned} I_1 + I_3 + I_8 =& \frac{1}{\tau}\left(\int_{\Omega_\tau^k}\frac{1}{2}\varrho_\tau^k\left|\mathbf{u}_\tau^k\right|^2\mathrm{d}x - \int_{\Omega_\tau^{k-1}}\frac{1}{2}\varrho_\tau^{k-1}\left|\mathbf{u}_\tau^{k-1}\right|^2\mathrm{d}x\right) + \int_\Sigma\delta_t\left(\frac{|z_\tau^k|^2}{2}\right)d\mathbf{r}. \\ &+ \frac{\tau}{2}\int_{\Omega_\tau^k}\varrho_\tau^{k-1}\circ\mathbf{X}_k^{k-1}\left|D_t^{\mathcal{A}}\mathbf{u}_\tau^k\right|^2\mathrm{d}x + \frac{\tau}{2}\int_\Sigma|\delta_t z_\tau^k|^2\,d\mathbf{r}. \end{aligned}$$

Term $I_2 + I_4$. For the convective terms, we have

$$I_2 + I_4 = \int_{\Omega_\tau^k} \left(-\mathrm{div}(\varrho_\tau^k \mathbf{v}_\tau^k) \frac{\left|\mathbf{u}_\tau^k\right|^2}{2} + \mathrm{div}\left(\varrho_\tau^k \mathbf{u}_\tau^k \otimes \mathbf{v}_\tau^k \right) \cdot \mathbf{u}_\tau^k \right) \mathrm{d}x$$
$$= \int_{\Omega_\tau^k} \left(\varrho_\tau^k \mathbf{v}_\tau^k \cdot \nabla \frac{\left|\mathbf{u}_\tau^k\right|^2}{2} - \varrho_\tau^k \mathbf{u}_\tau^k \otimes \mathbf{v}_\tau^k : \nabla \mathbf{u}_\tau^k \right) \mathrm{d}x = 0.$$

Pressure term I_5. Recalling the discrete internal energy equation (3.5), we can rewrite the pressure term as

$$I_5 = -\int_{\Omega_\tau^k} p(\varrho_\tau^k) \mathrm{div} \mathbf{u}_\tau^k \,\mathrm{d}x = \frac{1}{\tau} \left(\int_{\Omega_\tau^k} \mathcal{H}(\varrho_\tau^k) \,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} \mathcal{H}(\varrho_\tau^{k-1}) \,\mathrm{d}x \right) + \int_{\Omega_\tau^k} \tau \mathcal{J}_k^{k-1} \frac{\mathcal{H}''(\xi)}{2} \left| D_t^{\mathcal{A}} \varrho_\tau^k \right|^2 \mathrm{d}x.$$

Term $I_6 + I_7$. These terms don't change.
Term I_9. Applying again $a(a-b) = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2}$, we deduce

$$I_9 = \int_\Sigma \frac{1}{2} \delta_t \left(\alpha |\Delta \eta_\tau^k|^2 + \beta |\nabla \eta_\tau^k|^2 \right) d\mathbf{r} + \int_\Sigma \left(\frac{\tau\alpha}{2} \left| \delta_t(\Delta \eta_\tau^k) \right|^2 + \frac{\tau\beta}{2} \left| \delta_t(\nabla \eta_\tau^k) \right|^2 \right) d\mathbf{r}$$
$$= \int_\Sigma \frac{1}{2} \delta_t \left(\alpha |\Delta \eta_\tau^k|^2 + \beta |\nabla \eta_\tau^k|^2 \right) d\mathbf{r} + \frac{\tau}{2} \int_\Sigma \left(\alpha \left| \Delta z_\tau^k \right|^2 + \beta \left| \nabla z_\tau^k \right|^2 \right) d\mathbf{r}$$

Collecting all the above terms, we finish the proof, i.e.,

$$\frac{1}{\tau} \left(\int_{\Omega_\tau^k} E_f^k \,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} E_f^{k-1} \,\mathrm{d}x \right) + \int_\Sigma \delta_t E_s^k \,d\mathbf{r} + \frac{\tau}{2} \int_\Sigma \left(|\delta_t z_\tau^k|^2 + \alpha \left| \Delta z_\tau^k \right|^2 + \beta \left| \nabla z_\tau^k \right|^2 \right) d\mathbf{r}$$
$$+ \int_{\Omega_\tau^k} \left(2\mu |\mathbf{D}(\mathbf{u}_\tau^k)|^2 + \lambda |\mathrm{div} \mathbf{u}_\tau^k|^2 \right) \mathrm{d}x + \int_{\Omega_\tau^k} \tau \mathcal{J}_k^{k-1} \frac{\mathcal{H}''(\xi)}{2} \left| D_t^{\mathcal{A}} \varrho_\tau^k \right|^2 \mathrm{d}x + \int_{\Omega_\tau^k} \frac{\tau}{2} \varrho_\tau^{k-1} \circ \mathbf{X}_k^{k-1} \left| D_t^{\mathcal{A}} \mathbf{u}_\tau^k \right|^2 \mathrm{d}x$$
$$= \int_{\Omega_\tau^k} \varrho_\tau^k \mathbf{f}_\tau^k \cdot \mathbf{u}_\tau^k \,\mathrm{d}x + \int_\Sigma g_\tau^k z_\tau^k \,d\mathbf{r}.$$

□

3.3 Some a-priori estimates

Let us recall that all unknowns including the domain and the test functions are piecewise constant in time, see (2.2). We define $\overline{\eta_\tau}(t, \mathbf{r})$ as the affine linear interpolant of η_τ meaning that $\overline{\eta_\tau} \in C^0(0, T; \Sigma)$, such that $\overline{\eta_\tau}(t^k, \mathbf{r}) = \eta_\tau^k(\mathbf{r})$ and $\partial_t \overline{\eta_\tau}(t, \mathbf{r}) = z_\tau^k(\mathbf{r})$ for $t \in I^k = [t^k, t^{k+1})$.

With a little abuse of notation we use $[0, T] \times \Omega_\tau(\cdot) = \bigcup_{k=1}^{N_t} (t^{k-1}, t^k] \times \Omega_\tau(t^k)$. Accordingly we define for $s \in [0, \infty)$, $q \in [1, \infty]$

$$\begin{aligned} \|f_\tau\|_{L^p(0,T;W^{s,q}(\Omega_\tau(\cdot))} &:= \left(\sum_{l=0}^{N_t} \tau \left\| f_\tau^l \right\|_{W^{s,q}(\Omega_\tau(t_l))}^p \right)^{\frac{1}{p}} \text{ for } p \in [1, \infty), \\ \|f_\tau\|_{L^\infty(0,T;W^{s,q}(\Omega_\tau(\cdot))} &:= \max_k \left\| f_\tau^l \right\|_{W^{s,q}(\Omega_\tau(t_l))}. \end{aligned} \tag{3.7}$$

Note that the expressions above bound the respective norms for both the piecewise constant functions in time as well as the piecewise affine linear functions in time.

Then the energy estimate Theorem 3.7 implies the following a-priori estimates (for the piecewise constant functions $\eta_\tau, \varrho_\tau, \mathbf{u}_\tau$ that are uniform in τ:

$$\begin{aligned} \|\varrho_\tau\|_{L^\infty(0,T;L^\gamma(\Omega_\tau(\cdot)))} \le c, \quad \left\|\varrho_\tau |\mathbf{u}_\tau|^2\right\|_{L^\infty(0,T;L^1(\Omega_\tau(\cdot)))} \le c, \\ \|\mathbf{u}_\tau\|_{L^2(0,T;L^6(\Omega_\tau(\cdot)))} \le c, \quad \|\nabla \mathbf{u}_\tau\|_{L^2(0,T;L^2(\Omega_\tau(\cdot)))} \le c, \quad \|\mathrm{div} \mathbf{u}_\tau\|_{L^2(0,T;L^2(\Omega_\tau(\cdot)))} \le c, \\ \|z_\tau\|_{L^\infty(0,T;L^2(\Sigma))} \le c, \quad \alpha \|\Delta \eta_\tau\|_{L^\infty(0,T;L^2(\Sigma))} \le c, \quad \beta \|\nabla \eta_\tau\|_{L^\infty(0,T;L^2(\Sigma))} \le c, \end{aligned} \tag{3.8}$$

where c depends on the external force $\mathbf{f}$ and g as well as the initial data. Furthermore, for all $1 \le \beta < \gamma$ such that $\frac{\beta}{\gamma} + \frac{\beta}{a} = 1$ for some $a \in (1, \infty)$ we find

$$\begin{aligned}\int_{\Omega_\tau^k} (|\varrho_\tau^k|(|\mathbf{u}_\tau^k| + 1))^\beta \, dx &= \int_{\Omega_\tau^k} |\varrho_\tau^k|^\beta (|\mathbf{u}_\tau^k| + 1)^\beta \, dx \\ &\le \left\| \varrho_\tau^k \right\|_{L^\gamma(\Omega_\tau^k)}^\beta \left\| |\mathbf{u}_\tau^k| + 1 \right\|_{L^a(\Omega_\tau^k)}^\beta .\end{aligned} \tag{3.9}$$

Please observe, that in case $d = 2$ for every $\gamma > 1$ one finds an a such that the right hand side will be bounded. In case $d = 3$ we are restricted to $\gamma > \frac{6}{5}$. Indeed, in this case we find by Jensen's inequality that for $\beta \in (1, 2]$

$$\sum_k \tau^{\frac{\beta}{2}} \int_{\Omega_\tau^k} (|\varrho_\tau^k|(|\mathbf{u}_\tau^k| + 1))^\beta \, dx \le \|\varrho_\tau\|_{L^\infty(0,T;L^\gamma(\Omega_\tau))}^\beta \left\| |\mathbf{u}_\tau^k| + 1 \right\|_{L^2(0,T;L^a(\Omega_\tau^k))}^\beta . \tag{3.10}$$

In order to prove the consistency of the above scheme we need some additional a-priori estimates.

Lemma 3.8. *For all $s \in [0, \frac{1}{2})$ and all $q \in [1, 4)$ there is a constant independent of τ such that*

$$\max_k \left\| \delta_t \eta_\tau^k \right\|_{L^2(\Sigma)} + \sum_{l=1}^{N_t} \tau \Big(\left\| \delta_t \eta_\tau^l \right\|_{W^{s,2}(\Sigma)}^2 + \left\| \delta_t \eta_\tau^l \right\|_{L^q(\Sigma)}^2 \Big) \le C,$$

and

$$\max_k \left\| \mathbf{w}_\tau^k \right\|_{L^2(\Omega_\tau(t^k))} + \sum_{l=1}^{N_t} \tau \Big(\left\| \mathbf{w}_\tau^l \right\|_{W^{s,2}(\Omega_\tau(t^l))}^2 + \left\| \mathbf{w}_\tau^l \right\|_{L^q(\Omega_\tau(t^l))}^2 \Big) \le C.$$

The constant C depends on the initial values and the bounds of the energy estimates alone. Moreover, for all $\theta \in [0, \frac{1}{3})$ there exists a constant C depending on the energy estimates and θ, such that

$$\max_k \left\| \eta_\tau^k(x) - \eta_\tau^{k-1}(x) \right\|_{L^\infty(\Sigma)} \le C \tau^\theta. \tag{3.11}$$

Proof. The energy estimate (Theorem 3.7) implies that $\left\| z_\tau^k \right\|_{L^2(\Sigma)}$ is uniformly bounded, which implies the same bound for $\partial_t \eta_\tau^k$ by the definition of z_τ. Moreover, since z_τ^k is the trace of $\mathbf{u}_\tau^k$ which is in $W^{1,2}(\Omega_\tau^k)$, we find by the trace-theorem (see the related estimate in [35, Corollary 2.9]) that

$$\sum_{l=1}^{N_t} \tau \left\| \delta_t \eta_\tau^l \right\|_{W^{s,2}(\Sigma)}^2 = \sum_{l=1}^{N_t} \tau \left\| z_\tau^k \right\|_{W^{s,2}(\Sigma)}^2 \le c \sum_{l=1}^{N_t} \tau \left\| \widehat{\mathbf{u}}_\tau^l \right\|_{W^{s+\frac{1}{2},2}(\widehat{\Omega}_\tau)}^2 \le c \sum_{l=1}^{N_t} \tau \left\| \mathbf{u}_\tau^l \right\|_{W^{1,2}(\Omega_\tau(t^l))}^2 \left\| \eta_\tau^l \right\|_{W^{2,2}(\Sigma)}^2$$

which can be bounded by the energy as well. Due to the fact that for any $q \in [1, 4)$ there is an $s < 2$ such that $W^{s,2} \hookrightarrow L^q$ the first inequality is completed. The second inequality follows by the very definition of $\mathbf{w}_\tau$. We extend $\eta_\tau^k, \eta_\tau^{k-1}$ by zero to $\mathbb{R}^2$ and take $r < \tau$. We use the notation of $\fint_{B_r(x)} \eta_\tau^k(y) \, dy = \frac{1}{r^2 \pi} \int_{B_r(x)} \eta_\tau^k(y) \, dy$ for the mean value integral. Then by Sobolev embedding, we find that $\eta_\tau^k \in C^\alpha(\Sigma)$, for all $\alpha < 1$, and hence that

$$\begin{aligned}|\eta_\tau^k(\mathbf{r}) - \eta_\tau^{k-1}(\mathbf{r})| &\le \left| \eta_\tau^k(\mathbf{r}) - \fint_{B_r(\mathbf{r})} \eta_\tau^k(\mathbf{y}) \, d\mathbf{y} \right| + \left| \fint_{B_r(\mathbf{r})} \eta_\tau^k(\mathbf{y}) - \eta_\tau^{k-1}(\mathbf{y}) \, d\mathbf{y} \right| + \left| \eta_\tau^{k-1}(\mathbf{r}) - \fint_{B_r(\mathbf{r})} \eta_\tau^{k-1}(\mathbf{y}) \, d\mathbf{y} \right| \\ &\le C r^\alpha + \left| \fint_{B_r(\mathbf{r})} \eta_\tau^k(\mathbf{y}) - \eta_\tau^{k-1}(\mathbf{y}) \, d\mathbf{y} \right| \le C r^\alpha + \tau \fint_{B_r(\mathbf{r})} |z_\tau^k(\mathbf{y})| \, d\mathbf{y} \le C r^\alpha + C \frac{\tau}{r^2}.\end{aligned}$$

Now the result follows by choosing $r = \tau^{\frac{1}{\alpha+2}}$. □

The regularity can be used to guarantee a minimal existence interval in time in which the shell is not touching the bottom of the fluid domain. At first we have the following observation which is a direct consequence of (3.11) above.

Corollary 3.9 (Inductive prolongation principle)**.** *Let $\tau^\theta \le \frac{\delta_0}{C}$ and $\delta_1 \ge 2\delta_0$. Then, if for some $k \in \{1, ..., N_t\}$ we find that $\inf_\sigma \eta_\tau^k(\mathbf{r}) \ge \delta_1 - H$, the η_τ^{k+1} satisfies $\inf_\sigma \eta_\tau^k(\mathbf{r}) \ge \delta_1 - \delta_0 - H$*

Moreover, (3.11) implies the following lemma:

Lemma 3.10. *For every $\delta_0 \in (0, H/2)$ there exists a T_0 just depending on the bounds of the energy inequality and H, such that*

$$\inf_{[0,T_0]} \eta(t,\mathbf{r}) \geq \delta_0 - H.$$

Proof. The result essentially follows from (3.11) from which we import the constants C and θ. Let $(T_0+\tau)^\theta \leq \frac{H-\delta_0}{C}$. Then we choose N such that $(N-1)\tau < T_0 \leq N\tau$, then for $k \in \{1, ..., N\}$ we find by the fact that $\eta_\tau^0 \equiv 0$, by (3.11) and by Jensen's inequality (for the concave functions using $\theta \in (0,1]$) that

$$\begin{aligned}\eta_\tau^k(\mathbf{r}) = \eta_\tau^k(\mathbf{r}) - \eta_\tau^0(\mathbf{r}) &\geq - \left\| \eta_\tau^k - \eta_\tau^0 \right\|_L^\infty (\Sigma) \\ &\geq -C \sum_{i=0}^{N} \tau^\theta \geq -(T_0+\tau)^\theta C \sum_{i=0}^{N} \left(\frac{\tau}{N\tau} \right)^\theta \geq \delta_0 - H\end{aligned}$$

for all $\mathbf{r} \in \Sigma$. □

3.4 Consistency

In this subsection, we aim to show the consistency of the scheme, meaning the if the numerical solution converges, then it satisfies the weak formulation (2.1) in the limit of $\tau \to 0$.

Usually, for that one takes a fixed test function and shows that the error produced by the discretization vanishes in the limit. Due to the fact that the *domain of the test function is a part of the solution* we have to approximate the test function space as well. We will do this in the following. We recall that $\overline{\eta}_\tau : [0,T] \times \Sigma \to [\delta_0 - H, \infty)$ is defined as the affine linear function in time which satisfies $\overline{\eta}_\tau(k\tau) = \eta_\tau^k$ for all k.

Now, the a-priori estimates imply the following lemma:

Lemma 3.11. *For any $\alpha \in [0, \frac{1}{3})$ and any of the above approximation sequences there exists a sub-sequence, $\{\overline{\eta}_{\tau_j}\}_{j\in\mathbb{N}} \in C^\alpha([0,T]\times\Sigma)$ and a $\eta \in\in C^\alpha([0,T]\times\Sigma)$, such that*

$$\overline{\eta}_{\tau_j} \to \eta \textit{ with } j \to \infty$$

uniformly in $C^\alpha([0,T]\times\Sigma)$.

Proof. Sobolev embedding implies that $\overline{\eta}_\tau(t)$ is bounded in $C^\alpha(\Sigma)$ for all $\alpha \in (0,1)$ uniformly in t, τ. Combining that with (3.11) implies that $\overline{\eta}_\tau$ is bounded in $C^\alpha([0,T]\times\Sigma)$ for all $\alpha \in (0, \frac{1}{3})$ uniformly in τ. Hence the theorem of Arzela Ascoli implies the result. □

In the following we may assume that $\overline{\eta}_\tau \to \eta$ uniformly (omitting the index j). Now, we take a test function on the limit domain:

$$\begin{aligned}&(\psi, \Psi) \in C^\infty([0,T], C_0^\infty(\Sigma)) \times C^\infty([0,T]\times\Omega_\tau(t);\mathbb{R}^d) \text{ such that } \Psi(t)|_{\Gamma_D} = 0, \\ &\Psi(t,\cdot,\eta(t,\cdot)+H) = \psi(t,\cdot)\mathbf{e}_d \text{ on } \Sigma \text{ and } \Psi(t) \equiv 0 \equiv \psi(t) \text{ for all } t \geq T.\end{aligned} \tag{3.12}$$

In order to satisfy the coupling condition we introduce an approximating sequence introducing the new approximation parameter $\epsilon \in (0,1)$

$$\begin{aligned}&\Psi_\epsilon : C^\infty([0,T]\times\mathbb{R}^3;\mathbb{R}^d) \text{ such that } \Psi_\epsilon(t,\mathbf{r},x_d) = \psi(t,\mathbf{r})\mathbf{e}_d \text{ for all } \mathbf{r} \in \Sigma \\ &\text{and } x_d \in (\eta(t,\cdot)+H-\epsilon, \eta(t,\cdot)+H+\epsilon).\end{aligned} \tag{3.13}$$

Such an approximation can be made precise by taking a cut-off function. We take $\phi_\epsilon \in C^\infty[0,\infty)$, such that $\phi_\epsilon^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ and $\phi(x) \equiv 1$ for all $x \in [\epsilon, \infty)$ and $0 \leq \phi_\epsilon' \leq \frac{2}{\epsilon}$. Moreover, we take for a function $b : C^\alpha([0,T])$ the notation $(b)_\epsilon$ as the standard convolution function. Recall that since $\eta \in C^\alpha$ uniformly we find in particular $(\eta)_\epsilon \leq \eta + \epsilon^\alpha$. Then (for a fixed t) we define for $\epsilon < \min\{\frac{1}{3}\delta_0, 1\}$

$$\Psi_\epsilon(t,\mathbf{r},x^d) := (1 - \phi_\epsilon(H + (\eta)_\epsilon(t) - 2\epsilon^\alpha + x_d))\Psi(t,\mathbf{r},x^d) + \phi_\epsilon(H + (\eta)_\epsilon(t) - 2\epsilon^\alpha + x_d)\psi(t,\mathbf{r}).$$

For $\tau < \frac{1}{2}\epsilon$ the function $(\psi(t), \Psi_\epsilon(t))$ is now an admissible test function for all $t \in [0,T]$. For the continuity equation we do not need the extra approximation parameter for the test function since no boundary values are requested.

