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Abstract: We study classical solutions of the Robin problem for the Brinkman system and for the Darcy-Forchheimer-Brinkman system in a bounded domain with Ljapunov boundary.

Keywords: Brinkman system; Darcy-Forchheimer-Brinkman system; Robin problem; classical solution

1. Introduction

The paper is devoted to classical solutions of the Robin problem for the Darcy-Forchheimer-Brinkman system

$$\nabla p - \Delta \mathbf{v} + \lambda \mathbf{v} + a|\mathbf{v}|\mathbf{v} + b(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{F}, \quad \nabla\cdot\mathbf{v} = \psi \quad \text{in } \Omega, \tag{1.1}$$

$$T(\mathbf{v},p)\mathbf{n}^\Omega + h\mathbf{v} = \mathbf{g} \qquad \text{on } \partial\Omega. \tag{1.2}$$

Here $\Omega \subset R^m$ is a bounded domain with Ljapunov boundary, λ is a positive number, a, b and h are Hölder continuous functions with $h \geq 0$. The stress tensor $T(\mathbf{v},p)$ corresponding to the velocity $\mathbf{v}$ and the pressure p is given by

$$T(\mathbf{v},p) = 2\hat{\nabla}\mathbf{v} - pI, \qquad \hat{\nabla}\mathbf{v} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T].$$

The Darcy-Forchheimer-Brinkman system describes flows through porous media saturated with viscous incompressible fluids, where the inertia of such fluids is not negligible. The constants $\lambda, b > 0$ are determined by the physical properties of a porous medium. (For further details we refer the reader to the book [27, p. 17] and the references therein.)

Boundary value problems for the Darcy-Forchheimer-Brinkman system have been extensively studied in the recent years. The papers [7], [8], [12], [17], [22] and [25] concern the Dirichlet problem for the Darcy-Forchheimer-Brinkman system. The papers [1], [2], [10] and [13] are devoted to transmission problems that include the Darcy-Forchheimer-Brinkman system. M. Kohr at al. discussed in [11] the problem of Navier's type for the Darcy-Forchheimer-Brinkman system. The mixed Dirichlet-Robin problem and the mixed Dirichlet-Neumann problem for the Darcy-Forchheimer-Brinkman system (1.1) with $b = 0$ and $\psi = 0$ are studied in $H^{3/2}(\Omega,\mathbb{R}^3) \times H^{1/2}(\Omega)$ (see [11] and [9]). Here $\Omega \subset \mathbb{R}^3$ is a bounded creased domain with connected Lipschitz boundary. The paper [11] investigates the Robin problem (1.1), (1.2) with $b = 0$ and $\psi = 0$ in the space $H^s(\Omega,\mathbb{R}^m) \times H^{s-1}(\Omega)$, where $1 < s < 3/2$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary, $m \in \{2,3\}$. The paper [24] is devoted to the Robin problem (1.1), (1.2) in the Sobolev space $W^{1,q}(\Omega;\mathbb{R}^m) \times L^q(\Omega)$, where $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary, $m \in \{2,3\}$, and $3/2 < q \leq 3$.

We begin with the Brinkman system

$$-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = \psi \quad \text{in } \Omega \tag{1.3}$$

with the boundary condition (1.2). This problem has been studied in many papers, especially for $h \equiv 0$ (i.e. the Neumann problem for the Brinkman system). If $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $\mathbf{g} \in L^2(\partial\Omega; \mathbb{R}^m)$ then there exists a unique solution of the Neumann problem for the homogeneous Brinkman system in the sense of non-tangential limit. (See [3], [9], [16], [28].) (Remark that $\mathbf{v} \in W^{1,2}(\Omega; \mathbb{R}^m)$ and $p \in L^2(\Omega)$ for such solution $(\mathbf{v}, p)$.) If $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary, $s \in (0,1)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $\mathbf{g} \in H^{s-1}(\partial\Omega; \mathbb{R}^m)$, $\psi \in H^{s-1/2}(\Omega)$ and $\mathbf{f} \in H^{s-3/2}(\mathbb{R}^m; \mathbb{R}^m)$ is supported on $\overline{\Omega}$, then there exists a unique solution of the problem (1.3), (1.2) in $H^{s+1/2}(\Omega; \mathbb{R}^m) \times H^{s-1/2}(\Omega)$. (See [11], [12].) The papers [29], [30] prove the unique solvability of the Neumann problem for the Brinkman system in $W^{2,q}(\Omega; \mathbb{R}^m) \times W^{1,q}(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^m$ with boundary of class $\mathcal{C}^{2,1}$. The paper [24] is devoted to the Robin problem for the Brinkman system in Sobolev spaces $W^{1,q}(\Omega; \mathbb{R}^m) \times L^q(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary. Here one of the following three conditions is fulfilled: 1) $q = 2$; 2) $\partial\Omega$ is of class $\mathcal{C}^1$ and $1 < q < \infty$; 3) $2 \leq m \leq 3$ and $3/2 \leq q \leq 3$.