Theorem 3.12 (Consistency of the semi-discrete scheme (3.1)).
Let $(\varrho_\tau, \mathbf{u}_\tau, \eta_\tau)$ be a solution of the scheme (3.1). Then for any $\varphi \in C^2([0,T]\times\mathbb{R}^d)$ we have

$$-\int_{\Omega_\tau} \varrho_\tau^0 \varphi^0 \,\mathrm{d}x - \int_0^T \int_{\Omega_\tau(t)} (\varrho_\tau \partial_t \varphi + \varrho_\tau \mathbf{u}_\tau \cdot \nabla\varphi)\,\mathrm{d}x = \mathcal{O}(\tau^\vartheta), \tag{3.14}$$

If moreover, $\overline{\eta}_\tau \to \eta$ in $C^\alpha([0,T]\times\Sigma)$ (for some $\alpha\in(0,1)$, than for all pairs $(\Psi,\psi)\in C_0^2(0,T\times\mathbb{R}^d)\times C_0^2([0,T]\times\Sigma)$ as constructed in (3.12) we have uniformly in ϵ that for all $\tau\le\frac{1}{2}\epsilon$ and Ψ_ϵ satisfying (3.13) that

$$\begin{aligned}
&-\int_{\Omega_\tau} \varrho_\tau^0 \mathbf{u}_\tau^0 \cdot \Psi_\epsilon^0 \,\mathrm{d}x - \int_0^T \int_{\Omega_\tau(t)} (\varrho_\tau \mathbf{u}_\tau \cdot \partial_t \Psi_\epsilon + \varrho_\tau \mathbf{u}_\tau \otimes \mathbf{u}_\tau : \nabla\Psi_\epsilon)\,\mathrm{d}x \\
&\quad + \mu \int_0^T \int_{\Omega_\tau(t)} \nabla \mathbf{u}_\tau^k : \nabla\Psi_\epsilon \,\mathrm{d}x + (\mu+\lambda)\int_0^T \int_{\Omega_\tau(t)} \mathrm{div}\mathbf{u}_\tau^k \mathrm{div}\Psi_\epsilon \,\mathrm{d}x - \int_0^T \int_{\Omega_\tau(t)} p(\varrho_\tau)\mathrm{div}\Psi_\epsilon \,\mathrm{d}x \\
&\qquad - \int_\Sigma \partial_t\eta(0)\psi^0 \,d\mathbf{r} - \int_0^T \int_\Sigma \delta_t \eta_\tau \partial_t \psi \,d\mathbf{r}\,\mathrm{d}t + \int_0^T \int_\Sigma K'(\eta_\tau)\psi \,d\mathbf{r}\,\mathrm{d}t \\
&\qquad\qquad = \int_0^T \int_\Sigma g_\tau \psi \,d\mathbf{r} + \int_0^T \int_{\Omega_\tau(t)} \mathbf{f}_\tau \cdot \mathbf{\Psi}_\tau \,\mathrm{d}x + \mathcal{O}(\tau^\vartheta),
\end{aligned} \tag{3.15}$$

for some $\vartheta > 0$.

Proof. To prove the consistency, we must test the discrete problem (3.1) with piecewise constant in time test functions. Therefore, we have to apply the piecewise constant projection operator Π_t introduced in (2.4) to the smooth test functions φ, Ψ and ψ. Note that for any $\phi_\tau = \Pi_t[\phi]$, $\phi\in\{\varphi,\Psi_\epsilon,\psi\}$ and for any piecewise constant in time function r_τ it holds

$$\int_0^T r_\tau \phi_\tau \,\mathrm{d}t = \sum_{k=0}^{N_t-1} \int_{I^k} r_\tau \Pi_t[\phi]\,\mathrm{d}t = \sum_{k=0}^{N_t-1} \int_{I^k} r_\tau \phi \,\mathrm{d}t = \int_0^T r_\tau \phi \,\mathrm{d}t.$$

Thanks to this equality, hereafter, we will directly use smooth (in time) test functions to show the consistency of our numerical scheme.

The construction of the Ψ_ϵ is such that we may multiply (3.1b) with the couple (Ψ_ϵ, ψ). As we are dealing with continuous in space functional spaces, we only need to treat the consistency error of the time derivative terms.

First, for the time derivative term of the shell displacement, we have

$$\begin{aligned}
&\int_0^T \int_\Sigma \delta_t z_\tau \psi_\tau \,d\mathbf{r}\,\mathrm{d}t = \int_0^T \int_\Sigma \frac{z_\tau(t)-z_\tau(t-\tau)}{\tau}\psi(t)\,d\mathbf{r}\,\mathrm{d}t \\
&= \frac{1}{\tau}\int_0^T \int_\Sigma z_\tau(t)\psi(t)\,d\mathbf{r}\,\mathrm{d}t - \frac{1}{\tau}\int_{-\tau}^{T-\tau} \int_\Sigma z_\tau(t)\psi(t+\tau)\,d\mathbf{r}\,\mathrm{d}t \\
&= \int_0^T \int_\Sigma z_\tau(t)\frac{\psi(t)-\psi(t+\tau)}{\tau}\,d\mathbf{r}\,\mathrm{d}t + \frac{1}{\tau}\int_T^{T+\tau} \int_\Sigma z_\tau(t)\underbrace{\psi(t+\tau)}_{=0}\,d\mathbf{r}\,\mathrm{d}t - \frac{1}{\tau}\int_{-\tau}^0 \int_\Sigma z_\tau(t)\psi(t+\tau)\,d\mathbf{r}\,\mathrm{d}t \\
&= -\int_0^T \int_\Sigma z_\tau\big(\partial_t\psi + \frac{\tau}{2}\partial_t^2\psi|_{t^*}\big)\,d\mathbf{r}\,\mathrm{d}t - \int_\Sigma \psi^0 z_\tau^0 \,d\mathbf{r} \\
&= \int_0^T \int_\Sigma z_\tau \partial_t\psi \,d\mathbf{r}\,\mathrm{d}t - \int_\Sigma \psi^0 \partial_t\eta(0)\,d\mathbf{r} + c(\|\psi\|_{C^2}, \|z_\tau\|_{L^\infty(0,T;L^2(\Sigma))})\tau,
\end{aligned} \tag{3.16}$$

where $t^* \in (t, t+\tau)$ comes from Taylor's expansion in the last second equality. In the following we use r_τ as a substitute for either ϱ_τ or $\varrho_\tau\mathbf{u}_\tau$. We begin by the observation, that

$$\begin{aligned}
\int_{t_k}^{t^{k+1}} \int_{\Omega_\tau^k} r_\tau^{k-1}\circ \mathbf{X}_{k-1}^k \mathcal{J}_{k-1}^k \Psi_\epsilon(t)\,dx\,dt &= \int_{t_{k-1}}^{t^k} \int_{\Omega_\tau^k} r_\tau^{k-1}\circ \mathbf{X}_{k-1}^k \mathcal{J}_{k-1}^k \Psi_\epsilon(t+\tau)\,dx\,dt \\
&= \int_{t_{k-1}}^{t^k} \int_{\Omega_\tau^{k-1}} r_\tau^{k-1}\Psi_\epsilon\Big(t+\tau, \mathbf{r}, x_d \frac{\eta^k + H}{\eta^{k-1}+H}\Big)\,d\mathbf{r}\,dx_d,\ dt
\end{aligned}$$

We find

$$\int_0^T \int_{\Omega_\tau(t)} D_t r_\tau \Psi_\epsilon \, \mathrm{d}x \, \mathrm{d}t = \sum_{k=1}^{N_t} \int_{t_k}^{t^{k+1}} \int_{\Omega_\tau^k} \frac{r_\tau^k - r_\tau^{k-1} \circ \mathbf{X}_{k-1}^k \mathcal{J}_{k-1}^k}{\tau} \Psi_\epsilon(t) \, dx \, dt$$

$$= \sum_{k=2}^{N_t} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} r_\tau^k \frac{\Psi_\epsilon(t, \mathbf{r}, x_d) - \Psi_\epsilon\Big(t+\tau, \mathbf{r}, x_d \frac{\eta^{k+1}+H}{\eta^k+H}\Big)}{\tau} \, dx \, dt - \frac{1}{\tau} \int_0^\tau \int_{\widehat{\Omega}} r_\tau^0 \Psi_\epsilon(t) \, dx \, dt = I_1 + I_2.$$

Next observe that

$$\Psi_\epsilon\Big(t, \mathbf{r}, x_d \frac{\eta^{k+1}+H}{\eta^k+H}\Big) - \Psi_\epsilon\Big(t, \mathbf{r}, x_d\Big) = \int_0^1 \partial_d \Psi_\epsilon\Big(t, \theta x_d \frac{\eta^{k+1}+H}{\eta^k+H} + (1-\theta)x_d\Big) \, d\theta x_d \frac{\eta^{k+1}-\eta^k}{H+\eta^k}$$

$$= -\tau \mathbf{w}_\tau^k \cdot \nabla\Psi_\epsilon(t, \mathbf{r}, x_d) + \int_0^1 \partial_d \Big(\Psi_\epsilon\Big(t, \theta x_d \frac{\eta^{k+1}+H}{\eta^k+H} + (1-\theta)x_d\Big) - \Psi_\epsilon(t, \mathbf{r}, x_d)\Big) \, d\theta x_d \frac{\eta^{k+1}-\eta^k}{H+\eta^k}$$

$$=: -\tau \mathbf{w}_\tau^k \cdot \nabla\Psi_\epsilon(t, \mathbf{r}, x_d) + \mathcal{R}^k.$$

By Taylor expansion (and the bounds on η_τ) we find $|\mathcal{R}^k| \leq c \left\|\nabla^2 \Psi\right\|_\infty |\eta^{k+1} - \eta^k|^2$ which implies in particular that for α, such that $\alpha = \frac{2}{\beta'}$ (where β is defined via γ in (3.9)) and by Lemma 3.8 that

$$\frac{1}{\tau} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} |r_\tau^k| |\mathcal{R}^k| \, dx \leq c \left\|\nabla^2 \Psi\right\|_\infty \left\|\eta^{k+1} - \eta^k\right\|_\infty^{2-\alpha} \tau^\alpha \left\|z_\tau^{k+1}\right\|_{L^{\alpha\beta'}(\Sigma)}^\alpha \left\|r_\tau^k\right\|_{L^\beta(\Omega_\tau^k)} \leq C\tau^{\alpha+(2-\alpha)\theta} \left\|r_\tau^k\right\|_{L^\beta(\Omega_\tau^k)}.$$

Which implies by (3.10) and Lemma 3.8 that for $\frac{2}{\beta'} + \frac{2-\frac{2}{\beta'}}{3} - \frac{\beta}{2} > \vartheta > 0$ we have

$$\sum_{k=1}^{N_t} \frac{1}{\tau} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} |r_\tau^k| |\mathcal{R}^k| \, dx = O(\tau^\vartheta).$$

Actually

$$\frac{2 - \frac{2}{\beta'}}{3} - \frac{\beta}{2} > 0 \iff \alpha + \frac{2-\alpha}{3} - \frac{1}{2-\alpha} > 0$$

which is true for all $\alpha < 3/2$. Hence for all $\gamma > \frac{6}{5}$ there is a $\vartheta > 0$. The maximum is achieved for $\alpha = 2 - \sqrt{3/2}$, then $\vartheta = 2 - 2\sqrt{2/3}$ and $\beta = \sqrt{6}$ which is admissible for $\gamma > \frac{6}{5}(1+\sqrt{6})$ in 3d and all $\gamma > 1$ in 2d.

Now we calculate using Taylor's expansion (using the uniform bounds on $\left\|\partial_t^2 \Psi_\epsilon\right\|_\infty \tau \left\|r_\tau\right\|_{L^1(0,T;L^1(\Omega_\tau))}$) we find for a suitable $\theta > 0$ that

$$I_1 = \sum_{k=2}^{N_t} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} r_\tau^k \frac{\Psi_\epsilon(t, \mathbf{r}, x_d) - \Psi_\epsilon\Big(t+\tau, \mathbf{r}, x_d\Big)}{\tau} + \int_{\Omega_\tau^k} r_\tau^k \frac{\Psi_\epsilon(t+\tau, \mathbf{r}, x_d) - \Psi_\epsilon\Big(t+\tau, \mathbf{r}, x_d \frac{\eta^{k+1}+H}{\eta^k+H}\Big)}{\tau} \, dx \, dt$$

$$= -\sum_{k=2}^{N_t} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} r_\tau^k \partial_t \Psi_\epsilon \, dx \, dt + O(\tau) - \sum_{k=2}^{N_t} \int_{t^k}^{t^{k+1}} \int_{\Omega_\tau^k} r_\tau^k \mathbf{w}_\tau^k \cdot \nabla\Psi_\epsilon(t+\tau) \, dx \, dt + O(\tau^\theta).$$

Consequently we derive

$$\begin{aligned} \int_0^T \int_{\Omega_\tau(t)} D_t r_\tau \Psi_\epsilon \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_\tau(t=0)} r_\tau^0 \Psi^0 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\Omega_\tau(t)} r_\tau(t) \partial_t \Psi_\epsilon(t) \, \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \int_{\Omega_\tau(t)} r_\tau(t) \mathbf{w}_\tau(t) \cdot \nabla\Psi_\epsilon(t) \, \mathrm{d}x = \mathcal{O}(\tau^\vartheta), \ \vartheta > 0, \end{aligned} \tag{3.17}$$

for r_τ being ϱ_τ we may take $\Psi_\epsilon \equiv \varphi$ in case $r_\tau = \varrho_\tau \mathbf{u}_\tau$ we have to take the ϵ-approximation.

Finally, substituting (3.17) into the continuity method, (3.17) and (3.16) into the coupled momentum and structure method (3.1b), we finish the proof. □

Remark 3.13. In variable domain analysis and in particular in fluid structure interaction involving elastic solids it is unavoidable to approximate the space of test functions at some point. In our case we do this by introducing the parameter ϵ.

We wish to indicate what are the next steps in order to prove that a subsequence converges to a weak solution, which will be the content of a second paper (relaying on this work). The energy estimate allows to take weakly converging sub-sequences (in τ). In order to pass with $\tau \to 0$ one has to prove that the various non-linearities as the pressure and the convective terms do indeed decouple in the limit. This is a sophisticated analysis introduced in [4] and goes back to seminal works of Lions [41]. The last step is then to pass with $\epsilon \to 0$. This limit passage is how ever not as dramatic (essentially it uses Taylor expansion); but it depends sensitively on the regularity of $\partial_t \eta$ and in particular on the fact that $\gamma > \frac{12}{7}$. □

4 Fully-discrete scheme

In this section, we propose a fully discrete scheme for the FSI problem (1.2). For the time discretization, we take the method introduced in Section 3. Further, for the space discretization, we take a mixed finite volume-finite element method proposed by Karper [37] for the compressible Navier-Stoks part (1.2a)–(1.2b) and a standard finite element method for the shell part (1.2c). As in the last section we keep τ as the time discretization parameter. For the space discretization we introduce the value h which is assumed to be coupled to τ in a convenient manner[3]. In the following we will use $\varrho_{h,\tau}, \mathbf{u}_{h,\tau}, \mathbf{w}_{h,\tau}, \eta_{h,\tau}, z_{h,\tau}$ as functions which are discrete in space and piecewise constant in time. Similar notations with the same subscripts will be applied to all functions that will appear in this section. We shall write $a \stackrel{<}{\sim} b$ if $a \leq cb$ for some positive constant c (independent of h and τ), and also $a \approx b$ if $a \stackrel{<}{\sim} b$ and $b \stackrel{<}{\sim} a$.

4.1 Discretization

For the discretization in time, we follow the previous section and approximate all unknowns including the mesh and test functions by piecewise constant in time functions. For the space discretization, we start with the notations on the fixed reference domain.

Mesh for the fluid part. Let $\Omega^0 = \widehat{\Omega}$ (the reference domain) be a closed polygonal domain, and $\mathcal{T}_h^0$ be a triangulation of $\widehat{\Omega}$

$$\overline{\Omega^0} = \cup_{K \in \mathcal{T}_h^0} K.$$

The time evolution of the domain (or mesh) is described by the ALE mapping for $k \in \{0, .., N_t\}$ that

$$\Omega_{h,\tau}^k = \widehat{\Omega} \circ (\mathcal{A}_{h,\tau}^k)^{-1} \text{ and } \mathcal{T}_h^k = \mathcal{T}_h^0 \circ (\mathcal{A}_{h,\tau}^k)^{-1}$$

where the ALE mapping $\mathcal{A}_{h,\tau}^k$ will be given in (4.2) below, and

$$\Omega_{h,\tau}(t) = \widehat{\Omega} \circ \mathcal{A}_{h,\tau}^{-1}(t), \mathcal{T}_h^k = \mathcal{T}_h^0 \circ (\mathcal{A}_{h,\tau}^k)^{-1} \text{ and } \mathcal{T}_{h,\tau}(t) = \mathcal{T}_h^0 \circ \mathcal{A}_{h,\tau}^{-1}(t),$$

where the ALE mapping $\mathcal{A}_{h,\tau}$ is given in (4.2) below.

Further, we adopt the following notations and assumptions for the mesh of the fluid part.

- We denote by $|K|$ and $|\sigma|$ the Lebesgue measure of the element K and edge σ respectively. Further, we remark h_K as the diameter of K and $h = \max_{K \in \mathcal{T}_h^0} h_K$ as the size of the triangulation. The mesh is regular and quasi-uniform in the sense of [14], i.e. there exist positive real numbers θ_0 and c_0 independent of h such that

$$\theta_0 \leq \inf \left\{ \frac{\xi_K}{h_K}, K \in \mathcal{T}_h^0 \right\} \quad \text{and} \quad c_0 h \leq h_K,$$

  where ξ_K stands for the diameter of the largest ball included in K.

- The mesh is built by an extension of the $(d-1)$-dimensional bottom surface mesh in the d^{th} direction, i.e., the projection of any element in the d^{th} direction must coincide with an edge $\sigma \in \mathcal{E}_{\mathrm{E}}$ on the bottom surface. We give an example in two dimensions for illustrating such kind of mesh, see Figure 2. In particular $\mathcal{T}_h^k$ is assumed to be a conformal triangulation uniform in k, h, τ.

[3] For the consistency actually we will assume that $h \sim \tau$.

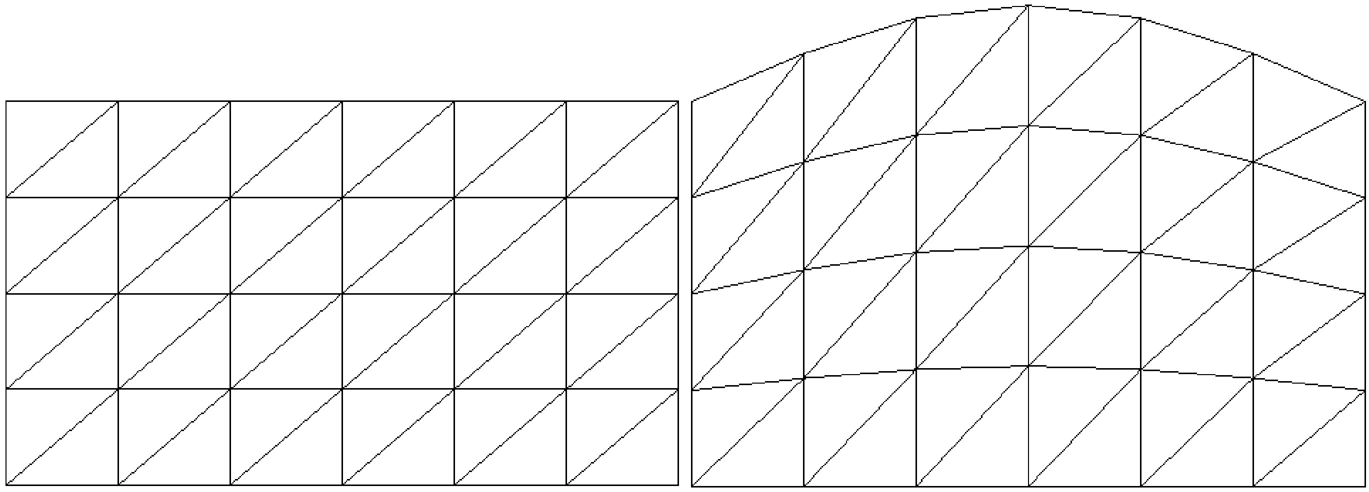

Figure 2: An example of mesh in two dimensions: left is the reference mesh and right is the deformed current mesh.