We study by the integral equation method the Robin problem for the homogeneous Brinkman system (1.3), i.e. for $\mathbf{f} \equiv 0$ and $\psi \equiv 0$. For this reason we define a Brinkman single layer potential and a Brinkman double layer potential and gather their properties. We look for a solution of the problem (1.3), (1.2) in an appropriate linear combination of a single layer potential and a double layer potential. Using boundary properties of potentials we obtain an integral equation on the boundary of the domain. We show the unique solvability of this integral equation and then the unique solvability of the problem (1.3), (1.2). Now we are able to concentrate on the Robin problem for the non-homogeneous Brinkman system. We prove that for a bounded domain $\Omega \subset \mathbb{R}^m$ with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \beta < \alpha < 1$, $\lambda \in (0, \infty)$, $h \in \mathcal{C}^\beta(\partial\Omega)$ with $h \geq 0$, $\mathbf{f} \in \mathcal{C}^\beta(\overline{\Omega}; \mathbb{R}^m)$, $\psi \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ and $\mathbf{g} \in \mathcal{C}^\beta(\partial\Omega; \mathbb{R}^m)$ there exists a unique solution $(\mathbf{v}, p)$ of the Robin problem (1.3), (1.2) in the space $\left[\mathcal{C}^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap \mathcal{C}^2(\Omega; \mathbb{R}^m)\right] \times \left[\mathcal{C}^\beta(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)\right]$. We obtain by the Fixed point theorem the existence of a classical solution of the Robin problem for the Darcy-Forchheimer-Brinkman system (1.1), (1.2) with Hölder continuous a and b.

2. Fundamental solution for the Brinkman system

We are going to study the Robin problem for the Brinkman system by the integral equation method. For this reason we need to define a fundamental solution for the Brinkman system and boundary layer potentials.

Let $0 \leq \lambda < \infty$. Suppose that $E^\lambda = (E^\lambda_{ij})$ is a matrix function of the type $m \times (m+1)$ and $Q^\lambda = (Q^\lambda_j)_{1 \leq j \leq m+1}$. We say that (E^λ, Q^λ) is a fundamental solution of the Brinkman system (1.3) in $\mathbb{R}^m$ if

$$-\Delta E^\lambda_{ij} + \lambda E^\lambda_{ij} + \partial_i Q^\lambda_j = \delta_{ij}\delta_0, \quad \partial_1 E^\lambda_{1j} + \dots \partial_m E^\lambda_{mj} = 0, \quad i, j \leq m, \tag{2.1}$$

$$-\Delta E^\lambda_{i,m+1} + \lambda E^\lambda_{i,m+1} + \partial_i Q^\lambda_{m+1} = 0, \quad \partial_1 E^\lambda_{1,m+1} + \dots \partial_m E^\lambda_{m,m+1} = \delta_0. \tag{2.2}$$

Here δ_0 is the unit measure concentrated at 0.

There exists a fundamental solution (E^λ, Q^λ) of the system (1.3) such that E^λ_{ij}, Q^λ_j are tempered distributions. If $\acute{E}^\lambda = (\acute{E}^\lambda_{ij})$, $\acute{Q}^\lambda = (\acute{Q}^\lambda_j)$ form another fundamental solution of the system (1.3) such that $\acute{E}^\lambda_{ij}$, $\acute{Q}^\lambda_j$ are tempered distributions, then $E^\lambda_{ij} - \acute{E}^\lambda_{ij}$, $Q^\lambda_j - \acute{Q}^\lambda_j$ are polynomials. (See [26, Theorem 10.3].) We use the fundamental solution introduced by W. Varnhorn in [33].