- By $\mathcal{E}(K)$ we denote the set of the edges σ of an element $K \in \mathcal{T}_{h,\tau}$. The set of all edges is denoted by $\mathcal{E}$. We distinguish exterior and interior edges:

$$\mathcal{E} = \mathcal{E}_{\rm I} \cup \mathcal{E}_{\rm E}, \ \mathcal{E}_{\rm E} = \left\{ \sigma \in \mathcal{E} \ \middle| \ \sigma \in \partial\Omega_{h,\tau} \right\}, \ \mathcal{E}_{\rm I} = \mathcal{E} \setminus \mathcal{E}_{\rm E}.$$

- We denote the set of all faces on the top boundary by $\mathcal{E}_{\rm S}$ $(\subset \mathcal{E}_{\rm E})$.
- For each $\sigma \in \mathcal{E}$ we denote $\mathbf{n}$ as the outer normal. Moreover, for any $\sigma = K|L$, we write $\mathbf{n}_{\sigma,K}$ as the normal vector that is oriented from K to L (so that $\mathbf{n}_{\sigma,K} = -\mathbf{n}_{\sigma,L}$), where $K|L$ denotes a common edge.

Mesh for the structure part. The mesh discretization of the time independent domain Σ coincides with that of the initial mesh of the top boundary $\Sigma_h = \mathcal{E}_{\rm S}^0$.

Remark 4.1. On one hand, the mesh is constructed by the extension of the mesh of the $(d-1)$-dimensional bottom boundary. On the other hand, we will define a linear function for the discrete ALE mapping $\mathcal{A}_{h,\tau}$, see (4.1) below. As a consequence, any triangle on the reference mesh is be kept to be a triangle in the current mesh, see a two dimensional mesh discretization in Figure 2.

Functional spaces Our scheme utilizes spaces of piecewise smooth functions, for which we define the traces

$$v^{\rm out} = \lim_{\delta\to 0} v(x + \delta\mathbf{n}), \ v^{\rm in} = \lim_{\delta\to 0} v(x - \delta\mathbf{n}), \quad x \in \sigma, \ \sigma \in \mathcal{E}.$$

Note that, $v_\sigma^{\rm out}$ is set according to the boundary condition for an exterior edges $\sigma \in \mathcal{E}_{\rm E}$. We also define

$$[\![v]\!]_\sigma = v^{\rm out} - v^{\rm in}, \ \overline{v}_\sigma = \frac{v^{\rm out} + v^{\rm in}}{2}, \ \langle v\rangle_\sigma = \frac{1}{|\sigma|}\int_\sigma v {\rm d}S(x).$$

Next, we introduce on the reference mesh the space of piecewise constant functions

$$\widehat{Q}_h = \left\{ \varphi \in L^1(\widehat{\Omega}) \ \middle| \ \varphi|_K = {\rm const} \in \mathbb{R} \text{ for any } K \in \mathcal{T}_h^0 \right\},$$

and the space of the linear Crouzeix–Raviart finite element

$$\widehat{V}_h = \left\{ v \in L^2(\widehat{\Omega}) \ \middle| \ v|_K = \text{affine function on } K \in \mathcal{T}_h^0, \ \int_\sigma [\![v]\!]_\sigma \, {\rm d}S(x) = 0 \text{ for } \sigma \in \mathcal{E}_{\rm I}^0 \right\},$$

and the space of piecewise quadratic functions on the shell Σ

$$\widehat{W}_h = \left\{ q \in C^1(\Sigma) \ \middle| q|_\sigma \in \mathcal{P}^2(\sigma) \text{ for } \sigma \in \Sigma_h \right\}.$$

In accordance with (2.3) we denote for all $t \in (0,T)$

$$v_{h,\tau} = \widehat{v}_{h,\tau} \circ \mathcal{A}_{h,\tau}^{-1}$$

for all unknowns including the test functions $v \in \{\varrho, \mathbf{u}, p, \eta, z, \mathbf{w}, \varphi, \boldsymbol{\Psi}, \psi\}$ and

$$Q_h(\Omega_{h,\tau}(t)) = \widehat{Q}_h \circ \mathcal{A}_{h,\tau}^{-1}(t), \quad V_h(\Omega_{h,\tau}(t)) = \widehat{V}_h \circ \mathcal{A}_{h,\tau}^{-1}(t), \quad W_h(\Sigma) \equiv \widehat{W}_h(\Sigma).$$

Thus it is clearly that

$$\varphi_{h,\tau} \in Q_h \Longleftrightarrow \widehat{\varphi}_{h,\tau} \in \widehat{Q}_h,$$
$$\boldsymbol{\Psi}_{h,\tau} \in V_h \Longleftrightarrow \widehat{\boldsymbol{\Psi}}_{h,\tau} \in \widehat{V}_h,$$

and

$$\psi_{h,\tau} \in W_h \Longleftrightarrow \widehat{\psi}_{h,\tau} \in \widehat{W}_h \text{ and } \psi_{h,\tau} \equiv \widehat{\psi}_{h,\tau}.$$

The associated projections of the functional spaces are

$$\Pi_{\mathcal{T}} : L^1(\Omega_{h,\tau}) \to Q_h, \ \Pi_{\mathcal{T}}[v] = \frac{1}{|K|}\int_K v \ \mathrm{d}x, \ K \in \mathcal{T}_h.$$

and (the uniquely defined interpolation operator [21])

$$\Pi_{\mathcal{E}} : W^{1,1}(\Omega_{h,\tau}) \to V_h, \text{such that } \int_\sigma \Pi_{\mathcal{E}}[v]\mathrm{d}S(x) = \int_\sigma v\mathrm{d}S(x) \text{ for any } \sigma \in \mathcal{E}.$$

We shall frequently write

$$\langle v \rangle_\sigma = \frac{1}{|\sigma|}\int_\sigma \Pi_{\mathcal{E}}[v]\mathrm{d}S(x).$$

Finally[4] we use the definition of $\widehat{\nabla}_h \widehat{\boldsymbol{\Psi}}_{h,\tau}$ as the discrete gradient for $\widehat{\boldsymbol{\Psi}}_{h,\tau} \in \widehat{V}_h$.

Coupling at the boundary and ALE mapping. Following the above notation we denote by $\Gamma_S^k = \Gamma_S(t^k)$ the piecewise quadratic boundary produced by η_h^k. Namely

$$\Gamma_S^k = \{(\mathbf{r}, x_d) \in [0,L]^{d-1} \times [0,\infty), \ \ x_d = H + \eta_{h,\tau}^k(\mathbf{r})\}.$$

As in the last section we define $\eta_{h,\tau}(t) = \eta_{h,\tau}^k$ on $[k\tau, (k+1)\tau)$. For the fluid domain we require the upper boundary of the mesh $\mathcal{T}_{h,\tau}$ to be of piecewise linear geometry. Therefore, we take a piecewise linear projection operator by keeping the values at the vertices of elements on Σ_h. Let ϕ_j be the standard piecewise linear basis function of the $(d-1)$-dimensional mesh Σ_h defined by the vertices $\{r_i\}_{i=1}^{N_p} \subset \Sigma_h$, where N_p is the total number of vertices on Σ_h. Then such a projection reads

$$\Pi_p : \mathcal{P}^2(\Sigma_h) \mapsto \mathcal{P}^1(\Sigma_h). \quad \Pi(\eta_{h,\tau}(r)) = \sum_{j=1}^{Np} \phi_j \eta_{h,\tau}(r_j).$$

We also illustrate such a projection in Figure 3 in the case of $d = 2$, where the red line is Γ_S while the dashed blue line is $\mathcal{E}_{\mathrm{S}}$ that determines the ALE mapping. Accordingly, the discrete ALE mapping (1.1) is redefined due to the space discretization proceedingly introduced

$$\mathbf{x} := \mathcal{A}_{h,\tau}(\widehat{\mathbf{x}}) \equiv \left(\widehat{\mathbf{r}}, \frac{\Pi_p(\eta_{h,\tau}) + H}{H}\widehat{x}_d\right), \text{ with } \mathbf{r} = \widehat{\mathbf{r}}, \ x_d = \frac{\Pi_p(\eta_{h,\tau}) + H}{H}\widehat{x}_d. \tag{4.1}$$

Moreover, we need to update the definitions in (2.6) and (2.8) due to the ALE mapping.

$$\begin{aligned}
&\widehat{\mathbf{w}}_{h,\tau}^k = \frac{\mathcal{A}_{h,\tau}^k - \mathcal{A}_{h,\tau}^{k-1}}{\tau} = \left(\mathbf{0}_{d-1}, \frac{\Pi_p(\eta_{h,\tau}^k - \eta_{h,\tau}^{k-1})}{\tau}\frac{\widehat{x}_d}{H}\right)^T, \quad \mathbf{w}_{h,\tau}^k = \left(\mathbf{0}_{d-1}, \frac{\Pi_p(\eta_{h,\tau}^k - \eta_{h,\tau}^{k-1})\ x_d}{\tau\left(\Pi_p(\eta_{h,\tau}^k) + H\right)}\right)^T, \\
&\mathcal{J}_i^j = \det\left(\frac{\partial \mathbf{X}_i^j(\mathbf{x}^i)}{\partial \mathbf{x}^i}\right) = \frac{\Pi_p(\eta_{h,\tau}^j) + H}{\Pi_p(\eta_{h,\tau}^i) + H}, \quad \text{and} \quad \tau \operatorname{div}\mathbf{w}_{h,\tau}^k = \frac{\Pi_p(\eta_{h,\tau}^k - \eta_{h,\tau}^{k-1})}{\Pi_p(\eta_{h,\tau}^k) + H} = 1 - \mathcal{J}_k^{k-1}.
\end{aligned} \tag{4.2}$$

Please observe, that here the domain $\Omega_{h,\tau}$ is defined via $\eta_{h,\tau}$ and its triangulation $\mathcal{T}_h$ is defined by $\Pi_p[\eta_{h,\tau}]$. Moreover, the Dirichlet boundary values of $\mathbf{u}_{h,\tau}$ will be defined by $\Pi_p[z_{h,\tau}]$.

[4] Please observe the difference between $\widehat{\nabla}_h \widehat{\boldsymbol{\Psi}}_{h,\tau}$ and $\widehat{\nabla \boldsymbol{\Psi}_{h,\tau}}$ which is defined in (2.12).

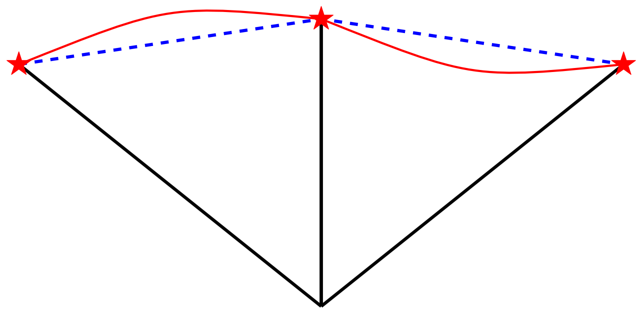

Figure 3: Part of Σ_h and $\mathcal{T}_h$ near top boundary: the red line is $\Gamma_S = \{\mathbf{x} \mid x_d = \eta_{h,\tau} + H\}$; the dashed blue line is $\mathcal{E}_{\rm S} = \{\mathbf{x} \mid x_d = H + \Pi_p(\eta_{h,\tau})\}$.

Upwind divergence

To approximate the convective terms, we apply a dissipative upwind operator

$$\mathrm{div}_\tau^{\rm up}(r_{h,\tau}, \mathbf{v}_{h,\tau})(\mathbf{x}) := \sum_{K\in\mathcal{T}_h} \frac{1_K}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \mathrm{Up}[r_{h,\tau}, \mathbf{v}_{h,\tau}] \mathrm{d}S(x), \tag{4.3}$$

where

$$\mathrm{Up}[r_{h,\tau}, \mathbf{v}_{h,\tau}] = \underbrace{r_{h,\tau}^{\rm up} \left\langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \right\rangle_\sigma}_{\text{standard upwind}} - \underbrace{h^\varepsilon \left[\!\left[r_h \right]\!\right]}_{\text{artificial diffusion}} = \underbrace{\Pi_{\mathcal{E}}[r_{h,\tau}] \left\langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \right\rangle_\sigma}_{\text{convective part}} - \underbrace{\left(h^\varepsilon + \frac{1}{2} |\left\langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \right\rangle_\sigma| \right) \left[\!\left[r_h \right]\!\right]}_{\text{dissipative part}}, \ \varepsilon > -1.$$

where

$$r_{h,\tau}^{\rm up}|_\sigma = r_{h,\tau}^{\rm in} \left\langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \right\rangle_\sigma^+ + r_{h,\tau}^{\rm out} \left\langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \right\rangle_\sigma^- \quad \text{and} \quad f^\pm = \frac{f \pm |f|}{2}.$$

As pointed out in [22], the additional artificial diffusion included in the above flux function is $h^{\varepsilon+1}$ which indicates $\varepsilon > -1$. For the consistency we will require

$$\varepsilon \in (0, 2(\gamma - 1)).$$

Observe, that the artificial diffusion has been introduced in order to allow to show that a weakly converging subsequence converges to a weak solution with $h, \tau \to 0$. Actually, up to today it is an unavoidable regularization with respect to the analysis. There are, however no numerical experiments that show the necessity of the artificial diffusion in this context. We wish to emphasize that the existence of the scheme, its stability, mass conservation and positivity of the density *do not depend* on the additional artificial diffusion term. However, it is important for deriving the unconditional consistency of our numerical scheme without any assumption on the regularity of the numerical solution, see Theorem 4.13.

In accordance with the relation (2.11), we introduce the upwind divergence on the reference domain as

$$\widehat{\mathrm{div}_\tau^{\rm up}(r_{h,\tau}, \mathbf{v}_{h,\tau})} := \mathrm{div}_\tau^{\rm up}(r_{h,\tau}, \mathbf{v}_{h,\tau}) \circ \mathcal{A}_{h,\tau} = \sum_{K\in\mathcal{T}_h^0} \frac{1_K}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \mathrm{Up}[\widehat{r}_\tau, \widehat{\mathbf{u}}_{h,\tau}] |\mathcal{J}\mathbb{J}^{-1}\widehat{\mathbf{n}}| \mathrm{d}S(\widehat{x}).$$

Preliminary inequalities

We assume the readers are familiar with the techniques in finite element method. For the sake of completeness, we report a few necessary inequalities. As is common the constant depend on the regularity of the mesh; i.e. on the constants θ_0 and c_0 above. As follows from our estimates (4.21) the numbers c_0 and θ_0 can be chosen independently of h and τ. Meaning that

$$\theta_0 \leq \inf \left\{ \frac{\xi_K}{h_K}, K \in \mathcal{T}_{h,\tau} \right\} \qquad \text{and} \qquad c_0 h \leq h_K. \tag{4.4}$$

where ξ_K stands for the diameter of the largest ball included in K. Moreover, it follows from (4.21) it follows that

$$h_K \leq c_1 h \text{ for all } K \in \mathcal{T}_{h,\tau}, \tag{4.5}$$

with a constant uniform in h and τ.

Since these constants can assumed to be uniform w.r.t the change of variables (due to bounds on $\eta_{h,\tau}$) the below estimates hold both on the reference domain, as well as on the current domain. For that reason we take $\Omega_{h,\tau} = \bigcup_{K\in\mathcal{T}_h} K$ as a (regular) polygonal domain. We define for the discrete in space function $\mathbf{u}_{h,\tau}$ the following norms:

$$\|\mathbf{u}_{h,\tau}\|_{1,\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K |\nabla \mathbf{u}_{h,\tau}|^2 \,\mathrm{d}x, \quad \|\mathbf{u}_{h,\tau}\|_{H^1_Y} := \sum_{\sigma\in\mathcal{E}_I} \int_\sigma \frac{1}{h} [\![\mathbf{u}_{h,\tau}]\!]^2 \,\mathrm{d}S(x).$$

Next we would like to introduce from Brenner the Korn inequality [7, equation (1.19)].

$$\|\mathbf{u}_{h,\tau}\|_{1,\mathcal{T}_h} \lesssim \|\mathbf{D}(\mathbf{u}_{h,\tau})\|_{L^2(\Omega_{h,\tau})} + \|\mathbf{u}_{h,\tau}\|_{L^2(\Gamma)} + \|\mathbf{u}_{h,\tau}\|_{H^1_Y},$$

and the Poincaré–Friedrichs inequality [6, equation(1.5)]

$$\|\mathbf{u}_{h,\tau}\|_{L^2(\Omega_{h,\tau})} \lesssim \|\mathbf{u}_{h,\tau}\|_{1,\mathcal{T}_h} + \|\mathbf{u}_{h,\tau}\|_{L^1(\Gamma)},$$

respectively for all $\mathbf{u}_{h,\tau} \in V_h$. Thus we deduce the following modified Korn inequality

$$\|\mathbf{u}_{h,\tau}\|_{1,\mathcal{T}_h} + \|\mathbf{u}_{h,\tau}\|_{L^2(\Omega_{h,\tau})} \lesssim C\left(\|\mathbf{D}(\mathbf{u}_{h,\tau})\|_{L^2(\Omega_{h,\tau})} + \|\mathbf{u}_{h,\tau}\|_{L^2(\Gamma)} + \|\mathbf{u}_{h,\tau}\|_{H^1_Y}\right) \tag{4.6}$$

Further, we need the following version of Sobolev's inequality [23, Lemma 2.3]

$$\|\mathbf{u}_{h,\tau}\|_{L^6(\Omega_{h,\tau})} \lesssim \|\mathbf{u}_{h,\tau}\|_{1,\mathcal{T}_h} + \|\mathbf{u}_{h,\tau}\|^2_{L^1(\Omega_{h,\tau})}, \ \forall\, \mathbf{u}_{h,\tau} \in V_h. \tag{4.7}$$

Next, due to scaling argument we introduce the trace theorem [23, equation (2.26)]

$$\|\mathbf{u}\|_{L^p(\partial K)} \lesssim h^{-1/p} \|\mathbf{u}\|_{L^p(K)}, \ \mathbf{u} \in \mathcal{P}^m(K),\ 1 \le p \le \infty,\ \forall\, K \in \mathcal{T}_h, \tag{4.8}$$

where $\mathcal{P}^m(K)$ denotes the space of polynomial degree not grater than m. It is worth mentioning the inverse estimate, see[14] and [33, Remark 2.1]

$$\|\mathbf{u}\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{h,\tau}))} \lesssim \tau^{\frac{1}{p_1}-\frac{1}{p_2}} h^{d(\frac{1}{q_1}-\frac{1}{q_2})} \|\mathbf{u}\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{h,\tau}))}, \ \forall\, 1 \le p_2 \le p_1 \le \infty,\ 1 \le q_2 \le q_1 \le \infty. \tag{4.9}$$

Finally, we recall the standard interpolation error estimates for $\phi \in C^1(\Omega_{h,\tau})$ [8] (see also [30, Appendix]).

$$[\![\Pi_{\mathcal{T}}[\phi]]\!] \lesssim h,\ \|\Pi_{\mathcal{T}}[\phi] - \phi\|_{L^p} \lesssim h,\ \|\Pi_{\mathcal{E}}[\phi] - \phi\|_{L^p} \lesssim h,\ \|\Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\phi]] - \phi\|_{L^p} \lesssim h,\ 1 \le p \le \infty. \tag{4.10}$$

and also from [37, Lemma 2.7] and [17] that

$$\begin{aligned} \|\mathbf{v} - \Pi_{\mathcal{T}}[\mathbf{v}]\|_{L^2(\Omega_{h,\tau})} &\lesssim h \|\nabla \mathbf{v}\|_{L^2(\Omega_{h,\tau})}, \ \forall \mathbf{v} \in V_h \text{ or } C^1(\Omega_{h,\tau}) \\ \|\mathbf{v} - \Pi_{\mathcal{E}}[\mathbf{v}]\|_{L^2(\Omega_{h,\tau})} + h \|\nabla(\mathbf{v} - \Pi_{\mathcal{E}}[\mathbf{v}])\|_{L^2(\Omega_{h,\tau})} &\lesssim h^2 \|\mathbf{v}\|_{W^{2,2}(\Omega_{h,\tau})}, \ \forall \mathbf{v} \in W^{2,2}(\Omega_{h,\tau}). \end{aligned} \tag{4.11}$$

4.2 The scheme

With the above notations, we are ready to present a mixed finite volume–finite element method for the FSI problem (1.2). First we present the scheme in the current domain.