If $j \in \{1, \dots, m\}$ then

$$Q^\lambda_j(x) = E^\lambda_{j,m+1}(x) = \frac{1}{\omega_n}\frac{x_j}{|x|^m},$$

$$Q^\lambda_{m+1} = \begin{cases} \delta_0(x) + (\lambda/\omega_m)\ln |x|^{-1}, & m = 2, \\ \delta_0(x) + (\lambda/\omega_m)(m-2)^{-1}|x|^{2-m}, & m > 2, \end{cases}$$

where ω_m is the area of the unit sphere in $\mathbb{R}^m$. (See [33, p. 60].)

For $\lambda = 0$ we obtain the fundamental solution of the Stokes system. If $i, j \in \{1, \dots, m\}$, the components of E^0 are given by

$$E^0_{ij}(x) = \frac{1}{2\omega_m}\left\{\frac{\delta_{ij}}{(m-2)|x|^{m-2}} + \frac{x_i x_j}{|x|^m}\right\}, \quad m \geq 3, \tag{2.3}$$

$$E^0_{ij}(x) = \frac{1}{4\pi}\left\{\delta_{ij}\ln\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right\}, \quad m = 2 \tag{2.4}$$

(see, e.g., [33, p. 16]). The expression of E^λ for $\lambda > 0$ can be found in [33, Chapter 2]. We omit it for the sake of brevity.

If $i, j \leq m$ then $E^\lambda_{ij} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \{0\})$,

$$E^\lambda_{ij} - E^0_{ij} \in \mathcal{C}(\mathbb{R}^m) \tag{2.5}$$

by [33, p. 66],

$$|\nabla E^\lambda_{ij}(x) - \nabla E^0_{ij}(x)| = O(|x|^{2-m}), \qquad |x| \to 0 \tag{2.6}$$

by [22, Lemma 4.1]. If $\lambda > 0$ and β is a multiindex, then

$$\partial^\beta E^\lambda(x) = O(|x|^{-m-|\beta|}), \qquad |x| \to \infty \tag{2.7}$$

by [15, Lemma 3.1].

3. Brinkman boundary layer potentials

We will look for a solution of the Robin problem for the Brinkman system in the form of a linear combination of a Brinkman single layer potential and a Brinkman double layer potential. Let us define these potentials.

We denote $Q(x) := (Q^0_1(x), \dots, Q^0_m(x)) = (Q^\lambda_1(x), \dots, Q^\lambda_m(x))$. By $\tilde{E}^\lambda$ we denote the matrix of the type $m \times m$, where $\tilde{E}^\lambda_{ij}(x) = E^\lambda_{ij}(x)$ for $i, j \leq m$.

Let now $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary. If $\mathbf{g} \in \mathcal{C}(\partial\Omega, \mathbb{R}^m)$ then the single layer potential $(E^\lambda_\Omega \mathbf{g}, Q_\Omega \mathbf{g})$ with the density $\mathbf{g}$ for the Brinkman system (1.3) is given by

$$E^\lambda_\Omega \mathbf{g}(x) := \int_{\partial\Omega} \tilde{E}^\lambda(x-y)\mathbf{g}(y)\, \mathrm{d}\sigma(y),$$

$$Q_\Omega \mathbf{g}(x) := \int_{\partial\Omega} Q(x-y)\mathbf{g}(y)\, \mathrm{d}\sigma(y).$$

Then $E^\lambda_\Omega \mathbf{g} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \partial\Omega, \mathbb{R}^m)$, $Q_\Omega \mathbf{g} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \partial\Omega)$ and $\nabla Q_\Omega \mathbf{g} - \Delta E^\lambda_\Omega \mathbf{g} + \lambda E^\lambda_\Omega \mathbf{g} = 0$, $\nabla \cdot E^\lambda_\Omega \mathbf{g} = 0$ in $\mathbb{R}^m \setminus \partial\Omega$.

Set

$$K^\lambda_\Omega(x, y) := -T_y(\tilde{E}^\lambda(y-x), Q(y-x))\mathbf{n}^\Omega(y).$$

Remark that

$$[K_\Omega^0(x,y)]_{ij} = \frac{m}{\omega_m}\frac{(x_i-y_i)(x_j-y_j)(x-y)\cdot\mathbf{n}^\Omega(y)}{|x-y|^{m+2}} \tag{3.1}$$

(see for example [16, p. 38, 39, 132]). For $\mathbf{\Psi}\in\mathcal{C}(\partial\Omega,\mathbb{R}^m)$ define in $\mathbb{R}^m\setminus\partial\Omega$ the velocity part of the double layer potential with density $\mathbf{\Psi}$ by

$$(D_\Omega^\lambda\mathbf{\Psi})(x) := \int_{\partial\Omega} K_\Omega^\lambda(x,y)\mathbf{\Psi}(y)\;\mathrm{d}\sigma(y), \tag{3.2}$$

and the corresponding pressure part by

$$(\Pi_\Omega^\lambda\mathbf{\Psi})(x) := \int_{\partial\Omega} \Pi_\Omega^\lambda(x,y)\mathbf{\Psi}(y)\;\mathrm{d}\sigma(y). \tag{3.3}$$

If $m>2$ then

$$\Pi_\Omega^\lambda(x,y) = \frac{1}{\omega_m}\left\{-(y-x)\frac{2m(y-x)\cdot\mathbf{n}^\Omega(y)}{|y-x|^{m+2}} + \frac{2\mathbf{n}^\Omega(y)}{|y-x|^m} - \lambda\frac{|x-y|^{2-m}}{m-2}\mathbf{n}^\Omega(y)\right\}.$$

If $m=2$ then

$$\Pi_\Omega^\lambda(x,y) = \frac{1}{2\pi}\left\{-(y-x)\frac{4(y-x)\cdot\mathbf{n}^\Omega(y)}{|y-x|^{4}} + \frac{2\mathbf{n}^\Omega(y)}{|y-x|^2} - \lambda\left(\ln\frac{1}{|x-y|}\right)\mathbf{n}^\Omega(y)\right\}.$$

(See [33, pp. 61–62].) One has $D_\Omega^\lambda\mathbf{\Psi}\in\mathcal{C}^\infty(\mathbb{R}^m\setminus\partial\Omega,\mathbb{R}^m)$, $\Pi_\Omega^\lambda\mathbf{\Psi}\in\mathcal{C}^\infty(\mathbb{R}^m\setminus\partial\Omega)$ and $\nabla\Pi_\Omega^\lambda\mathbf{\Psi}-\Delta D_\Omega^\lambda\mathbf{\Psi}+\lambda D_\Omega^\lambda\mathbf{\Psi}=0$, $\nabla\cdot D_\Omega^\lambda\mathbf{\Psi}=0$ in $\mathbb{R}^m\setminus\partial\Omega$.

Proposition 3.1. *Let $\Omega\subset\mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$, $0<\beta<\alpha<1$ and $0\le\lambda<\infty$. Then $Q_\Omega:\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^\beta(\overline{\Omega})$, $E_\Omega^\lambda:\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ and $E_\Omega^\lambda-E_\Omega^0:\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^m)$ are bounded linear operators.*

Proof. For $\lambda=0$ see [20, Lemma 3.12].

Let now $\lambda>0$. Fix $\mathbf{\Psi}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$. Then $E_\Omega^\lambda\mathbf{\Psi}, E_\Omega^0\mathbf{\Psi}\in\mathcal{C}(\mathbb{R}^m;\mathbb{R}^m)$ by [34, Theorem 2.3]. Define the distributions $F_1,\dots,F_m$ by

$$\langle F_j,\varphi\rangle := \int_{\partial\Omega}\Psi_j\varphi\;\mathrm{d}\sigma.$$

Set $\mathbf{F}=(F_1,\dots,F_m)$. Since $E_\Omega^\lambda\mathbf{\Psi}=\tilde{E}^\lambda*\mathbf{F}$, properties of fundamental solutions give

$$-\Delta E_\Omega^\lambda\mathbf{\Psi}+\lambda E_\Omega^\lambda\mathbf{\Psi}-\nabla Q_\Omega\mathbf{\Psi}=\mathbf{F},$$

$$-\Delta E_\Omega^0\mathbf{\Psi}-\nabla Q_\Omega\mathbf{\Psi}=\mathbf{F}.$$

Subtracting

$$\Delta(E_\Omega^\lambda\mathbf{\Psi}-E_\Omega^0\mathbf{\Psi})=\lambda E_\Omega^\lambda\mathbf{\Psi}.$$