Definition 4.2 (Fully discrete scheme on the current domain)**.**
We seek a solution $(\eta^k_{h,\tau}, \varrho^k_{h,\tau}, \mathbf{u}^k_{h,\tau}) \in (W_h, Q_h(\Omega^k_{h,\tau}), V_h(\Omega^k_{h,\tau}))$ for all $k \in \{1,\dots,N_t\}$, such that for all $(\varphi^k_{h,\tau}, \mathbf{\Psi}^k_{h,\tau}, \psi^k_{h,\tau}) \in (Q_h(\Omega^k_{h,\tau}), V_h(\Omega^k_{h,\tau}), W_h(\Sigma))$ with $\Pi_{\mathcal{E}}[\mathbf{\Psi}^k_{h,\tau} \circ \mathcal{A}^k_{h,\tau}] = \Pi_{\mathcal{E}}[\Pi_p[\psi^k_{h,\tau}]]\mathbf{e}_d$ (for all $k \in \{1,\dots,N_t\}$) the following hold:

$$\int_{\Omega^k_\tau} D_t \varrho^k_{h,\tau} \varphi_{h,\tau} \,\mathrm{d}x + \int_{\Omega^k_\tau} \mathrm{div}^{\mathrm{up}}_\tau(\varrho^k_{h,\tau}, \mathbf{v}^k_{h,\tau}) \varphi_{h,\tau} \,\mathrm{d}x = 0; \tag{4.12a}$$

$$
\begin{aligned}
&\int_{\Omega_\tau^k} D_t\left(\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right)\cdot \mathbf{\Psi}_{h,\tau} + \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k], \mathbf{v}_{h,\tau}^k)\cdot \mathbf{\Psi}_{h,\tau}\,\mathrm{d}x \\
&+ 2\mu \int_{\Omega_\tau^k} \mathbf{D}(\mathbf{u}_{h,\tau}^k) : \nabla \mathbf{\Psi}_{h,\tau}\,\mathrm{d}x + \lambda \int_{\Omega_\tau^k} \mathrm{div}\mathbf{u}_{h,\tau}^k \mathrm{div}\mathbf{\Psi}_{h,\tau}\,\mathrm{d}x + 2\mu \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h}\left[\!\left[\mathbf{u}_{h,\tau}^k\right]\!\right]\cdot [\![\mathbf{\Psi}_{h,\tau}]\!]\,\mathrm{d}S(x) \\
&- \int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{\Psi}_{h,\tau}\,\mathrm{d}x + \int_\Sigma \delta_t z_{h,\tau}^k \psi_{h,\tau}\,d\mathbf{r} + \int_\Sigma \Delta\eta_{h,\tau}^k \Delta\psi_{h,\tau}\,d\mathbf{r} = \int_{\Omega_\tau^k} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{\Psi}_{h,\tau}\,\mathrm{d}x + \int_\Sigma g_\tau^k \psi_{h,\tau}\,d\mathbf{r}
\end{aligned}
\tag{4.12b}
$$

where

$$
z_{h,\tau}^k = \delta_t \eta_{h,\tau}^k, \quad \mathbf{v}_{h,\tau}^k = \mathbf{u}_{h,\tau}^k - \mathbf{w}_{h,\tau}^k, \quad \mathbf{w}_{h,\tau}^k(\mathbf{x}) = \left(\mathbf{0}_{d-1}, \frac{\Pi_p[z_{h,\tau}^k]x_d}{\Pi_p[\eta_{h,\tau}^k] + H}\right)^T,
$$

and

$$
g_\tau^k := \frac{1}{\tau}\int_{I^k} \Pi_{\mathcal{E}}[g]\,dt \text{ and } \mathbf{f}_\tau^k := \frac{1}{\tau}\int_{I^k} \Pi_{\mathcal{E}}[\mathbf{f}]\,dt.
$$

The scheme is supplemented with the initial data

$$
\varrho_h^0 = \Pi_{\mathcal{T}}[\varrho_0], \quad \mathbf{u}_{h,\tau}^0 \in \Pi_{\mathcal{T}}[\mathbf{u}_0], \quad \eta_{h,\tau}^0 = 0, \quad z_{h,\tau}^0 = 0,
$$

and the boundary conditions

$$
\langle \mathbf{v}_{h,\tau}\rangle_\sigma = 0, \quad \left[\!\left[\varrho_{h,\tau}^k\right]\!\right]_\sigma = 0, \ \forall\, \sigma \in \mathcal{E}_{\mathrm{E}}.
$$

Pleas observe that by construction

$$
\left\langle \mathbf{u}_{h,\tau}^k \right\rangle_\sigma = \left\langle \mathbf{w}_\tau^k \right\rangle_\sigma = \left\langle \Pi_p[z_{h,\tau}^k] \right\rangle_\sigma \text{ for all } \sigma \in \Sigma_h.
$$

Recalling (2.3) allows to transfer the scheme in the following way:

Definition 4.3 (Fully discrete scheme on reference domain)**.**
We seek the solution $\eta_{h,\tau}^k \in W_h(\Sigma)$ and $(\widehat{\varrho}_{h,\tau}^k, \widehat{\mathbf{u}}_{h,\tau}^k) \in Q_h(\widehat{\Omega}) \times V_h(\widehat{\Omega})$for all $k \in \{1,\ldots,N_t\}$, such that for all $(\widehat{\varphi}_{h,\tau}^k, \widehat{\mathbf{\Psi}}_{h,\tau}^k, \psi_{h,\tau}^k) \in Q_h(\widehat{\Omega}) \times V_h(\widehat{\Omega}) \times W_h(\Sigma)$, with $\Pi_{\mathcal{E}}[\widehat{\mathbf{\Psi}}_{h,\tau}^k]|_\Sigma = \Pi_{\mathcal{E}}[\Pi_p[\psi_{h,\tau}^k(\mathbf{r})]]\mathbf{e}_d$ (for all $\mathbf{r} \in \Sigma$ and all $k \in \{1,\ldots,N_t\}$) the following holds:

$$
\int_{\widehat{\Omega}} \frac{\widehat{\varrho}_{h,\tau}^k \mathcal{J}_0^k - \widehat{\varrho}_{h,\tau}^{k-1}\mathcal{J}_0^{k-1}}{\tau}\widehat{\varphi}_{h,\tau}\,\mathrm{d}\widehat{x} + \int_{\widehat{\Omega}} \widehat{\mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k)}\widehat{\varphi}_{h,\tau}\,\mathrm{d}\widehat{x} = 0; \tag{4.13a}
$$

$$
\begin{aligned}
&\int_{\widehat{\Omega}} \frac{\widehat{\varrho}_{h,\tau}^k \Pi_{\mathcal{T}}[\widehat{\mathbf{u}}_{h,\tau}^k]\mathcal{J}_0^k - \widehat{\varrho}_{h,\tau}^{k-1}\Pi_{\mathcal{T}}[\widehat{\mathbf{u}}_{h,\tau}^{k-1}]\mathcal{J}_0^{k-1}}{\tau}\cdot \widehat{\mathbf{\Psi}}_{h,\tau}\,\mathrm{d}\widehat{x} + \int_{\widehat{\Omega}} \widehat{\mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k], \mathbf{v}_{h,\tau}^k)}\cdot \widehat{\mathbf{\Psi}}_{h,\tau}\mathcal{J}_0^k\,\mathrm{d}\widehat{x} \\
&+ 2\mu \int_{\widehat{\Omega}} \widehat{\mathbf{D}(\mathbf{u}_{h,\tau}^k)} : \widehat{\nabla \mathbf{\Psi}_{h,\tau}}\mathcal{J}_0^k\,\mathrm{d}\widehat{x} + 2\mu \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}}\int_\sigma \frac{1}{h}\left[\!\left[\widehat{\mathbf{u}}_{h,\tau}^k\right]\!\right]\cdot\left[\!\left[\widehat{\mathbf{\Psi}}_{h,\tau}\right]\!\right]|\mathcal{J}_0^k(\mathbb{J}_0^k)^{-T}\widehat{\mathbf{n}}|\mathrm{d}S(\widehat{x}) \\
&+ \lambda \int_{\widehat{\Omega}} \widehat{\mathrm{div}\mathbf{u}_{h,\tau}^k}\widehat{\mathrm{div}\mathbf{\Psi}_{h,\tau}}\mathcal{J}_0^k\,\mathrm{d}\widehat{x} - \int_{\widehat{\Omega}} p(\widehat{\varrho}_{h,\tau}^k)\widehat{\mathrm{div}\mathbf{\Psi}_{h,\tau}}\mathcal{J}_0^k\,\mathrm{d}\widehat{x} + \int_\Sigma \delta_t z_{h,\tau}^k \psi\,d\mathbf{r} + \int_\Sigma \Delta\eta_{h,\tau}^k \Delta\psi\,d\mathbf{r} \\
&\quad= \int_{\widehat{\Omega}} \widehat{\varrho}_{h,\tau}^k \widehat{\mathbf{f}}_\tau^k \cdot \widehat{\mathbf{\Psi}}_{h,\tau}\mathcal{J}_0^k\,\mathrm{d}\widehat{x} + \int_\Sigma g_\tau^k \psi_{h,\tau}\,d\mathbf{r}
\end{aligned}
\tag{4.13b}
$$

supplemented with the initial data

$$
\varrho_{h,\tau}^0 = \Pi_{\mathcal{T}}[\varrho_0], \quad \mathbf{u}_{h,\tau}^0 = \Pi_{\mathcal{T}}[\mathbf{u}_0], \quad \eta_{h,\tau}^0 = 0, \quad z_{h,\tau}^0 = 0, \tag{4.14}
$$

and boundary conditions

$$
\left\langle \widehat{\mathbf{v}}_{h,\tau}^k \right\rangle_\sigma := \left\langle \widehat{\mathbf{u}}_{h,\tau}^k - \widehat{\mathbf{w}}_\tau^k \right\rangle_\sigma = 0, \quad \left[\!\left[\widehat{\varrho}_{h,\tau}^k\right]\!\right]_\sigma = 0, \ \forall\, \sigma \in \mathcal{E}_{\mathrm{E}}. \tag{4.15}
$$

4.3 Stability

Analogously as the semi-discrete case, the fully-discrete scheme (4.12) (or (4.13)) satisfies the following internal energy balance.

Lemma 4.4 (Discrete internal energy balance)**.**
Let $(\varrho_{h,\tau}, \mathbf{u}_{h,\tau}) \in Q_h \times V_h$ *satisfy the discrete continuity equation* (4.12a)*. Then there exists* $\xi \in \mathrm{co}\{\varrho_{h,\tau}^{k-1} \circ \mathbf{X}^k, \varrho_{h,\tau}^k\}$ *and* $\zeta \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ *for any* $\sigma = K|L \in \mathcal{E}_{\mathrm{I}}^k$ *such that*

$$\frac{1}{\tau}\left(\int_{\Omega_\tau^k} \mathcal{H}(\varrho_{h,\tau}^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} \mathcal{H}(\varrho_{h,\tau}^{k-1})\,\mathrm{d}x\right) + \int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{u}_{h,\tau}^k\,\mathrm{d}x = -D_1 - D_2, \tag{4.16}$$

where

$$D_1 = \int_{\Omega_\tau^k} \tau\mathcal{J}^k \frac{\mathcal{H}''(\xi)}{2}\left|D_t^{\mathcal{A}}\varrho_{h,\tau}^k\right|^2 \mathrm{d}x, \quad D_2 = \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma \mathcal{H}''(\zeta)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\left(h^\varepsilon + \left|\left\langle \mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\right|\right)\mathrm{d}S(x). \tag{4.17}$$

Remark 4.5 (Renormalized discrete continuity equation)**.** It is apparent from the argument below that a respective discrete continuity equation is valid for $B(\varrho_{h,\tau})$ for an arbitrary functions $B : C^1([0,\infty))$. Indeed, by replacing $\mathcal{H}$ by B one derives an analogous result to Lemma 3.4.

Proof of Lemma 4.4. Firstly, we set $\varphi_{h,\tau} = \mathcal{H}'(\varrho)$ in (4.12a) and obtain

$$\int_{\Omega_\tau^k} D_t\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\,\mathrm{d}x + \int_{\Omega_\tau^k} \mathrm{div}_\tau^{\mathrm{up}}\left(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k\right)\mathcal{H}'(\varrho_{h,\tau}^k)\,\mathrm{d}x = 0.$$

Next, recalling (3.4), we know there exist $\xi \in \mathrm{co}\{\varrho_{h,\tau}^{k-1}\circ\mathbf{X}^k, \varrho_{h,\tau}^k\}$ such that

$$\begin{aligned}
&\int_{\Omega_\tau^k} D_t\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\,\mathrm{d}x = \int_{\Omega_\tau^k} \frac{\varrho_{h,\tau}^k - \varrho_{h,\tau}^{k-1}\circ\mathbf{X}^k\mathcal{J}^k}{\tau}\mathcal{H}'(\varrho_{h,\tau}^k)\,\mathrm{d}x \\
&= \frac{1}{\tau}\left(\int_{\Omega_\tau^k}\mathcal{H}(\varrho_{h,\tau}^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}}\mathcal{H}(\varrho_{h,\tau}^{k-1})\,\mathrm{d}x\right) + \int_{\Omega_\tau^k} p_{h,\tau}^k\mathrm{div}\mathbf{w}_{h,\tau}^k\,\mathrm{d}x + \int_{\Omega_\tau^k}\tau\mathcal{J}^k\frac{\mathcal{H}''(\xi)}{2}\left|D_t^{\mathcal{A}}\varrho_{h,\tau}^k\right|^2\mathrm{d}x
\end{aligned}$$

Further, by recalling the definition of the upwind flux (4.3), and using again the Taylor expansion, we reformulate the convective term as

$$\begin{aligned}
&\int_{\Omega_\tau^k}\mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k)\mathcal{H}'(\varrho_{h,\tau}^k)\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h^k}\int_K\mathcal{H}'(\varrho_{h,\tau}^k)\sum_{\sigma\in\partial K}\frac{|\sigma|}{|K|}\left(\varrho_{h,\tau}^{k,up}\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma - h^\varepsilon\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]\right)\,\mathrm{d}x \\
&= \int_{\Omega_\tau^k}\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{v}_{h,\tau}^k\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h^k}\int_K\mathcal{H}'(\varrho_{h,\tau}^k)\sum_{\sigma\in\partial K}\frac{|\sigma|}{|K|}(\varrho_{h,\tau}^{k,up} - \varrho_K^k)\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\,\mathrm{d}x \\
&\quad - h^\varepsilon\sum_{K\in\mathcal{T}_h^k}\sum_{\sigma\in\partial K}\int_\sigma\left(\left[\!\left[\mathcal{H}(\varrho_{h,\tau}^k)\right]\!\right] - \frac{\mathcal{H}''(\zeta)}{2}\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\right)\mathrm{d}S(x) \\
&= \int_{\Omega_\tau^k}\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{v}_{h,\tau}^k\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h^k}\sum_{\sigma\in\partial K}\int_\sigma\mathcal{H}'(\varrho_K^k)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]\left[\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\right]^-\mathrm{d}S(x) \\
&\quad + h^\varepsilon\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\partial K}\int_\sigma\frac{\mathcal{H}''(\zeta)}{2}\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\mathrm{d}S(x) \\
&= \sum_{K\in\mathcal{T}_h^k}\int_K\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{v}_{h,\tau}^k\,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\mathcal{H}''(\zeta)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2 h^\varepsilon\mathrm{d}S(x) \\
&\quad + \sum_{K\in\mathcal{T}_h^k}\sum_{\sigma\in\partial K}\int_\sigma\left(\left[\!\left[\mathcal{H}(\varrho_{h,\tau}^k)\right]\!\right] - \frac{\mathcal{H}''(\zeta)}{2}\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\right)\frac{\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma - \left|\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\right|}{2}\mathrm{d}S(x) \\
&= \sum_{K\in\mathcal{T}_h^k}\int_K\varrho_{h,\tau}^k\mathcal{H}'(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{v}_{h,\tau}^k\,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\mathcal{H}''(\zeta)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\left(h^\varepsilon + \left|\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\right|\right)\mathrm{d}S(x)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{K\in\mathcal{T}_h^k} \sum_{\sigma\in\partial K} \int_\sigma \mathcal{H}(\varrho_K^k)\mathbf{v}_{h,\tau}^k \cdot \mathbf{n} \mathrm{d}S(x) \\
&= \int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k) \mathrm{div}\mathbf{v}_{h,\tau}^k \,\mathrm{d}x + D_2,
\end{aligned}
$$

where D_2 is given in (4.17) above, and $\zeta \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}_\mathrm{I}$.

Consequently, collecting the above terms and seeing $\mathbf{v}_{h,\tau} = \mathbf{u}_{h,\tau} - \mathbf{w}_{h,\tau}$, we complete the proof, i.e.,

$$
\begin{aligned}
0 &= \frac{1}{\tau}\left(\int_{\Omega_\tau^k} \mathcal{H}(\varrho_{h,\tau}^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} \mathcal{H}(\varrho_{h,\tau}^{k-1})\,\mathrm{d}x \right) + \int_{\Omega_\tau^k} p_\tau^k \mathrm{div}\mathbf{w}_{h,\tau}^k \,\mathrm{d}x + D_1 + \int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{v}_{h,\tau}^k \,\mathrm{d}x + D_2 \\
&= \frac{1}{\tau}\left(\int_{\Omega_\tau^k} \mathcal{H}(\varrho_{h,\tau}^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} \mathcal{H}(\varrho_{h,\tau}^{k-1})\,\mathrm{d}x \right) + \int_{\Omega_\tau^k} p_\tau^k \mathrm{div}\mathbf{u}_{h,\tau}^k \,\mathrm{d}x + D_1 + D_2.
\end{aligned}
$$

□

Analogously as the semi-discrete case, the fully-discrete scheme (4.12) (or (4.13)) dissipates the total energy.