Since $E_\Omega^\lambda\mathbf{\Psi}\in\mathcal{C}(\mathbb{R}^m;\mathbb{R}^m)$ and $(E_\Omega^\lambda\mathbf{\Psi}-E_\Omega^0\mathbf{\Psi})\in\mathcal{C}(\mathbb{R}^m;\mathbb{R}^m)$, [23, Proposition 3.18.1] gives that $(E_\Omega^\lambda\mathbf{\Psi}-E_\Omega^0\mathbf{\Psi})\in\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^m)$. If $\mathbf{f}_k\to\mathbf{f}$ in $\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, then $E_\Omega^\lambda\mathbf{f}_k(x)-E_\Omega^0\mathbf{f}_k(x)\to E_\Omega^\lambda\mathbf{f}(x)-E_\Omega^0\mathbf{f}(x)$ for all $x\in\Omega$. The Closed graph theorem forces that $E_\Omega^\lambda-E_\Omega^0:\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^m)$ is a bounded operator. Since $\mathcal{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^m)\hookrightarrow\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$, the operator $E_\Omega^\lambda:\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ is bounded. $\square$

4. Boundary integral operators

If we study boundary behavior of potentials at boundary then crucial role is played by the integral operators $K_{\Omega,\lambda}$ and $K'_{\Omega,\lambda}$. Let us define them.

Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary and $0 \le \lambda < \infty$. For $\mathbf{\Psi} \in \mathcal{C}(\partial\Omega, \mathbb{R}^m)$ and $x \in \partial\Omega$ define

$$K_{\Omega,\lambda}\mathbf{\Psi}(x) := \lim_{\epsilon\searrow 0} \int_{\partial\Omega\setminus B(x;\epsilon)} K^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y),$$

$$K'_{\Omega,\lambda}\mathbf{\Psi}(x) := \lim_{\epsilon\searrow 0} \int_{\partial\Omega\setminus B(x;\epsilon)} K^\lambda_\Omega(y,x)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y),$$

where $B(x;\epsilon) = \{y; |x-y| < \epsilon\}$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$ and $0 \le \lambda < \infty$. Then there exists a constant C such that*

$$|K^\lambda_\Omega(x,y)| \le C|x-y|^{\alpha+1-m}, \qquad x,y \in \partial\Omega. \tag{4.1}$$

Proof. If $\lambda = 0$ then (4.1) holds by (3.1) and [23, Lemma 1.17.9].

Let now $\lambda > 0$. According to (2.6) there exists a constant C_1 such that

$$|K^\lambda_\Omega(x,y) - K^0_\Omega(x,y)| \le C_1|x-y|^{2-m}, \qquad x,y \in \partial\Omega.$$

This and the inequality (4.1) for $\lambda = 0$ give (4.1) for general λ. □

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ and $0 \le \lambda < \infty$. If $\mathbf{\Psi} \in \mathcal{C}(\partial\Omega, \mathbb{R}^m)$ and $x \in \partial\Omega$ then*

$$K_{\Omega,\lambda}\mathbf{\Psi}(x) = \int_{\partial\Omega} K^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y), \tag{4.2}$$

$$K'_{\Omega,\lambda}\mathbf{\Psi}(x) = \int_{\partial\Omega} K^\lambda_\Omega(y,x)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y) \tag{4.3}$$

are well defined. The operators $K_{\Omega,\lambda}$ and $K'_{\Omega,\lambda}$ are bounded on $\mathcal{C}(\partial\Omega;\mathbb{R}^m)$.

Proof. According to Lemma 4.1 there exists a constant C such that (4.1) holds. The rest is a consequence of [18, Theorem 2.22]. □

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \beta < \alpha < 1$ and $0 \le \lambda < \infty$. If $\mathbf{\Psi} \in \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$ and $z \in \partial\Omega$ then*

$$T(E^\lambda_\Omega\mathbf{\Psi}(z), Q_\Omega\mathbf{\Psi}(z))\mathbf{n}^\Omega(z) = \frac{1}{2}\mathbf{\Psi}(z) - K'_{\Omega,\lambda}\mathbf{\Psi}(z). \tag{4.4}$$