Theorem 4.6 (Energy stability of the fully-discrete scheme (4.12))**.**
Let $\left(\varrho_{h,\tau}^k, \mathbf{u}_{h,\tau}^k, \eta_{h,\tau}^k\right)_{k=1}^{N_T}$ *be a family of numerical solutions obtained by the scheme* (4.12) *(or* (4.13)*). Then for any* $N = 1, \dots, N_T$ *the energy is stable in the following sense*

$$
\begin{aligned}
&\int_{\Omega_{h,\tau}^N} E_f^N \,\mathrm{d}x + \int_\Sigma E_s^N \,d\mathbf{r} + \tau \sum_{k=1}^N \int_{\Omega_\tau^k} \left(2\mu|\mathbf{D}(\mathbf{u}_{h,\tau}^k)|^2 + \lambda |\mathrm{div}\mathbf{u}_{h,\tau}^k|^2 \right) \mathrm{d}x + 2\mu\tau \sum_{k=1}^N \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right]^2 \mathrm{d}S(x) \\
&+ \frac{\tau^2}{2} \sum_{k=1}^N \int_\Sigma \left(|\delta_t z_{h,\tau}^k|^2 + \alpha \left| \Delta z_{h,\tau}^k \right|^2 + \beta \left| \nabla z_{h,\tau}^k \right|^2 \right) d\mathbf{r} + \tau \sum_{k=1}^N \int_{\Omega_\tau^k} \frac{\tau}{2} \varrho_{h,\tau}^{k-1} \circ \mathbf{X}^k \left| D_t \Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k] \right|^2 \mathrm{d}x \\
&+ \tau \sum_{k=1}^N (D_1 + D_2) + \tau \sum_{k=1}^N \sum_{\sigma\in\mathcal{E}_\mathrm{I}^k} \int_\sigma \left(\frac{1}{2} \varrho_{h,\tau}^{k,up} |\mathbf{v}_{h,\tau}^k \cdot \mathbf{n}| + h^\varepsilon \left\langle \varrho_{h,\tau}^k \right\rangle_\sigma \right) \left[\!\left[\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k] \right]\!\right]^2 \mathrm{d}S(x) \\
&= \int_{\widehat{\Omega}} E_f^0 \,\mathrm{d}\widehat{x} + \int_\Sigma E_s^0 \,d\mathbf{r} + \tau \sum_{k=1}^N \int_{\Omega_\tau^k} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{u}_{h,\tau}^k \,\mathrm{d}x + \tau \sum_{k=1}^N \int_\Sigma g_\tau^k z_{h,\tau}^k \,d\mathbf{r}
\end{aligned}
$$

where D_1, D_2 *are given in* (4.17), *and*

$$
E_f^k = \frac{1}{2} \varrho_{h,\tau}^k \left| \Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k] \right|^2 + \mathcal{H}(\varrho_{h,\tau}^k), \quad E_s^k = \frac{1}{2}(|z_{h,\tau}^k|^2 + \alpha|\Delta\eta_{h,\tau}^k|^2 + \beta|\nabla\eta_{h,\tau}^k|^2), \quad z_{h,\tau}^k = \delta_t \eta_{h,\tau}^k.
$$

Proof. Setting $\varphi_{h,\tau} = -\frac{|\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k]|^2}{2}$ in (4.12a), and $(\mathbf{\Psi}_{h,\tau}, \psi_{h,\tau}) = (\mathbf{u}_{h,\tau}^k, z_{h,\tau}^k)$ in (4.12b) we have

$$
\sum_{i=1}^2 I_i = 0, \text{ and } \sum_{i=3}^9 I_i = 0, \tag{4.18}
$$

respectively, where

$$
\begin{aligned}
I_1 &= -\int_{\Omega_\tau^k} D_t \varrho_{h,\tau}^k \frac{\left|\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k]\right|^2}{2} \mathrm{d}x, \; I_2 = -\int_{\Omega_\tau^k} \mathrm{div}_\mathcal{T}^{\mathrm{up}} \left(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k \right) \frac{\left|\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k]\right|^2}{2} \mathrm{d}x, \\
I_3 &= \int_{\Omega_\tau^k} D_t \left(\varrho_{h,\tau}^k \Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k] \right) \cdot \mathbf{u}_{h,\tau}^k \,\mathrm{d}x, \; I_4 = \int_{\Omega_\tau^k} \mathrm{div}_\mathcal{T}^{\mathrm{up}} \left(\varrho_{h,\tau}^k \Pi_\mathcal{T}[\mathbf{u}]_h^k, \mathbf{v}_{h,\tau}^k \right) \cdot \mathbf{u}_{h,\tau}^k \,\mathrm{d}x, \\
I_5 &= -\int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k) \mathrm{div}\mathbf{u}_{h,\tau}^k \,\mathrm{d}x, I_6 = \int_{\Omega_\tau^k} \left(2\mu |\mathbf{D}(\mathbf{u}_{h,\tau}^k)|^2 + \lambda |\mathrm{div}\mathbf{u}_{h,\tau}^k|^2 \right) \mathrm{d}x + 2\mu \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right]^2 \mathrm{d}S(x) \\
I_7 &= \int_{\Omega_\tau^k} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{u}_{h,\tau}^k \,\mathrm{d}x + \int_\Sigma g_\tau^k z_{h,\tau}^k \,d\mathbf{r}, \; I_8 = \int_\Sigma \frac{z_{h,\tau}^k - z_{h,\tau}^{k-1}}{\tau} z_{h,\tau}^k \,d\mathbf{r} \\
I_9 &= \int_\Sigma \left(\alpha \Delta\eta_{h,\tau}^k \Delta z_{h,\tau}^k + \beta \nabla\eta_{h,\tau}^k \cdot \nabla z_{h,\tau}^k \right) d\mathbf{r}.
\end{aligned}
$$

Now we proceed with the summation of all the I_i terms for $i = 1, \dots, 9$.

Term $(I_1 + I_3 + I_8) + (I_6 + I_7) + I_9$**.** Firstly, analogously as in the proof of Theorem 3.7 we have

$$
\begin{aligned}
&(I_1 + I_3 + I_8) + (I_6 + I_7) + I_9 \\
=&\frac{1}{\tau}\left(\int_{\Omega_\tau^k} \frac{1}{2}\varrho_{h,\tau}^k \left|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2 \mathrm{d}x - \int_{\Omega_\tau^{k-1}} \frac{1}{2}\varrho_{h,\tau}^{k-1} \left|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^{k-1}]\right|^2 \mathrm{d}x\right) + \frac{\tau}{2}\int_{\Omega_\tau^k} \varrho_{h,\tau}^{k-1} \circ \mathbf{X}_k^{k-1} \left|D_t^{\mathcal{A}}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2 \mathrm{d}x \\
&+ \int_{\Sigma_h} \left(\delta_t\left(\frac{|z_{h,\tau}^k|^2}{2}\right) + \frac{\tau}{2}|\delta_t z_{h,\tau}^k|^2\right) d\mathbf{r} \\
&+ \int_{\Omega_\tau^k} \left(2\mu|\mathbf{D}(\mathbf{u}_{h,\tau}^k)|^2 + \lambda|\mathrm{div}\mathbf{u}_{h,\tau}^k|^2\right) \mathrm{d}x + 2\mu\sum_{\sigma\in\mathcal{E}_{\mathrm{I}}}\int_\sigma \frac{1}{h}\left[\!\left[\mathbf{u}_{h,\tau}^k\right]\!\right]^2 \mathrm{d}S(x) + \int_{\Omega_\tau^k} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{u}_{h,\tau}^k \,\mathrm{d}x \\
&+ \int_\Sigma g_\tau^k z_{h,\tau}^k \,d\mathbf{r} + \int_\Sigma \frac{1}{2}\delta_t\left(\alpha|\Delta\eta_{h,\tau}^k|^2 + \beta|\nabla\eta_{h,\tau}^k|^2\right) d\mathbf{r} + \int_\Sigma \left(\frac{\tau\alpha}{2}\left|\Delta z_{h,\tau}^k\right|^2 + \frac{\tau\beta}{2}\left|\nabla z_{h,\tau}^k\right|^2\right) d\mathbf{r}.
\end{aligned}
$$

Term I_2+I_4**.** For the convective terms, we have using the fact that $\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]$ and $\mathrm{div}_{\mathcal{T}}^{\mathrm{up}}\left(\varrho_{h,\tau}^k\Pi_{\mathcal{T}}[\mathbf{u}]_h^k, \mathbf{v}_{h,\tau}^k\right)$ are constant on each $K \in \mathcal{T}_{h,\tau}$ and the upwind divergence

$$
\begin{aligned}
I_2 + I_4 =& \int_{\Omega_\tau^k} -\mathrm{div}_{\mathcal{T}}^{\mathrm{up}}\left(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k\right)\frac{\left|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2}{2}\mathrm{d}x + \int_{\Omega_\tau^k}\mathrm{div}_{\mathcal{T}}^{\mathrm{up}}\left(\varrho_{h,\tau}^k\Pi_{\mathcal{T}}[\mathbf{u}]_h^k, \mathbf{v}_{h,\tau}^k\right)\cdot\mathbf{u}_{h,\tau}^k\,\mathrm{d}x \\
=& \sum_{K\in\mathcal{T}_h^k}\sum_{\sigma\in\partial K}\int_\sigma\left(\varrho_{h,\tau}^{k,up}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]^{k,up}\cdot\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k] - \varrho_{h,\tau}^{k,up}\frac{1}{2}\left|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2\right)\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\mathrm{d}S(x) \\
&- h^\varepsilon\sum_{K\in\mathcal{T}_h^k}\sum_{\sigma\in\partial K}\int_\sigma\left(\left[\!\left[\varrho_{h,\tau}^k\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right]\!\right]\cdot\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k] - \left[\!\left[\varrho_{h,\tau}^k\right]\!\right]\frac{1}{2}\left|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2\right)\mathrm{d}S(x) \\
=& \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\frac{1}{2}\left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right]\!\right]^2\left(\varrho_K^k[\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}_{\sigma,K}]^+ + \varrho_L^k[\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}_{\sigma,L}]^+\right)\mathrm{d}S(x) \\
&+ h^\varepsilon\sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\left\langle\varrho_{h,\tau}^k\right\rangle_\sigma\left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right]\!\right]^2\mathrm{d}S(x) \\
=& \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\left(\frac{1}{2}\varrho_{h,\tau}^{k,up}|\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}| + h^\varepsilon\left\langle\varrho_{h,\tau}^k\right\rangle_\sigma\right)\left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right]\!\right]^2\mathrm{d}S(x).
\end{aligned}
$$

Pressure term I_5**.** Recalling the discrete internal energy equation (3.5), we can rewrite the pressure term as

$$
I_5 = -\int_{\Omega_\tau^k} p(\varrho_{h,\tau}^k)\mathrm{div}\mathbf{u}_{h,\tau}^k\,\mathrm{d}x = \frac{1}{\tau}\left(\int_{\Omega_\tau^k}\mathcal{H}(\varrho_{h,\tau}^k)\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}}\mathcal{H}(\varrho_{h,\tau}^{k-1})\,\mathrm{d}x\right) + D_1 + D_2,
$$

where D_1 and D_2 are given in (4.17). Collecting all the above terms, we get

$$
\begin{aligned}
&\frac{1}{\tau}\left(\int_{\Omega_\tau^k} E_f^k\,\mathrm{d}x - \int_{\Omega_\tau^{k-1}} E_f^{k-1}\,\mathrm{d}x\right) + \int_{\Sigma_h}\delta_t\left(\frac{|z_{h,\tau}^k|^2}{2} + \alpha\frac{|\Delta\eta_{h,\tau}^k|^2}{2} + \beta\frac{|\nabla\eta_{h,\tau}^k|^2}{2}\right)d\mathbf{r} \\
&+ \frac{\tau}{2}\int_\Sigma\left(|\delta_t z_{h,\tau}^k|^2 + \alpha\left|\Delta z_{h,\tau}^k\right|^2 + \beta\left|\nabla z_{h,\tau}^k\right|^2\right)d\mathbf{r} + \int_{\Omega_\tau^k}\left(2\mu|\mathbf{D}(\mathbf{u}_{h,\tau}^k)|^2 + \lambda|\mathrm{div}\mathbf{u}_{h,\tau}^k|^2\right)\mathrm{d}x \\
&+ 2\mu\sum_{\sigma\in\mathcal{E}_{\mathrm{I}}}\int_\sigma\frac{1}{h}\left[\!\left[\mathbf{u}_{h,\tau}^k\right]\!\right]^2\mathrm{d}S(x) \\
=& -D_1 - D_2 - \int_{\Omega_\tau^k}\frac{\tau}{2}\varrho_h^{k-1}\circ\mathbf{X}^k\left|D_t\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right|^2\mathrm{d}x + \int_{\Omega_\tau^k}\varrho_{h,\tau}^k\mathbf{f}_\tau^k\cdot\mathbf{u}_{h,\tau}^k\,\mathrm{d}x + \int_\Sigma g_\tau^k z_{h,\tau}^k\,d\mathbf{r} \\
&- \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}^k}\int_\sigma\left(\frac{1}{2}\varrho_{h,\tau}^{k,up}|\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}| + \left\langle\varrho_{h,\tau}^k\right\rangle_\sigma h^\varepsilon\right)\left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k]\right]\!\right]^2\mathrm{d}S(x).
\end{aligned}
$$

We finish the proof by summing up the above equation for $k = 1, \dots, N$ and multiplying with τ. $\square$

4.4 A-priori estimates

Before proving the consistency of the scheme (4.12) (or equivalently (4.13)) we derive some useful estimates. Due to the coherence of the argument we use the notation of Subsection 3.4. In particular we use the same definition of the piecewise constant in time functions (as defined in (2.2)) and the piecewise constant domain $\Omega_{h,\tau}$.

Applying the modified Korn inequality (4.6) and the Sobolev inequality (4.7) to the energy estimates (Theorem 4.6) and the definition of D_1 and D_2 (see (4.17)) directly imply the following uniform bounds on the numerical solutions:

$$
\begin{aligned}
\|\varrho_{h,\tau}\|_{L^\infty(0,T;L^\gamma(\Omega_{h,\tau}(\cdot)))} \le c, \quad \left\|\varrho_{h,\tau}|\mathbf{u}_{h,\tau}|^2\right\|_{L^\infty(0,T;L^1(\Omega_{h,\tau}(\cdot)))} \le c,\\
\|\mathbf{D}(\mathbf{u}_{h,\tau})\|_{L^2(0,T;L^2(\Omega_{h,\tau}(\cdot)))} \le c, \quad \|\mathrm{div}\mathbf{u}_{h,\tau}\|_{L^2(0,T;L^2(\Omega_{h,\tau}(\cdot)))} \le c, \quad \int_0^T \sum_{\sigma\in\mathcal{E}_I}\int_\sigma \frac{1}{h}\left[\!\left[\mathbf{u}_{h,\tau}\right]\!\right]^2 \mathrm{d}S(x) \le c\\
\|\nabla\mathbf{u}_{h,\tau}\|_{L^2(0,T;L^2(\Omega_{h,\tau}(\cdot)))} \le c, \quad \|\mathbf{u}_{h,\tau}\|_{L^2(0,T;L^6(\Omega_{h,\tau}(\cdot)))} \le c,\\
\|z_{h,\tau}\|_{L^\infty(0,T;L^2(\Sigma))} \le c, \quad \|\Delta\eta_{h,\tau}\|_{L^\infty(0,T;L^2(\Sigma))} \le c,\\
\int_0^T \sum_{\sigma\in\mathcal{E}_I}\int_\sigma \left(\frac{1}{2}\varrho_{h,\tau}^{up}|\mathbf{v}_{h,\tau}\cdot\mathbf{n}| + h^\varepsilon\left\langle\varrho_{h,\tau}\right\rangle_\sigma\right)\left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\right]\!\right]^2 \mathrm{d}S(x) \le c,\\
\int_0^T \sum_{\sigma\in\mathcal{E}_I^k}\int_\sigma \mathcal{H}''(\zeta)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2\left(h^\varepsilon + \left|\left\langle\mathbf{v}_{h,\tau}^k\cdot\mathbf{n}\right\rangle_\sigma\right|\right)\mathrm{d}S(x) \le c,\\
\left\|\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\right\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega_{h,\tau}(\cdot)))} \le c, \quad \left\|\varrho_{h,\tau}\mathbf{u}_{h,\tau}\right\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega_{h,\tau}(\cdot)))} \le c.
\end{aligned}
\tag{4.19}
$$

where c depends on the external force $\mathbf{f}_\tau$ and g_τ as well as the initial data. Further, since the discretization of the displacement $\eta_{h,\tau}$ is conformal we find for $\eta_{h,\tau}$

$$
\|\nabla\eta_{h,\tau}\|_{L^\infty(0,T;L^2(\Sigma))} \le c, \quad \|\eta_{h,\tau}\|_{L^\infty(0,T;L^\infty(\Sigma))} \le c,
$$

Moreover, by precisely the same argument as in Lemma 3.8 we find for all $\theta\in[0,\frac{1}{3})$ there exists a constant C depending on the energy estimates and θ, such that

$$
\max_k \left\|\eta_{h,\tau}^k(\mathbf{r}) - \eta_{h,\tau}^{k-1}(\mathbf{r})\right\|_{L^\infty(\Sigma)} \le C\tau^\theta. \tag{4.20}
$$

which implies the following corollary by the very same argument as in the semi-discrete case.

Corollary 4.7 (Exclusion of self-touching). *Let $\tau^\theta \le \frac{\delta_0}{C}$ and $\delta_1 \ge 2\delta_0$. Then, if for some $k\in\{0,...,N_t\}$ we find that $\inf_\sigma \eta_{h,\tau}^k(\mathbf{r}) \ge \delta_1 - H$, the $\eta_{h,\tau}^{k+1}$ satisfies $\inf_\sigma \eta_{h,\tau}^k(\mathbf{r}) \ge \delta_1 - \delta_0 - H$. Moreover, for every $\delta_0\in(0,H/2)$ there exists a T_0 just depending on the bounds of the energy inequality H, such that*

$$
\inf_{[0,T_0]} \eta(t,\mathbf{r}) \ge \delta_0 - H.
$$

From the above and the L^∞ bound of $\eta_{h,\tau}$, we may assume in the following that there exist two positive constants $\delta_2 > \delta_1 > 0$ such that

$$
0 < \delta_1 \le \mathcal{J}_0^k = \frac{\eta_{h,\tau}+H}{H} \le \delta_2. \tag{4.21}
$$

Remark 4.8. Note the uniform upper and lower bounds on the Jacobian (4.21) imply that all uniform bounds in Lebesgue spaces appeared in this paper hold both on the reference domain and the current time-dependent domain. We emphasize this fact in the following by denoting L^pL^q, L^q for the norms $L^p(0,T;L^q(\Omega_{h,\tau}(\cdot)))$.

Moreover, by the same reasoning all estimates on integrals over the jumps, as well as on area-integrals that have been shown on the reference mesh are also valid on the push forwarded mesh.

Due to the fact that the trace theorem for $\mathbf{u}_{h,\tau}$ is not available we have to use a different estimate on $\mathbf{w}_{h,\tau}$. First by the L^2-Stability of Π_p we find $\|\mathbf{w}_{h,\tau}\|_{L^\infty L^2}$ uniformly bounded. Second, since the projection Π_p is conformal we actually do have a proper gradient of $\mathbf{w}_{h,\tau}$ and may interpolate to find

$$
\left\|\nabla\mathbf{w}_\tau^k\right\|_{L^2(\Omega_\tau)}^2 \approx \left\|\widehat{\nabla}\widehat{\mathbf{w}}_\tau^k\right\|_{L^2(\widehat{\Omega}_\tau)}^2 \le c\int_\Sigma |\nabla\Pi_p[z_{h,\tau}^k]|^2\,d\mathbf{r} + c\int_\Sigma |\Pi_p[z_{h,\tau}^k]|^2 d\mathbf{r}
$$

$$
\begin{aligned}
&\le \frac{c}{\tau^2}\left\|\nabla(\eta_{h,\tau}^k-\eta_{h,\tau}^{k-1})\right\|_{L^2}^2 + c\left\|z_{h,\tau}^k\right\|_{L^2}^2\\
&= -\frac{c}{\tau^2}\int_\Sigma \Delta(\eta_{h,\tau}^k-\eta_{h,\tau}^{k-1})\cdot(\eta_{h,\tau}^k-\eta_{h,\tau}^{k-1}) + c\left\|z_{h,\tau}^k\right\|_{L^2}^2\\
&\le \frac{c}{\tau}(\left\|\Delta\eta_{h,\tau}^k\right\|_{L^2}+\left\|\Delta\eta_{h,\tau}^{k-1}\right\|_{L^2})\left\|z_{h,\tau}^k\right\|_{L^2} + c\left\|z_{h,\tau}^k\right\|_{L^2}^2
\end{aligned}
$$

But this implies by Sobolev embedding (using the fact that $\widehat{\mathbf{w}}_{h,\tau}\equiv 0$ on Γ_D) that

$$
\left\|\mathbf{w}_{h,\tau}^k\right\|_{L^6(\Omega_{h,\tau}^k)} \sim \left\|\widehat{\mathbf{w}}_{h,\tau}^k\right\|_{L^6(\widehat{\Omega}_\tau)} \le c\left\|\widehat{\nabla}\widehat{\mathbf{w}}_{h,\tau}^k\right\|_{L^2(\widehat{\Omega}_\tau)} \lesssim \tau^{-\frac{1}{2}}. \tag{4.22}
$$

Interpolation (Hölder's inequality) implies for $\beta\in[0,1]$ and $q=6\beta+2(1-\beta)$ that

$$
\|\mathbf{w}_{h,\tau}\|_{L^\infty(0,T;L^q(\Omega_{h,\tau}))} \sim \left\|\widehat{\mathbf{w}}_{h,\tau}^k\right\|_{L^\infty(0,T;L^q(\widehat{\Omega}_\tau))} \lesssim \tau^{\frac{-\beta}{2}}.
$$

Further, from the definition of the $\mathbf{w}_{h,\tau}$ (4.2) and the uniform lower and upper bounds of $\eta_{h,\tau}$, we notice

$$
\|\mathrm{div}\mathbf{w}_{h,\tau}\|_{L^\infty L^2} = \left\|\frac{z_{h,\tau}}{H+\eta_{h,\tau}}\right\|_{L^\infty L^2} = H\left\|\frac{z_{h,\tau}}{H+\eta_{h,\tau}}\right\|_{L^\infty(0,T;L^2(\Sigma))} \lesssim c \tag{4.23}
$$

Seeing $\mathbf{v}_{h,\tau}=\mathbf{u}_{h,\tau}-\mathbf{w}_{h,\tau}$, we discover the bound on $\mathbf{v}_{h,\tau}$ that

$$
\|\mathrm{div}\mathbf{v}_{h,\tau}\|_{L^2L^2}\le c. \tag{4.24}
$$

Before going further, we recall a negative estimate from [22, Lemma 3.5].

Lemma 4.9. *Let $\varrho_{h,\tau},\mathbf{v}_{h,\tau}$ be a numerical solution of (4.12a) with $h\in(0,1)$ and satisfy the estimates*

$$
\|\varrho_{h,\tau}\|_{L^\infty L^\gamma}\le c,\qquad \left\|\varrho_{h,\tau}|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]|^2\right\|_{L^\infty L^1}\le c,\quad h^\varepsilon\int_0^T\sum_{\sigma\in\mathcal{E}_I^k}\int_\sigma \mathcal{H}''(\zeta)\left[\!\left[\varrho_{h,\tau}^k\right]\!\right]^2 \mathrm{d}S(x)\le c
$$

Then the following holds

$$
\|\varrho_{h,\tau}\|_{L^2L^2}\le ch^{-\frac{\varepsilon+2}{2\gamma}} \text{ and } \|\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\|_{L^2L^2}\le ch^{-\frac{\varepsilon+2}{2\gamma}},
$$

where $c>0$ depends on the external force $\mathbf{f}$ and g and the initial data, but independent of the parameters τ and h.

Next, we report an estimate on the jump of the density from [30, Lemma 4.3].

Lemma 4.10. *Let $\varrho_{h,\tau},\mathbf{v}_{h,\tau}$ be a numerical solution of (4.12a) with $\gamma\ge 2$ and satisfies the estimates*

$$
\|\varrho_{h,\tau}\|_{L^\infty L^\gamma}\le c,\quad \|\mathrm{div}\mathbf{v}_{h,\tau}\|_{L^2L^2}\le c.
$$

Then the following holds

$$
\int_0^T\sum_{K\in\mathcal{T}_{h,\tau}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma \frac{[\![\varrho_{h,\tau}]\!]^2}{\max\{\varrho_{h,\tau}^{\mathrm{in}},\varrho_{h,\tau}^{\mathrm{out}}\}}|\left\langle\mathbf{v}_{h,\tau}\cdot\mathbf{n}\right\rangle_\sigma|\mathrm{d}S(x)\,\mathrm{d}t\le c,
$$

where $c>0$ depends on the external force $\mathbf{f}$ and g and the initial data, but independent of the parameters τ and h.

Finally, we show the following estimates that shall be used in the analysis of the convective terms.

Lemma 4.11. *Let $\varrho_{h,\tau},\mathbf{u}_{h,\tau},\mathbf{w}_{h,\tau}$ satisfy the estimates in (4.19). Then the following hold*

$$
\int_0^T\sum_{K\in\mathcal{T}_{h,\tau}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma |\,[\![\varrho_{h,\tau}]\!]\left\langle\mathbf{v}_{h,\tau}\cdot\mathbf{n}\right\rangle_\sigma^-|\mathrm{d}S(x)\,\mathrm{d}t\le c\tau^{-\frac{1}{4}}h^\theta, \tag{4.25a}
$$

$$
\int_0^T\sum_{K\in\mathcal{T}_{h,\tau}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\left\langle\mathbf{v}_{h,\tau}\cdot\mathbf{n}\right\rangle_\sigma^-|\mathrm{d}S(x)\,\mathrm{d}t\le c\tau^{-1/4}\,h^\zeta, \tag{4.25b}
$$

where $c>0$ depends on the external force $\mathbf{f}$ and g and the initial data, but independent of the parameters τ and h and

$$
\theta=\begin{cases}-\frac{1}{2} & \text{if } \gamma\ge\frac{6}{5},\\ \frac{3\gamma-6}{4\gamma} & \text{if } \gamma\in(1,\frac{6}{5}),\end{cases}\qquad \zeta=\begin{cases}-\frac{1}{2} & \text{if } \gamma\ge\frac{4}{3},\\ \frac{7\gamma-12}{4\gamma} & \text{if } \gamma\in(1,\frac{4}{3}).\end{cases}
$$

Proof. All the estimates below depend on the a-priori estimates on $\mathbf{v}_{h,\tau}$. In particular on the estimate

$$\|\mathbf{v}_{h,\tau}\|_{L^\infty(L^6)} \leq \|\mathbf{u}_{h,\tau}\|_{L^\infty(L^6)} + \|\mathbf{w}_{h,\tau}\|_{L^\infty(L^6)} \lesssim \tau^{-\frac{1}{2}},$$

where we used (4.9) for $\mathbf{u}_{h,\tau}$ and (4.22) for $\mathbf{w}_{h,\tau}$. On one hand, for $\gamma \geq 2$, we employ Lemma 4.10 to get

$$\begin{aligned}
&\int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma^-\,|\mathrm{d}S(x)\,\mathrm{d}t \\
&\lesssim \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{[\![\varrho_{h,\tau}]\!]^2}{\max\{\varrho_{h,\tau}^{\mathrm{in}},\varrho_{h,\tau}^{\mathrm{out}}\}}|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \max\{\varrho_{h,\tau}^{\mathrm{in}},\varrho_{h,\tau}^{\mathrm{out}}\}|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \\
&\lesssim h^{-1/2}\left(\|\varrho_{h,\tau}\|_{L^2L^2}\,\|\mathbf{v}_{h,\tau}\|_{L^2L^2}\right)^{1/2} \lesssim h^{-1/2}.
\end{aligned}$$

On the other hand, it is easy to check for $\gamma \in (1,2)$ that $\mathcal{H}''(r) = ar^{\gamma-2} \geq a$ if $r \leq 1$ and $r\mathcal{H}''(r) = ar^{\gamma-1} \geq a$ if $r \geq 1$. Therefore

$$\mathcal{H}''(r)(1+r) \geq a \text{ for all } r \in (0,\infty)$$

Applying these inequalities together with Hölder's inequality, and the estimate (4.19) we derive (by choosing $\varrho_{h,\tau}^\dagger$ conveniently and (4.8)) that

$$\begin{aligned}
&\int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma^-\,|\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq \frac{2}{\sqrt{a}} \int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \sqrt{\mathcal{H}''(\varrho_{h,\tau}^\dagger)}|\,[\![\varrho_{h,\tau}]\!]\,|\sqrt{|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|}\sqrt{(1+\varrho_{h,\tau}^\dagger)|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|}\mathrm{d}S(x) \\
&\leq \frac{2}{\sqrt{a}} \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \mathcal{H}''(\varrho_{h,\tau}^\dagger)\,[\![\varrho_{h,\tau}]\!]^2\,|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \\
&\quad\times \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma |\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,| + |\varrho_{h,\tau}^\dagger\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \\
&\lesssim h^{-1/2}\left(\|\mathbf{v}_{h,\tau}\|_{L^1L^1}^{1/2} + \|\varrho_{h,\tau}\|_{L^2L^{6/5}}\,\|\mathbf{v}_{h,\tau}\|_{L^2L^6}\right)^{1/2} \lesssim h^{-\frac{1}{2}}\tau^{-\frac{1}{4}}\,\|\varrho_{h,\tau}\|_{L^\infty L^{6/5}}^{1/2} =: I_1.
\end{aligned}$$

Then, for $\gamma \in [6/5,2)$ we have

$$I_1 \lesssim h^{-\frac{1}{2}}\tau^{-\frac{1}{4}}.$$

Concerning $\gamma \in (1,6/5)$ we deduce by inverse estimate (4.9) that

$$I_1 \lesssim h^{-\frac{1}{2}}\tau^{-\frac{1}{4}}h^{\frac{3}{2}(\frac{5}{6}-\frac{1}{\gamma})}\,\|\varrho_{h,\tau}\|_{L^\infty L^\gamma}^{1/2} \lesssim h^{\frac{5\gamma-6}{4\gamma}-\frac{1}{2}}\tau^{-\frac{1}{4}},$$

which completes the proof of the first estimate (4.25a).

Similarly, we prove the second estimate (4.25b) in two steps. First for $\gamma \geq 2$ we may derive it due to Hölder's inequality, trace theorem, and the inverse estimate (4.9) that

$$\begin{aligned}
&\int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}]\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma^-\,|\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{[\![\varrho_{h,\tau}]\!]^2}{\max\{\varrho_{h,\tau}^{\mathrm{in}},\varrho_{h,\tau}^{\mathrm{out}}\}}|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \\
&\quad\times \left(\int_0^T \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \max\{\varrho_{h,\tau}^{\mathrm{in}},\varrho_{h,\tau}^{\mathrm{out}}\}\,(\Pi_\mathcal{T}[\mathbf{u}_{h,\tau}])^2\,|\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2} \\
&\lesssim h^{-1/2}\left(\|\varrho_{h,\tau}\|_{L^\infty L^2}\,\|\mathbf{u}_{h,\tau}\|_{L^2L^6}^2\,\|\mathbf{v}_{h,\tau}\|_{L^\infty L^6}\right)^{1/2} \lesssim h^{-1/2}\tau^{-\frac{1}{4}}.
\end{aligned}$$

Next, we proceed to show the second estimates for $\gamma \in (1,2)$.

$$
\begin{aligned}
&\int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma^- \,|\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq \left(\int_0^T \sum_{\sigma\in\mathcal{E}_I}\int_\sigma \left[\!\!\left[\varrho_{h,\tau}^{\gamma/2}\right]\!\!\right]^2 |\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\mathrm{d}S(x)\right)^{1/2}\left(\int_0^T \sum_{\sigma\in\mathcal{E}_I}\int_\sigma \left\langle \varrho_{h,\tau}^{1-\gamma/2}\right\rangle_\sigma^2 |\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\,|\,(\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])^2\,\mathrm{d}S(x)\right)^{1/2} \\
&\lesssim h^{-1/2}\left(\int_0^T \|\varrho_{h,\tau}\|_{L^{2(2-\gamma)}}^{2-\gamma}\,\|\mathbf{v}_{h,\tau}\|_{L^6}\,\|\mathbf{u}_{h,\tau}\|_{L^6}^2\,\mathrm{d}t\right)^{1/2} \lesssim h^{-1/2}\,\|\varrho_{h,\tau}\|_{L^\infty L^{2(2-\gamma)}}^{(2-\gamma)/2}\,\|\mathbf{v}_{h,\tau}\|_{L^\infty L^6}^{1/2}\,\|\mathbf{u}_{h,\tau}\|_{L^2L^6} \\
&\lesssim h^{-1/2}\tau^{-\frac{1}{4}}\,\|\varrho_{h,\tau}\|_{L^\infty L^{2(2-\gamma)}}^{(2-\gamma)/2} =: I_2,
\end{aligned}
$$

where we have used the algebraic inequality for $\gamma \in (1,2)$ that

$$
[\![\varrho_{h,\tau}]\!]^2 \leq \left[\!\!\left[(\varrho_{h,\tau}^{\gamma/2})\right]\!\!\right]^2 \left(\left\langle \varrho_\tau^{1-\gamma/2}\right\rangle_\sigma\right)^2.
$$

If $\frac{4}{3} \leq \gamma$ it follows (as before) that

$$
I_2 \lesssim h^{-1/2}\tau^{-\frac{1}{4}}.
$$

On the other hand, if $1 < \gamma < \frac{4}{3}$ we complete the proof by the inverse estimates (4.9) and find

$$
I_2 \leq h^{-1/2}\tau^{-\frac{1}{4}}\,\|\varrho_{h,\tau}\|_{L^\infty(L^{2(2-\gamma)})}^{\frac{2-\gamma}{2}} \leq h^{-1/2}\tau^{-\frac{1}{4}}\left(h^{\frac{3}{2(2-\gamma)}-\frac{3}{\gamma}}\,\|\varrho_{h,\tau}\|_{L^\infty(L^\gamma)}\right)^{\frac{2-\gamma}{2}} = h^{-1/2}\tau^{-\frac{1}{4}}h^{\frac{9\gamma-12}{4\gamma}}
$$

which finishes the estimate. □

Finally, it is useful to reformulate the convective term as following, see [24, Chaper 9, Lemma 7].

Lemma 4.12. *([24, Chaper 9, Lemma 7]) Let $r, F \in Q(\mathcal{T}_{h,\tau}), \mathbf{v} \in V(\mathcal{T}_{h,\tau})$ and $\phi \in C^1(\mathcal{T}_{h,\tau})$. Then it holds*

$$
\int_{\Omega_\tau} r\mathbf{v}\cdot\nabla\phi\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K F\mathrm{div}_\tau^{\mathrm{up}}[r,\mathbf{v}]\;\mathrm{d}x + \sum_{i=1}^4 E_i(r)
$$

where

$$
\begin{aligned}
&E_1(r) = \sum_{K\in\mathcal{T}_{h,\tau}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma (F-\phi)\,[\![r]\!]\,\langle\mathbf{v}\cdot\mathbf{n}\rangle_\sigma^-\,\mathrm{d}S(x), \quad E_2(r) = \sum_{K\in\mathcal{T}_{h,\tau}}\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma \phi r\big(\mathbf{v}\cdot\mathbf{n} - \langle\mathbf{v}\cdot\mathbf{n}\rangle_\sigma\big)\mathrm{d}S(x), \\
&E_3(r) = \int_{\Omega_\tau} r(F-\phi)\mathrm{div}\mathbf{v}\,\mathrm{d}x, \quad E_4(r) = h^\varepsilon\sum_{\sigma\in\mathcal{E}_I}\int_\sigma [\![r]\!]\,[\![F]\!]\,\mathrm{d}S(x).
\end{aligned}
$$

4.5 Consistency

With the a-priori estimates derived in the last subsection, we are ready to show the consistency of the fully-discrete scheme (4.12) (or equivalently (4.13)). For the momentum equation we have to introduce the ϵ-layer again.

Theorem 4.13 (Consistency of the fully discrete scheme (4.12))**.**
Let $(\varrho_{h,\tau}, \mathbf{u}_{h,\tau}, \eta_{h,\tau})$ be the numerical solution of the scheme (4.12) with $\tau \approx h$, $\gamma > \frac{6}{5}$ and $\varepsilon \in (0, 2(\gamma-1))$. Then for any $\varphi \in C_0^2(0,T;\mathbb{R}^d)$ we have

$$
-\int_{\Omega_\tau}\varrho_{h,\tau}^0\varphi^0\,\mathrm{d}x - \int_0^T\int_{\Omega_\tau(t)}\left(\varrho_{h,\tau}\partial_t\varphi + \varrho_{h,\tau}\mathbf{u}_{h,\tau}\cdot\nabla\varphi\right)\mathrm{d}x \lesssim \mathcal{O}(h). \tag{4.26}
$$

If moreover, $\eta_{h,\tau} \to \eta$ in $C^\alpha([0,T]\times\Sigma)$ (for some $\alpha \in (0,1)$), then there exists a positive ϑ such that for all pairs $(\Psi,\psi) \in C_0^2(0,T\times\mathbb{R}^d)\times C_0^2([0,T]\times\Sigma)$ as constructed in (3.12) we have uniformly in ϵ that for all $\tau \leq \frac{1}{2}\epsilon$ and Ψ_ϵ satisfying (3.13) that

$$
\begin{aligned}
&-\int_{\Omega_\tau} \varrho_{h,\tau}^0 \mathbf{u}_{h,\tau}^0 \cdot \Psi_\epsilon^0 \,\mathrm{d}x - \int_0^T \int_{\Omega_\tau(t)} \left(\varrho_{h,\tau}\mathbf{u}_{h,\tau} \cdot \partial_t \Psi_\epsilon + \varrho_{h,\tau}\mathbf{u}_{h,\tau} \otimes \mathbf{u}_{h,\tau} : \nabla \Psi_\epsilon\right) \mathrm{d}x \\
&\qquad + \mu \int_0^T \int_{\Omega_\tau(t)} \nabla \mathbf{u}_{h,\tau}^k : \nabla \Psi_\epsilon \,\mathrm{d}x + (\mu+\lambda) \int_{\Omega_\tau(t)} \mathrm{div}\mathbf{u}_{h,\tau}^k \mathrm{div}\Psi_\epsilon \,\mathrm{d}x - \int_{\Omega_\tau(t)} p(\varrho_{h,\tau})\mathrm{div}\Psi_\epsilon \,\mathrm{d}x\,\mathrm{d}t \\
&-\int_\Sigma \partial_t \eta(0)\psi^0 \,\mathrm{d}\mathbf{r} - \int_0^T \int_\Sigma \delta_t \eta_{h,\tau} \partial_t \psi \,\mathrm{d}\mathbf{r} + \int_\Sigma K'(\eta_{h,\tau})\psi \,\mathrm{d}\mathbf{r}\,\mathrm{d}t - \int_0^T \int_\Sigma g_\tau \psi \,\mathrm{d}\mathbf{r}\,\mathrm{d}t - \int_0^T \int_{\Omega_\tau(t)} \mathbf{f}_\tau \cdot \Psi_\epsilon \,\mathrm{d}x\,\mathrm{d}t \lesssim \mathcal{O}(h^\vartheta).
\end{aligned} \tag{4.27}
$$

Proof. To show the consistency of the numerical scheme, we take $\Psi_{\epsilon,h} = \Pi_{\mathcal{T}}[\Psi_\epsilon]$ and the pair $(\Psi_{\epsilon,h}, \psi_{h,\tau}) = (\Pi_{\mathcal{E}}[\Psi_\epsilon], \Pi_{\mathcal{E}}[\psi])$ as the test functions in the discrete density and momentum equation, respectively. As mentioned already before due to the uniform conformity of the mesh with respect to time change we have bounds on the projection error independent of the time-step. And as before we will use below all quantities that are related to the triangulation like $\mathcal{T}_{h,\tau}, K, \sigma, \mathcal{E}$ as quantities that change from time-step to time-step.

We deal with each term separately:

Step 1 – time derivative terms. The consistency of the time derivative terms have been done in Theorem 3.12. Indeed, by recalling (3.17) and (3.16) and using the uniform in τ bounds on the spatial projection (4.10), we find that

$$
\begin{aligned}
\int_0^T \int_{\Omega_\tau(t)} D_t \varrho_{h,\tau} \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau(t=0)} \varrho_{h,\tau}^0 \varphi^0 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_\tau(t)} \varrho_{h,\tau}(t) \partial_t \varphi(t) \,\mathrm{d}x\,\mathrm{d}t& \\
+ \int_0^T \int_{\Omega_\tau(t)} \varrho_{h,\tau}(t) \mathbf{w}_{h,\tau}(t) \cdot \nabla \varphi(t) \,\mathrm{d}x = \mathcal{O}(\tau^\theta) + \mathcal{O}(h),\ \theta > 0,&
\end{aligned} \tag{4.28a}
$$

$$
\begin{aligned}
\int_0^T \int_{\Omega_\tau(t)} D_t(\varrho_{h,\tau}\mathbf{u}_{h,\tau}) \cdot \Psi_\epsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\tau(t=0)} \varrho_{h,\tau}^0 \mathbf{u}_{h,\tau}^0 \cdot \Psi_\epsilon^0 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_\tau(t)} \varrho_{h,\tau}\mathbf{u}_{h,\tau} \cdot \partial_t \Psi_\epsilon(t) \,\mathrm{d}x\,\mathrm{d}t& \\
+ \int_0^T \int_{\Omega_\tau(t)} (\varrho_{h,\tau}\mathbf{u}_{h,\tau} \otimes \mathbf{w}_{h,\tau}) : \nabla \Psi_\epsilon(t) \,\mathrm{d}x = \mathcal{O}(\tau^\theta),\ \theta > 0,&
\end{aligned} \tag{4.28b}
$$

$$
\int_0^T \int_\Sigma \delta_t z_{h,\tau} \psi \,\mathrm{d}\mathbf{r}\,\mathrm{d}t = \int_0^T \int_\Sigma z_{h,\tau} \partial_t \psi \,\mathrm{d}\mathbf{r}\,\mathrm{d}t - \int_\Sigma \psi^0 \partial_t \eta(0) \,\mathrm{d}\mathbf{r} + \mathcal{O}(\tau) + \mathcal{O}(h), \tag{4.28c}
$$

Step 2 – convective terms. We first deal with convective terms of the discrete density problem by setting $r = \varrho_{h,\tau}$, $\mathbf{v} = \mathbf{v}_{h,\tau}$, $\phi = \varphi$, and $F = \Pi_{\mathcal{T}}[\varphi]$ in Lemma 4.12

$$
\begin{aligned}
\int_0^T \int_{\Omega_\tau} \varrho_{h,\tau} \mathbf{v}_{h,\tau} \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \sum_{K \in \mathcal{T}_h} \int_K \Pi_{\mathcal{T}}[\varphi] \mathrm{div}_\tau^{\mathrm{up}}[\varrho_{h,\tau}, \mathbf{v}_{h,\tau}] \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^4 E_i \\
&= -\int_0^T \int_{\Omega_\tau} \varphi \mathrm{div}_\tau^{\mathrm{up}}[\varrho_{h,\tau}, \mathbf{v}_{h,\tau}] \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^4 E_i
\end{aligned}
$$

where

$$
\begin{aligned}
E_1(\varrho_{h,\tau}) &= \int_0^T \sum_{K \in \mathcal{T}_{h,\tau}} \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma (\Pi_{\mathcal{T}}[\varphi] - \varphi) [\![\varrho_{h,\tau}]\!] \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- \,\mathrm{d}S(x)\,\mathrm{d}t, \\
E_2(\varrho_{h,\tau}) &= \int_0^T \sum_{K \in \mathcal{T}_{h,\tau}} \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \varphi \varrho_{h,\tau} \big(\mathbf{v}_{h,\tau} \cdot \mathbf{n} - \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma\big) \mathrm{d}S(x)\,\mathrm{d}t \\
E_3(\varrho_{h,\tau}) &= \int_0^T \int_{\Omega_\tau} \varrho_{h,\tau} (\Pi_{\mathcal{T}}[\varphi] - \varphi) \mathrm{div}\mathbf{v}_{h,\tau} \,\mathrm{d}x\,\mathrm{d}t, \\
E_4(\varrho_{h,\tau}) &= h^\varepsilon \int_0^T \sum_{\sigma \in \mathcal{E}_I} \int_\sigma [\![\varrho_{h,\tau}]\!] [\![\Pi_{\mathcal{T}}[\varphi]]\!] \,\mathrm{d}S(x)\,\mathrm{d}t
\end{aligned}
$$

Next, we estimate the terms $\sum_{i=1}^4 E_i$.