Proof. For $\lambda > 0$ see [34, Theorem 2.3]. Let now $\lambda = 0$. Then $E^0_\Omega\mathbf{\Psi} \in \mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$, $Q_\Omega\mathbf{\Psi} \in \mathcal{C}^\beta(\overline{\Omega})$ by Proposition 3.1. This forces $T(E^0_\Omega\mathbf{\Psi}.Q_\Omega\mathbf{\Psi})\mathbf{n}^\Omega \in \mathcal{C}(\partial\Omega;\mathbb{R}^m)$. The relation (4.4) holds for almost all $z \in \partial\Omega$ by [14, Lemma 3.1]. Since $\frac{1}{2}\mathbf{\Psi} - K'_{\Omega,0}\mathbf{\Psi} \in \mathcal{C}(\partial\Omega;\mathbb{R}^m)$ by Lemma 4.2, we deduce that (4.4) is true for all $z \in \partial\Omega$. □

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \beta < \alpha < 1$ and $0 \le \lambda < \infty$. Then the operators*

$$K'_{\Omega,\lambda} : \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m) \to \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m), \tag{4.5}$$

$$K'_{\Omega,\lambda} - K'_{\Omega,0} : \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m) \to \mathcal{C}^\alpha(\partial\Omega, \mathbb{R}^m) \tag{4.6}$$

are bounded.

Proof. If $\mathbf{\Psi} \in \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$ then

$$K'_{\Omega,\lambda}\mathbf{\Psi} = \frac{1}{2}\mathbf{\Psi} - T(E^\lambda_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi})\mathbf{n}^\Omega \tag{4.7}$$

by (4.4). The mapping $\mathbf{\Psi} \mapsto T(E^\lambda_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi})$ is bounded from $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$ to $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m \times \mathbb{R}^m)$ by Proposition 3.1. Since $\mathbf{n}^\Omega \in \mathcal{C}^\alpha(\partial\Omega; \mathbb{R}^m) \subset \mathcal{C}^\beta(\partial\Omega; \mathbb{R}^m)$, [23, Lemma 1.16.8] and (4.7) give that $K'_{\Omega,\lambda}$ is a bounded operator on $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$.

$$K'_{\Omega,\lambda} - K'_{\Omega,0} = T(E^0_\Omega, Q_\Omega)\mathbf{n}^\Omega - T(E^\lambda_\Omega, Q_\Omega)\mathbf{n}^\Omega = T(E^0_\Omega - E^\lambda_\Omega, 0)\mathbf{n}^\Omega \tag{4.8}$$

by (4.7). Proposition 3.1 gives that $T(E^0_\Omega - E^\lambda_\Omega, 0)$ is a bounded mapping from $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$ to $\mathcal{C}^\alpha(\partial\Omega, \mathbb{R}^m \times \mathbb{R}^m)$. Remember that $\mathbf{n}^\Omega \in \mathcal{C}^\alpha(\partial\Omega; \mathbb{R}^m)$. According to [23, Lemma 1.16.8] and (4.8) we infer that the operator (4.6) is bounded. □

We want to prove compactness of the operator $K'_{\Omega,\lambda}$ on the space $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$. For this we need the following auxiliary lemmas:

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \alpha < 1$. If $0 < \beta < 1$ then $K_{\Omega,0}$ is a compact operator on $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$.*

Proof. V. Maz'ya, M. Mitrea and T. Shaposhnikova proved in [21, p. 232] that $K_{\Omega,0}$ is a compact operator on some scale of Besov spaces including $B^{\infty,\infty}_\beta(\partial\Omega, \mathbb{R}^m)$. Remember that $B^{\infty,\infty}_\beta(\partial\Omega, \mathbb{R}^m) = \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$. (See for example [32, §3.5.3, Theorem] or [31, §2.5.7, Theorem].) □

Lemma 4.6. *Let $f \in \mathcal{C}^1(R^m \setminus \{0\})$ with $|\nabla f(x)| \le C_1|x|^\alpha$. Then there exists a constant C_2 such that $|f(x) - f(y)| \le C_2|x-y||x|^\alpha$ for $|x| > 2|x-y|$.*