Term $E_1(\varrho_{h,\tau})$ Applying Lemma 4.11 we get

$$|E_1(\varrho_{h,\tau})| \le h\,\|\varphi\|_{C^1} \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,\langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma^-\,|\mathrm{d}S(x)\,\mathrm{d}t \le ch^{\zeta_1},$$

where

$$\zeta_1 = \begin{cases} \frac{1}{4} & \text{if } \gamma \ge \frac{6}{5}, \\ \frac{3(\gamma-1)}{2\gamma} & \text{if } \gamma \in (1, \frac{6}{5}), \end{cases} \tag{4.29}$$

Obviously $\zeta_1 > 0$ for all $\gamma > 1$.

Term $E_2(\varrho_{h,\tau})$ It is easy to get from Hölder's inequality, the estimates (4.19), the fact that $\varrho_{h,\tau}$ is piece wise constant, Gauss theorem and Lemma 4.9 that

$$\begin{aligned}
|E_2(\varrho_{h,\tau})| &= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \varrho_{h,\tau}(\varphi - \langle\varphi\rangle_\sigma)\big(\mathbf{v}_{h,\tau}\cdot\mathbf{n} - \langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\big)\mathrm{d}S(x)\,\mathrm{d}t \right| \\
&= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \varrho_K \int_{\partial K} (\varphi - \langle\varphi\rangle_{\partial K})\big(\mathbf{v}_{h,\tau}\cdot\mathbf{n} - \langle \mathbf{v}_{h,\tau}\rangle_K\cdot\mathbf{n}\big) \right| \mathrm{d}S(x)\,\mathrm{d}t \\
&= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \varrho_K \int_K (\varphi - \langle\varphi\rangle_{\partial K})\mathrm{div}\mathbf{v}_{h,\tau} + \nabla\varphi\cdot(\mathbf{v}_{h,\tau} - \langle\mathbf{v}_{h,\tau}\rangle_K)\,\mathrm{d}x \right| \mathrm{d}t \\
&= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \varrho_K \int_K (\varphi - \langle\varphi\rangle_{\partial K})\mathrm{div}\mathbf{v}_{h,\tau} + (\nabla\varphi - \langle\nabla\varphi\rangle_K)\cdot\mathbf{v}_{h,\tau}\,\mathrm{d}x \right| \mathrm{d}t \\
&\lesssim h\,\|\varphi\|_{C^2}\,\|\varrho_{h,\tau}\|_{L^2L^2}\,(\|\mathrm{div}\mathbf{v}_{h,\tau}\|_{L^2L^2} + \|\mathbf{v}_{h,\tau}\|_{L^2L^2}) \le h^{\zeta_2},
\end{aligned} \tag{4.30}$$

where ζ_2 reads

$$\zeta_2 = \begin{cases} 1 - \frac{\varepsilon+2}{2\gamma} & \text{if } \gamma \in (1,2), \\ 1 & \text{if } \gamma \ge 2. \end{cases} \tag{4.31}$$

Obviously $\zeta_2 > 0$ as $\varepsilon < 2(\gamma-1)$.

Term $E_3(\varrho_{h,\tau})$ Applying Hölder's inequality and inverse estimate (4.9) we get

$$|E_3(\varrho_{h,\tau})| \le h\,\|\varphi\|_{C^1}\,\|\varrho_{h,\tau}\|_{L^2L^2}\,\|\mathrm{div}\mathbf{v}_{h,\tau}\|_{L^2L^2} \le h\,\|\varrho_{h,\tau}\|_{L^2L^2} \le h^{\zeta_2},$$

where $\zeta_2 > 0$ is the same as in (4.31).

Term $E_4(\varrho_{h,\tau})$ Applying Hölder's inequality, the interpolation estimate (4.10), the uniform bounds (4.19) and the fact the fact ($|a-b| \le a+b$) for $a, b \ge 0$ we get

$$\begin{aligned}
|E_4(\varrho_{h,\tau})| &= h^\varepsilon \Big| \int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma [\![\varrho_{h,\tau}]\!]\,[\![\Pi_\mathcal{T}[\varphi]]\!]\,\mathrm{d}S(x)\,\mathrm{d}t \Big| \\
&\le h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma |\,[\![\varrho_{h,\tau}]\!]\,|\mathrm{d}S(x)\,\mathrm{d}t \le h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma 2\overline{\varrho}_\sigma \mathrm{d}S(x)\,\mathrm{d}t \\
&\le h^\varepsilon\,\|\varrho_{h,\tau}\|_{L^1(0,T;\Omega_{h,\tau})} \le h^\varepsilon.
\end{aligned}$$

Consequently, we derive

$$\int_0^T \int_{\Omega_\tau(\cdot)} \varrho_{h,\tau}\mathbf{v}_{h,\tau}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \sum_K \int_K \Pi_\mathcal{T}[\varphi]\mathrm{div}_\tau^{\mathrm{up}}[\varrho_{h,\tau}, \mathbf{v}_{h,\tau}]\ \mathrm{d}x\,\mathrm{d}t \le h^\theta, \quad \theta = \min\{\zeta_1, \zeta_2, \varepsilon\}. \tag{4.32}$$

Clearly, $\theta > 0$ for $\varepsilon \in (0, 2(\gamma-1))$ and $\gamma > 1$.

Next, we deal with convective terms in the discrete momentum problem by recalling Lemma 4.12 with $r = \varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]$, $\mathbf{v} = \mathbf{v}_{h,\tau}$, $\phi = \Psi_\epsilon$, $F = \Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\varphi]]$

$$
\begin{aligned}
\int_0^T \int_{\Omega_\tau} \varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \otimes \mathbf{v}_{h,\tau} : \nabla \Psi_\epsilon \, \mathrm{d}x \, \mathrm{d}t &= -\int_0^T \sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}_{\mathcal{T}}^{\mathrm{up}}(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}], \mathbf{v}_{h,\tau}) \cdot \Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\Psi_\epsilon]] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \sum_{i=1}^4 E_i(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) \\
= -\int_0^T \int_{\Omega_\tau} \mathrm{div}_{\mathcal{T}}^{\mathrm{up}}(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}], \mathbf{v}_{h,\tau}) \cdot \Pi_{\mathcal{E}}[\Psi_\epsilon] \ \mathrm{d}x \ \ \mathrm{d}t + \sum_{i=1}^4 E_i(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])
\end{aligned}
$$

where

$$
\begin{aligned}
E_1(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) &= \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (\Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\Psi_\epsilon]] - \Psi_\epsilon) \left[\!\left[\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right] \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- \, \mathrm{d}S(x) \, \mathrm{d}t, \\
E_2(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) &= \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \Psi_\epsilon \cdot (\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) \big(\mathbf{v}_{h,\tau} \cdot \mathbf{n} - \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma \big) \mathrm{d}S(x) \, \mathrm{d}t \\
E_3(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) &= \int_0^T \int_{\Omega_\tau} \varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \cdot (\Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\Psi_\epsilon]] - \Psi_\epsilon) \mathrm{div} \mathbf{v}_{h,\tau} \, \mathrm{d}x \, \mathrm{d}t, \\
E_4(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) &= h^\varepsilon \frac{1}{2} \int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma \left[\!\left[\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right] \cdot \left[\!\left[\Pi_{\mathcal{T}}[\Pi_{\mathcal{E}}[\Psi_\epsilon]] \right]\!\right] \mathrm{d}S(x) \, \mathrm{d}t
\end{aligned}
$$

Next, we estimate the terms $\sum_{i=1}^4 E_i(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])$.

Term $E_1(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])$ By Hölder's inequality and the interpolation estimate (4.10) we get

$$
\begin{aligned}
&|E_1(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])| \leq h \|\Psi_\epsilon\|_{C^1} \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma | \left[\!\left[\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right] \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- | \mathrm{d}S(x) \, \mathrm{d}t \\
&\leq h \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma | \big(\left[\!\left[\varrho_{h,\tau} \right]\!\right] \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] + \varrho_{h,\tau}{}^{\mathrm{out}} \left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right] \big) \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- | \mathrm{d}S(x) \, \mathrm{d}t =: I_1 + I_2,
\end{aligned}
$$

where we have also applied the chain rule

$$
\left[\!\left[uv \right]\!\right]_\sigma = u_\sigma^{\mathrm{in}} \left[\!\left[v \right]\!\right]_\sigma + \left[\!\left[u \right]\!\right]_\sigma v_\sigma^{\mathrm{out}} \text{ for all } u, v \in Q.
$$

Applying the estimate (4.25b) we get the estimates of the first term

$$
I_1 = h \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma | \left[\!\left[\varrho_{h,\tau} \right]\!\right] \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma | \mathrm{d}S(x) \, \mathrm{d}t \leq h^{\zeta_3}
$$

where

$$
\zeta_3 = \begin{cases} 1/4 & \text{if } \gamma \in [\frac{4}{3}, \infty), \\ (5\gamma - 6)/(2\gamma) & \text{if } \gamma \in (1, \frac{4}{3}), \end{cases} \quad \zeta_3 > 0 \text{ provided } \gamma > \frac{6}{5}.
$$

The second term I_2 can be estimates by

$$
\begin{aligned}
I_2 &= h \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma |\varrho_{h,\tau}^{\mathrm{out}} \left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right] \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- | \mathrm{d}S(x) \, \mathrm{d}t \\
&\leq h \left(\int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma \varrho_{h,\tau}^{\mathrm{out}} | \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- | \left[\!\left[\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \right]\!\right]^2 \mathrm{d}S(x) \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}_I} \int_\sigma \varrho_{h,\tau}^{\mathrm{out}} | \langle \mathbf{v}_{h,\tau} \cdot \mathbf{n} \rangle_\sigma^- | \mathrm{d}S(x) \, \mathrm{d}t \right)^{1/2} \\
&\leq h^{1/2} \|\varrho_{h,\tau}\|_{L^1 L^{6/5}}^{1/2} \|\mathbf{v}_{h,\tau}\|_{L^\infty L^6}^{1/2} \lesssim h^{1/4} \|\varrho_{h,\tau}\|_{L^\infty L^{6/5}}^{1/2},
\end{aligned}
$$

where we have used the estimate $(4.19)_4$. It is obvious that $I_2 \lesssim h^{\frac{1}{4}}$ for $\gamma \geq \frac{6}{5}$. Further by the inverse estimate we derive for $\gamma \in (1, \frac{6}{5})$ that

$$I_2 \lesssim h^{\frac{1}{4}} h^{\frac{3}{2}(\frac{5}{6}-\frac{1}{\gamma})} \left\| \varrho_{h,\tau} \right\|_{L^\infty L^\gamma}^{1/2} \lesssim h^{\frac{3(\gamma-1)}{2\gamma}}.$$

Consequently, $I_2 \lesssim h^{\zeta_1}$, and $\zeta_1 > 0$ for all $\gamma > 1$, see (4.29).

Combining the estimates of the terms I_1 and I_2 we get

$$E_1(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]) \leq h^{\zeta_3} + h^{\zeta_1}.$$

Term $E_2(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])$ We proceed as in (4.30) using the fact that $\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]$ is constant on each K. Hence we find analogously

$$\begin{aligned}
&|E_2(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])| = |\int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \cdot (\Psi_\epsilon - \langle\Psi_\epsilon\rangle_\sigma)\big(\mathbf{v}_{h,\tau}\cdot\mathbf{n} - \langle \mathbf{v}_{h,\tau}\cdot\mathbf{n}\rangle_\sigma\big)\mathrm{d}S(x)\,\mathrm{d}t| \\
&= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{j=1}^d \varrho_K \Pi_{\mathcal{T}}[(u_K)_j] \int_{\partial K} (\Psi_\epsilon^j - \langle\Psi_\epsilon^j\rangle_{\partial K})\big(\mathbf{v}_{h,\tau}\cdot\mathbf{n} - \langle\mathbf{v}_{h,\tau}\rangle_K\cdot\mathbf{n}\big)\mathrm{d}S(x) \right| \mathrm{d}t \\
&= \left| \int_0^T \sum_{K\in\mathcal{T}_{h,\tau}} \sum_{j=1}^d \varrho_K \Pi_{\mathcal{T}}[(u_K)_j] \int_K (\Psi_\epsilon^j - \langle\Psi_\epsilon^j\rangle_{\partial K})\mathrm{div}\mathbf{v}_{h,\tau} + (\nabla\Psi_\epsilon^j - \langle\nabla\Psi_\epsilon^j\rangle_K)\cdot\mathbf{v}_{h,\tau}\,\mathrm{d}x \right| \mathrm{d}t \\
&\leq h \left\|\Psi_\epsilon\right\|_{C^2} \left\|\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\right\|_{L^2L^2} \left(\left\|\mathrm{div}\mathbf{v}_{h,\tau}\right\|_{L^2L^2} + \left\|\mathbf{v}_{h,\tau}\right\|_{L^2L^2}\right) \lesssim h^{\zeta_2},
\end{aligned}$$

where $\zeta_2 > 0$ is given in (4.31).

Term $E_3(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])$ Employing Hölder's inequality, the interpolation estimate (4.10) and the estimate (4.19) we derive

$$|E_3(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])| \leq h \left\|\Psi_\epsilon\right\|_{C^1} \left\|\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\right\|_{L^2L^2} \left\|\mathrm{div}\mathbf{v}_{h,\tau}\right\|_{L^2L^2} \leq h^{\zeta_2}$$

where $\zeta_2 > 0$ is given in (4.31).

Term $E_4(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])$ Using Hölder's inequality, the interpolation estimate (4.10), and the estimate (4.19) we derive

$$\begin{aligned}
&|E_4(\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}])| = h^\varepsilon |\int_0^T \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma [\![\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]]\!]\, [\![\Pi_{\mathcal{T}}[\Psi_\epsilon]]\!]\,\mathrm{d}S(x)\,\mathrm{d}t| \\
&\leq h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma |\, [\![\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]]\!]\, |\mathrm{d}S(x)\,\mathrm{d}t \leq h^{\varepsilon+1} \int_0^T \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma 2\overline{\varrho_{h,\tau}|\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]|}\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq h^\varepsilon \left\|\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}]\right\|_{L^1(0,T;\Omega_{h,\tau})} \leq h^\varepsilon.
\end{aligned}$$

Consequently, we derive

$$\int_0^T \int_{\Omega_\tau} (\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}] \otimes \mathbf{v}_{h,\tau}) : \nabla\Psi_\epsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}_{\mathcal{T}}^{\mathrm{up}}[\varrho_{h,\tau}\Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}], \mathbf{v}_{h,\tau}] \cdot \Pi_{\mathcal{E}}[\Psi_\epsilon]\ \mathrm{d}x\,\mathrm{d}t \leq h^\theta \quad (4.33)$$

where $\theta = \min\{\zeta_1, \zeta_2, \zeta_3, \varepsilon\} > 0$ provided $\gamma > \frac{6}{5}$ and $\varepsilon \in (0, 2(\gamma-1))$.

Step 3 – pressure and diffusion terms. First, it is easy to calculate

$$\int_{\Omega_\tau} p_{h,\tau}\mathrm{div}\Pi_{\mathcal{E}}[\Psi_\epsilon]\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} p_K \int_K \mathrm{div}\Pi_{\mathcal{E}}[\Psi_\epsilon]\ \mathrm{d}x = \sum_{K\in\mathcal{T}_h} p_K \int_{\mathcal{E}(K)} \Psi_\epsilon\cdot\mathbf{n} = \int_{\Omega_\tau} p_{h,\tau}\mathrm{div}\Psi_\epsilon\,\mathrm{d}x. \quad (4.34)$$

Similarly for the physical diffusion term we have

$$\int_{\Omega_\tau} \mathrm{div}\mathbf{u}_{h,\tau}\mathrm{div}\Pi_{\mathcal{E}}[\Psi_\epsilon]\,\mathrm{d}x = \int_{\Omega_\tau} \mathrm{div}\mathbf{u}_{h,\tau}\mathrm{div}\Psi_\epsilon\,\mathrm{d}x, \quad \int_{\Omega_\tau} \mathbf{D}(\mathbf{u}_{h,\tau}) : \nabla\Pi_{\mathcal{E}}[\Psi_\epsilon]\,\mathrm{d}x = \int_{\Omega_\tau} \mathbf{D}(\mathbf{u}_{h,\tau}) : \nabla\Psi_\epsilon\,\mathrm{d}x. \tag{4.35}$$

Concerning the penalty diffusion term, we control it as follows

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} [\![\mathbf{u}_{h,\tau}]\!] \cdot [\![\Pi_{\mathcal{E}}[\Psi_\epsilon]]\!]\,\mathrm{d}S(x) &= \sum_{\sigma\in\mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} [\![\mathbf{u}_{h,\tau}]\!] \cdot [\![\Pi_{\mathcal{E}}[\Psi_\epsilon] - \Psi_\epsilon]\!]\,\mathrm{d}S(x) \lesssim \|\mathbf{u}_{h,\tau}\|_{H^1_Y} \|\Pi_{\mathcal{E}}[\Psi_\epsilon] - \Psi_\epsilon\|_{H^1_Y} \\ &\lesssim h\,\|\mathbf{u}_{h,\tau}\|_{H^1_Y} \|\Psi_\epsilon\|_{W^{2,2}}\end{aligned}$$

where we have used Hölder's inequality, the interpolation error (4.11) and the fact $[\![\Psi_\epsilon]\!] \equiv 0$.

Step – 4 rest of the structure part and external forces. By the standard interpolation error, we have

$$\begin{aligned}\int_0^T \int_\Sigma \Delta\eta_{h,\tau}\Delta\psi_{h,\tau}\,\mathrm{d}\mathbf{r} &= \int_0^T \int_\Sigma \Delta\eta_{h,\tau}\Delta\psi\,\mathrm{d}\mathbf{r} + \mathcal{O}(h), \\ \int_0^T \int_{\Omega_\tau} \varrho_{h,\tau}\mathbf{f}_{h,\tau}\cdot\Psi_\epsilon\,\mathrm{d}x + \int_0^T \int_\Sigma g_{h,\tau}\psi_{h,\tau}\,\mathrm{d}\mathbf{r} &= \int_0^T \int_{\Omega_\tau} \varrho_{h,\tau}\mathbf{f}_{h,\tau}\cdot\Psi_\epsilon\,\mathrm{d}x + \int_0^T \int_\Sigma g_\tau\psi\,\mathrm{d}\mathbf{r} + \mathcal{O}(h).\end{aligned} \tag{4.36}$$

Finally, collecting all the above terms we finish the proof. □

4.6 Existence of numerical solution

We aim to show the existence of the numerical solution for the nonlinear scheme (4.13) in this section. We closely follow previous approaches from [29] which we adapt for the the situation of time-dependent domain. To this end, we first introduce an abstract theorem, see [29, Theorem A.1].

Theorem 4.14. *([29, Theorem A.1])*
Let M and N be positive integers. Let $C_1 > \epsilon > 0$ and $C_2 > 0$ be real numbers. Let V and W be defined as follows:

$$V = \{(x, y) \in R^M \times R^N,\ x > 0\},$$
$$W = \{(x, y) \in R^M \times R^N,\ \epsilon < x < C_1 \text{ and } \|y\| \le C_2\},$$

where the notation $x > c$ means that each component of y is greater than c, and $\|\cdot\|$ is a norm defined over R^N. Let F be a continuous function from $V \times [0,1]$ to $R^M \times R^N$ satisfying:

1. *$\forall \zeta \in [0,1]$, if $v \in V$ is such that $F(v,\zeta) = 0$ then $v \in W$,*
2. *The equation $F(v, 0) = 0$ is a linear system on v and has a solution in W.*

Then there exists at least a solution $v \in W$ such that $F(v, 1) = 0$.