Proof. Let $|x| > 2|x-y|$. Then there exists z in the interval xy such that $f(x) - f(y) = (x-y)\cdot\nabla f(z)$. Since $\frac{1}{2}|x| \le |z| \le \frac{3}{2}|x|$ we infer that

$$|f(x) - f(y)| = |(x-y)\cdot\nabla f(z)| \le C_1|x-y|[(1/2)^\alpha + (3/2)^\alpha]|x|^\alpha.$$

□

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \beta < \alpha < 1$. Then $K_{\Omega,0} + K'_{\Omega,0} : L^\infty(\partial\Omega, \mathbb{R}^m) \to \mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$ is a bounded operator.*

Proof. (3.1) yields

$$[K^0_\Omega(x,y) + K^0_\Omega(y,x)]_{ij} = \frac{m}{\omega_m}\frac{(x_i - y_i)(x_j - y_j)(x-y)\cdot(\mathbf{n}^\Omega(y) - \mathbf{n}^\Omega(x))}{|x-y|^{m+2}}. \tag{4.9}$$

If $x, y \in \partial\Omega$ then $|n^\Omega(x) - n^\Omega(y)| \le M_1|x-y|^\alpha$ for some constant M_1. Therefore there exists a constant C_1 such that

$$|K^0_\Omega(x,y) + K^0_\Omega(y,x)| \le C_1|x-y|^{\alpha+1-m} \qquad x, y \in \partial\Omega.$$

Hence there is a constant M_2 such that

$$\int\limits_{B(x;r)\cap\partial\Omega} |K^0_\Omega(x,y) + K^0_\Omega(y,x)| \, \mathrm{d}\sigma(y) \le M_2 r^\alpha \tag{4.10}$$

for each $x \in \partial\Omega$ and $r > 0$. (See [23, Lemma 1.26.1].) So,

$$|(K_{\Omega,0} + K'_{\Omega,0})\mathbf{f}(x)| \le C_2\|\mathbf{f}\|_{L^\infty(\partial\Omega;\mathbb{R}^m)} \qquad \forall x \in \partial\Omega$$

for some constant C_2.

If $\|\mathbf{f}\|_{L^\infty(\partial\Omega;\mathbb{R}^m)} \leq 1$ and $x, z \in \partial\Omega$ with $|x-z| \geq 1$ then

$$|(K_{\Omega,0} + K'_{\Omega,0})\mathbf{f}(x) - (K_{\Omega,0} + K'_{\Omega,0})\mathbf{f}(z)| \leq 2C_2 \leq 2C_2|x-z|^\beta.$$

According to Lemma 4.6 there exists a constant M_3 such that

$$|w^\gamma |w|^{-m-2} - u^\gamma |u|^{-m-2}| \leq M_3|w-u||w|^{-m} \qquad \text{for } |w| > 2|w-u| \tag{4.11}$$

for each multiindex γ with $|\gamma| = 3$. If $x, y, z \in \partial\Omega$ with $|y-z| > 2|x-z| = 2|(x-y)-(z-y)|$, then (4.9) and (4.11) yield

$$|[K^0_\Omega(x,y) + K^0_\Omega(y,x)]_{ij} - [K^0_\Omega(z,y) + K^0_\Omega(y,z)]_{ij}| \leq \frac{m}{\omega_m}|x-y|^{1-m}|\mathbf{n}^\Omega(z) - \mathbf{n}^\Omega(x)|$$

$$+\frac{m}{\omega_m}\left|\frac{(x_i-y_i)(x_j-y_j)(x-y)}{|x-y|^{m+2}} - \frac{(z_i-y_i)(z_j-y_j)(z-y)}{|z-y|^{m+2}}\right| \; |\mathbf{n}^\Omega(z) - \mathbf{n}^\Omega(y)|$$

$$\leq M_1\frac{m}{\omega_m}|x-y|^{1-m}|x-z|^\alpha + M_3\frac{m}{\omega_m}|x-z| \; |z-y|^{-m}M_1|z-y|^\alpha.$$

Therefore

$$|[K^0_\Omega(x,y) + K^0_\Omega(y,x)] - [K^0_\Omega(z,y) + K^0_\Omega(y,z)]| \leq \frac{C_3|x-z|^\alpha}{|x-y|^{m-1}} + \frac{C_3|x-z|}{|z-y|^{m-\alpha}} \tag{4.12}$$

for some constant C_3.