Now we are ready to state the existence of numerical solution for the scheme (4.12).

Theorem 4.15 (Existence of a numerical solution and positivity of the density.)**.**
Let $0 < \varrho^{k-1}_{h,\tau} \in Q_h(\Omega^{k-1}_{h,\tau})$, $(\mathbf{u}^{k-1}_{h,\tau}, \eta^{k-1}_{h,\tau}, z^{k-1}_{h,\tau}) \in V_h(\Omega^{k-1}_{h,\tau}) \times W_h(\Sigma) \times W_h(\Sigma)$ be given. For simplicity, we denote $\Omega^k_{h,\tau}$ as $\Omega_{h,\tau}$. Then there exists $0 < \varrho^k_{h,\tau} \in Q_h(\Omega_{h,\tau})$ and $(\mathbf{u}^k_{h,\tau}, \eta^k_{h,\tau}, z^k_{h,\tau} := \frac{\eta^k_{h,\tau}-\eta^{k-1}_{h,\tau}}{\tau}) \in V_h(\Omega_{h,\tau}) \times W_h(\Sigma) \times W_h(\Sigma)$ satisfying the discrete problem (4.12), where $\eta^k_{h,\tau} = \eta^{k-1}_{h,\tau} + \tau z^k_{h,\tau}$.

Proof. Let us denote $U^k_{h,\tau} = (\mathbf{u}^k_{h,\tau}, z^k_{h,\tau})$, $\mathbb{Q}(\Omega_{h,\tau}) = \{(\mathbf{\Psi}, \phi) \in V_h(\Omega_{h,\tau}) \times W_h(\Sigma) | \mathbf{\Psi}|_\Sigma = \psi\mathbf{e}_d\}$, and define

$$V = \{(\varrho^k_{h,\tau}, U^k_{h,\tau}) \in Q_h(\Omega_{h,\tau}) \times \mathbb{Q},\ \varrho^k_{h,\tau} > 0\}.$$

It is obvious that the degrees of freedom of the spaces $Q_h(\Omega_{h,\tau})$ and $\mathbb{Q}(\Omega_{h,\tau})$ are finite. Indeed, the space $Q_h(\Omega_{h,\tau})$ can be identified by the set of values ϱ_K for all $K \in \mathcal{T}^k_h$, therefore $Q_h(\Omega_{h,\tau}) \subset \mathbb{R}^M$, where M is the total number of elements of $\mathcal{T}^k_h$. Analogously, $\mathbb{Q}(\Omega_{h,\tau}) \subset \mathbb{R}^N$, where N is the sum of d times degrees of freedom of $\mathcal{E}^k$ and the degrees of freedom of Σ.

Let us consider the mapping

$$
\begin{aligned}
F : V \times [0,1] &\longrightarrow Q_h \times \mathbb{Q} \\
(\varrho_{h,\tau}^k, U_{h,\tau}^k, \zeta) &\longmapsto (\varrho^\star, U^\star) = F(\varrho_{h,\tau}^k, U_{h,\tau}^k, \zeta),
\end{aligned}
$$

where $(\varrho^\star, U^\star) \in Q_h \times \mathbb{Q}$ is such that

$$
\int_{\Omega_\tau} \varrho^\star \varphi_{h,\tau} \,\mathrm{d}x = \int_{\Omega_\tau} \frac{\varrho_{h,\tau}^k - \varrho_{h,\tau}^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \varphi_{h,\tau} \,\mathrm{d}x + \zeta \int_{\Omega_\tau} \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k) \varphi_{h,\tau} \,\mathrm{d}x; \tag{4.37a}
$$

$$
\begin{aligned}
\int_{\Omega_\tau} U^\star \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x &= \int_{\Omega_\tau} \frac{\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k] - (\varrho_{h,\tau}^{k-1} \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^{k-1}]) \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x \\
&+ \int_\Sigma \frac{z_{h,\tau}^k - z_{h,\tau}^{k-1}}{\tau} \psi_{h,\tau} \, d\mathbf{r} + \int_\Sigma \Delta \eta_{h,\tau}^k \Delta \psi_{h,\tau} \, d\mathbf{r} - \int_{\Omega_\tau} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \int_\Sigma g_\tau^k \psi_{h,\tau} \, d\mathbf{r} \\
&+ \zeta \int_{\Omega_\tau} \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k], \mathbf{v}_{h,\tau}^k) \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x - \zeta \int_{\Omega_\tau} p(\varrho_{h,\tau}^k) \mathrm{div} \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \zeta \lambda \int_{\Omega_\tau} \mathrm{div} \mathbf{u}_{h,\tau}^k \mathrm{div} \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x \\
&+ \zeta 2\mu \int_{\Omega_\tau} \mathbf{D}(\mathbf{u}_{h,\tau}^k) : \nabla \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + (1-\zeta) 2\mu \int_{\Omega_\tau} \left((\mathbb{J}_0^k)^{-1} \mathbf{D}(\mathbf{u}_{h,\tau}^k) \right) : \left((\mathbb{J}_0^k)^{-1} \nabla \mathbf{\Psi}_{h,\tau} \right) \,\mathrm{d}x \\
&+ \zeta 2\mu \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right] \cdot [\![\mathbf{\Psi}_{h,\tau}]\!] \,\mathrm{d}S(x) + (1-\zeta) 2\mu \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right] \cdot [\![\mathbf{\Psi}_{h,\tau}]\!] \frac{|\sigma \circ \mathcal{A}|}{|\sigma|} \mathrm{d}S(x);
\end{aligned} \tag{4.37b}
$$

where

$$
\mathbf{\Psi}_{h,\tau} = (\mathbf{\Psi}_{h,\tau}, \psi_{h,\tau}), \quad \eta_{h,\tau}^k = \eta_{h,\tau}^{k-1} + \tau z_{h,\tau}^k, \quad \mathbf{u}_{h,\tau}^k|_\Sigma = z_{h,\tau}^k \mathbf{e}_d, \quad \mathcal{J}_k^{k-1} = \left(H + \eta_{h,\tau}^{k-1} \right) / \left(H + \eta_{h,\tau}^k \right).
$$

It is easy to check that F is continuous. Indeed, it is a one to one mapping, since the values of $\varrho^\star$ and $U^\star$ can be determined by setting $\varphi_{h,\tau} = 1_K$ in (4.37a), and $(\Phi_\tau)_i = 1_{D_\sigma}, (\Phi_\tau)_j = 0$ for $j \neq i$ in (4.37b).

Let $(\varrho_{h,\tau}^k, U_{h,\tau}^k) \in Q_h \times \mathbb{Q}$ and $\zeta \in [0,1]$ such that $F(\varrho_{h,\tau}^k, U_{h,\tau}^k, \zeta) = (0,0)$ (in particular $\varrho_{h,\tau}^k > 0$). Then for any $\left(\varphi_{h,\tau}, \mathbf{\Phi}_{h,\tau} = (\mathbf{\Psi}_{h,\tau}, \psi_{h,\tau})\right) \in Q_h \times \mathbb{Q}$

$$
\int_{\Omega_\tau} \frac{\varrho_{h,\tau}^k - \varrho_{h,\tau}^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \varphi_{h,\tau} \,\mathrm{d}x + \zeta \int_{\Omega_\tau} \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k, \mathbf{v}_{h,\tau}^k) \varphi_{h,\tau} \,\mathrm{d}x = 0; \tag{4.38a}
$$

$$
\begin{aligned}
&\int_{\Omega_\tau} \frac{\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k] - (\varrho_{h,\tau}^{k-1} \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^{k-1}]) \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \int_\Sigma \frac{z_{h,\tau}^k - z_{h,\tau}^{k-1}}{\tau} \psi_{h,\tau} \, d\mathbf{r} \\
&\qquad + \int_\Sigma \Delta \eta_{h,\tau}^k \Delta \psi_{h,\tau} \, d\mathbf{r} - \int_{\Omega_\tau} \varrho_{h,\tau}^k \mathbf{f}_\tau^k \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \int_\Sigma g_\tau^k \psi_{h,\tau} \, d\mathbf{r} \\
&+ \zeta \int_{\Omega_\tau} \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k \Pi_{\mathcal{T}}[\mathbf{u}_{h,\tau}^k], \mathbf{v}_{h,\tau}^k) \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x - \zeta \int_{\Omega_\tau} p(\varrho_{h,\tau}^k) \mathrm{div} \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \zeta \lambda \int_{\Omega_\tau} \mathrm{div} \mathbf{u}_{h,\tau}^k \mathrm{div} \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x \\
&+ \zeta 2\mu \int_{\Omega_\tau} \mathbf{D}(\mathbf{u}_{h,\tau}^k) : \nabla \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + (1-\zeta) 2\mu \int_{\Omega_\tau} \frac{1}{\mathcal{J}_0^k} \left((\mathbb{J}_0^k)^T \mathbf{D}(\mathbf{u}_{h,\tau}^k) \right) : \left((\mathbb{J}_0^k)^T \nabla \mathbf{\Psi}_{h,\tau} \right) \,\mathrm{d}x \\
&+ \zeta 2\mu \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right] \cdot [\![\mathbf{\Psi}_{h,\tau}]\!] \,\mathrm{d}S(x) + (1-\zeta) 2\mu \sum_{\sigma \in \mathcal{E}_{\mathrm{I}}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right] \cdot [\![\mathbf{\Psi}_{h,\tau}]\!] \frac{1}{|\mathcal{J}_0^k (\mathbb{J}_0^k)^{-T} \widehat{\mathbf{n}}|} \mathrm{d}S(x).
\end{aligned} \tag{4.38b}
$$

Taking $\varphi_{h,\tau} = 1$ as a test function in (4.38a) we obtain

$$
\left\| \varrho_{h,\tau}^k \right\|_{L^1(\Omega_{h,\tau})} = \int_{\Omega_\tau^k} \varrho_{h,\tau}^k \,\mathrm{d}x = \int_{\Omega_\tau^{k-1}} \varrho_{h,\tau}^{k-1} \,\mathrm{d}x > 0, \tag{4.39}
$$

which indicates the boundedness of $\varrho_{h,\tau}^k$ in the L^1 norm, and thus in all norms as the problem is of finite dimension. Following the same argument as Lemma 3.5 we know that $\varrho_{h,\tau}^k \geq 0$ provided $\varrho_{h,\tau}^{k-1} \geq 0$.

Taking $\mathbf{\Phi}_{h,\tau} = (\mathbf{u}_{h,\tau}^k, z_{h,\tau}^k)$ as the test function in (4.38b) and follow the proof of Theorem (3.7) gives

$$\left\| U_{h,\tau}^k \right\| := \left\| \nabla \mathbf{u}_{h,\tau}^k \right\|_{L^2(\Omega_{h,\tau})} + \left\| z_{h,\tau}^k \right\|_{L^2(\Sigma)} \leq C_1 \tag{4.40}$$

where C_1 depends on the data of the problem.

Further, let $K \in \mathcal{T}_h^k$ be such that ϱ_K^k is the smallest, i.e., $\varrho_K^k \leq \varrho_L^k$ for all $L \in \mathcal{T}_h^k$. We denote $K' = \mathcal{A}_{h,\tau}^{k-1} \circ (\mathcal{A}_{h,\tau}^k)^{-1}(K)$. Then a straightforward computation gives

$$\begin{aligned}
\frac{\varrho_K^k |K| - \varrho_{K'}^{k-1}|K'|}{\tau\zeta} &= -\int_K \mathrm{div}_\tau^{\mathrm{up}}(\varrho_{h,\tau}^k, \mathbf{u}_{h,\tau}^k) = -\sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_{h,\tau}^{k,up} \left\langle \mathbf{u}_{h,\tau}^k \cdot \mathbf{n} \right\rangle_\sigma + \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \underbrace{h^\varepsilon \left[\!\left[\varrho_{h,\tau}^k \right]\!\right]}_{\geq 0} \\
&\geq -\sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_K^k \left\langle \mathbf{u}_{h,\tau}^k \cdot \mathbf{n} \right\rangle_\sigma + \sum_{\sigma\in\mathcal{E}(K)} |\sigma| (\varrho_K^k - \varrho_{h,\tau}^{k,up}) \left\langle \mathbf{u}_{h,\tau}^k \cdot \mathbf{n} \right\rangle_\sigma \\
&= -|K| \varrho_K^k (\mathrm{div} \mathbf{u}_{h,\tau}^k)_K - \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \left[\!\left[\varrho_{h,\tau}^k \right]\!\right] \left\langle \mathbf{u}_{h,\tau}^k \cdot \mathbf{n} \right\rangle_\sigma^- \\
&\geq -|K| \varrho_K^k (\mathrm{div} \mathbf{u}_{h,\tau}^k)_K \geq -|K| \varrho_K^k |(\mathrm{div} \mathbf{u}_{h,\tau}^k)_K|.
\end{aligned}$$

Thus

$$\varrho_{h,\tau}^k \geq \varrho_K^k \geq \frac{|K'|}{|K|} \frac{\varrho_{K'}^{k-1}}{1 + \tau\zeta |(\mathrm{div} \mathbf{u}_{h,\tau}^k)_K|} > 0.$$

Consequently, by virtue of (4.40)

$$\varrho_{h,\tau}^k > \epsilon,$$

where ϵ depends only on the data of the problem.

Further, we get from (4.39) that

$$\varrho_{h,\tau}^k \leq \frac{\int_{\Omega_\tau^{k-1}} \varrho_{h,\tau}^{k-1} \,\mathrm{d}x}{\min_{K\in\mathcal{T}_h^k} |K|},$$

which indicates the existence of $C_2 > 0$ such that

$$\varrho_{h,\tau}^k < C_2.$$

Therefore, the Hypothesis 1 of Theorem 4.14 is satisfied.

Next, we proceed to show that the Hypothesis 2 of Theorem 4.14 is satisfied. Let $\zeta = 0$ then the system $F(\varrho_{h,\tau}^k, U_{h,\tau}^k) = 0$ reads

$$\varrho_{h,\tau}^k = \varrho_{h,\tau}^{k-1} \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}; \tag{4.41a}$$

$$\begin{aligned}
&\int_{\Omega_\tau} \frac{\varrho_{h,\tau}^k \Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^k] - (\varrho_{h,\tau}^{k-1} \Pi_\mathcal{T}[\mathbf{u}_{h,\tau}^{k-1}]) \circ \mathbf{X}_k^{k-1} \mathcal{J}_k^{k-1}}{\tau} \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \int_\Sigma \frac{z_{h,\tau}^k - z_{h,\tau}^{k-1}}{\tau} \psi_{h,\tau} \,\mathrm{d}\mathbf{r} \\
&\quad + 2\mu \int_{\Omega_\tau} \frac{1}{\mathcal{J}_0^k} \left((\mathbb{J}_0^k)^T \mathbf{D}(\mathbf{u}_{h,\tau}^k) \right) : \left((\mathbb{J}_0^k)^T \nabla \mathbf{\Psi}_{h,\tau} \right) \mathrm{d}x + 2\mu \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{1}{h} \left[\!\left[\mathbf{u}_{h,\tau}^k \right]\!\right] \cdot [\![\mathbf{\Psi}_{h,\tau}]\!] \frac{1}{|\mathcal{J}_0^k (\mathbb{J}_0^k)^{-T} \widehat{\mathbf{n}}|} \mathrm{d}S(x) \\
&\qquad + \int_\Sigma \left(\alpha \Delta \eta_{h,\tau}^k \Delta \psi_{h,\tau} + \beta \nabla \eta_{h,\tau}^k \nabla \psi_{h,\tau} \right) \mathrm{d}\mathbf{r} - \int_{\Omega_\tau} \varrho_{h,\tau}^k \mathbf{f}_{h,\tau}^k \cdot \mathbf{\Psi}_{h,\tau} \,\mathrm{d}x + \int_\Sigma g_\tau^k \psi_{h,\tau} \,\mathrm{d}\mathbf{r} = 0.
\end{aligned} \tag{4.41b}$$

To solve the above system (4.41), we further reformulate it on the reference domain according to (2.11)

$$\widehat{\varrho}_{h,\tau}^k \mathcal{J}_0^k = \widehat{\varrho}_{h,\tau}^{k-1} \mathcal{J}_0^{k-1}; \tag{4.42a}$$

$$\begin{aligned}
&\int_{\widehat{\Omega}} \widehat{\varrho}_{h,\tau}^{k-1} \mathcal{J}_0^{k-1} \frac{\Pi_\mathcal{T}[\widehat{\mathbf{u}}_{h,\tau}^k] - \Pi_\mathcal{T}[\widehat{\mathbf{u}}_{h,\tau}^{k-1}]}{\tau} \cdot \widehat{\mathbf{\Psi}}_{h,\tau} \,\mathrm{d}\widehat{x} + \int_\Sigma \frac{z_{h,\tau}^k - z_{h,\tau}^{k-1}}{\tau} \psi_{h,\tau} \,\mathrm{d}\mathbf{r} \\
&\quad + 2\mu \int_{\widehat{\Omega}} \widehat{\mathbf{D}}(\widehat{\mathbf{u}}_{h,\tau}^k) : \widehat{\nabla} \widehat{\mathbf{\Psi}}_{h,\tau} \,\mathrm{d}\widehat{x} + 2\mu \sum_{\sigma\in\mathcal{E}_\mathrm{I}} \int_\sigma \frac{1}{h} \left[\!\left[\widehat{\mathbf{u}}_{h,\tau}^k \right]\!\right] \cdot \left[\!\left[\widehat{\mathbf{\Psi}}_{h,\tau} \right]\!\right] \mathrm{d}S(\widehat{x})
\end{aligned}$$

$$+\int_{\Sigma}\left(\alpha\Delta\eta^k_{h,\tau}\Delta\psi_{h,\tau}+\beta\nabla\eta^k_{h,\tau}\nabla\psi_{h,\tau}\right)d\mathbf{r}-\int_{\widehat{\Omega}}\widehat{\mathbf{f}}^k_{\tau}\cdot\widehat{\mathbf{\Psi}}_{h,\tau}\widehat{\varrho}^{k-1}_{h,\tau}\mathcal{J}_0^{k-1}\,\mathrm{d}\widehat{x}+\int_{\Sigma}g^k\psi_{h,\tau}\,d\mathbf{r}=0, \quad (4.42b)$$

where $\mathcal{J}_0^{k-1}=1+\eta^{k-1}_{h,\tau}/H$ and $\mathcal{J}_0^{k}=1+\eta^{k}_{h,\tau}/H$. Realizing that (4.41b) is a linear system with a matrix being block-wise symmetric positive definite, we know that there exists exactly one solution $\widehat{U}^k_{h,\tau}=(\widehat{\mathbf{u}}^k_{h,\tau},z^k_{h,\tau})$. Then using the fact $\eta^k_{h,\tau}=\eta^{k-1}_{h,\tau}+\tau z^k_{h,\tau}$ we get $\eta^k_{h,\tau}$ and $\mathcal{A}^k_{h,\tau}$. Further, it is straightforward that $\mathbf{u}^k_{h,\tau}=\widehat{\mathbf{u}}^k_{h,\tau}\circ\mathcal{A}^k_{h,\tau}(\widehat{\mathbf{x}})$. Finally, substituting $\eta^k_{h,\tau}$ into (4.41a) we obtain the solution for $\varrho^k_{h,\tau}$. Obviously, $\varrho^k_{h,\tau}>0$ as long as no self touching. Thus the solution $(\varrho^k_{h,\tau},U^k_{h,\tau})$ belongs to V.

We have shown that both Hypothesis of Theorem 4.14 hold. Applying Theorem 4.14 finishes the proof. □

Conclusion

We have studied the fluid–structure interaction problem involving compressible viscous fluids. We have firstly proposed an energy stable time discretization scheme (3.1), see Theorem 3.7. Moreover, we have shown that the numerical solutions satisfy the renormalized equation and they are consistent with respect to the weak solutions, see Lemma 3.4 and Theorem 3.12, respectively. Further, we have developed a fully discretized mixed finite volume–finite element method (4.12). We have shown that the numerical solutions of (4.12) satisfy the renormalized equations (see Lemma 4.4) and they are consistent to the weak solutions as well (see Theorem 4.13). Finally, we have proven the existence of a numerical solution to the scheme (4.12) in Theorem 4.15 as well as the positivity of density.

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