Suppose that $\|\mathbf{f}\|_{L^\infty(\partial\Omega;\mathbb{R}^m)} \leq 1$ and $x, z \in \partial\Omega$ with $0 < |x-z| < 1$. According to (4.10) and (4.12)

$$|(K_{\Omega,0} + K'_{\Omega,0})\mathbf{f}(x) - (K_{\Omega,0} + K'_{\Omega,0})\mathbf{f}(z)| \leq \int_{B(z;2|x-z|)\cap\partial\Omega} |K^0_\Omega(z,y)$$

$$+K^0_\Omega(y,z)| \, \mathrm{d}\sigma(y) + \int_{B(z;2|x-z|)\cap\partial\Omega} |K^0_\Omega(x,y) + K^0_\Omega(y,x)| \, \mathrm{d}\sigma(y)$$

$$+\int_{\partial\Omega\setminus B(z;2|x-z|)} |(K^0_\Omega(x,y) + K^0_\Omega(y,x)) - (K^0_\Omega(z,y) + K^0_\Omega(y,z))| \, \mathrm{d}\sigma(y)$$

$$\leq 2M_2|x-z|^\alpha + \int_{B(x;3|x-z|)\cap\partial\Omega} |K^0_\Omega(x,y) + K^0_\Omega(y,x)| \, \mathrm{d}\sigma(y)$$

$$+\int_{\partial\Omega\setminus B(z;2|x-z|)} \left(\frac{C_3|x-z|^\alpha}{|x-y|^{m-1}} + \frac{C_3|x-z|}{|z-y|^{m-\alpha}}\right) \, \mathrm{d}\sigma(y)$$

$$\leq 5M_2|x-z|^\alpha + C_3|x-z|^\beta \int_{\partial\Omega\setminus B(x;|x-z|)} |x-y|^{1-m+\alpha-\beta} \, \mathrm{d}\sigma(y)$$

$$+C_3|x-z|^\beta \int_{\partial\Omega\setminus B(z;2|x-z|)} |z-y|^{1-m+\alpha-\beta} \, \mathrm{d}\sigma(y)$$

$$\leq |x-z|^\beta \left[5M_2 + C_3\int_{\partial\Omega} (|x-y|^{1-m+\alpha-\beta} + |z-y|^{1-m+\alpha-\beta}) \, \mathrm{d}\sigma(y)\right] \leq C_4|x-z|^\beta$$

with C_4 independent on x and z (see [23, Lemma 1.26.1]). □

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ and $0 < \beta < \alpha < 1$. Suppose that $0 \leq \lambda < \infty$. Then $K'_{\Omega,\lambda}$ is a compact operator on $\mathcal{C}^\beta(\partial\Omega, \mathbb{R}^m)$.*

Proof. $K_{\Omega,0}+K'_{\Omega,0}: L^\infty(\partial\Omega,\mathbb{R}^m)\to\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$ is a bounded operator by Lemma 4.7. Since $\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)\hookrightarrow L^\infty(\partial\Omega,\mathbb{R}^m)$ compactly by [18, Theorem 7.4], the operator $K_{\Omega,0}+K'_{\Omega,0}$ is compact on $\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$. Thus $K'_{\Omega,0}=(K_{\Omega,0}+K'_{\Omega,0})-K_{\Omega,0}$ is a compact operator on $\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$ by Lemma 4.5.

The operator $K'_{\Omega,\lambda}-K'_{\Omega,0}:\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)\to\mathcal{C}^\alpha(\partial\Omega,\mathbb{R}^m)$ is bounded by Proposition 4.4. Since $\mathcal{C}^\alpha(\partial\Omega,\mathbb{R}^m)\hookrightarrow\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$ compactly by [18, Theorem 7.4], the operator $K'_{\Omega,\lambda}-K'_{\Omega,0}$ is compact on $\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$. Hence $K'_{\Omega,\lambda}=(K'_{\Omega,\lambda}-K'_{\Omega,0})+K'_{\Omega,0}$ is compact on $\mathcal{C}^\beta(\partial\Omega,\mathbb{R}^m)$. □

5. Robin problem for the Brinkman system

In this section we study the Brinkman system (1.3) with the Robin boundary condition (1.2). First we study the homogeneous Brinkman system, i.e. with $\mathbf{f}\equiv 0$ and $\psi\equiv 0$.

Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0<\beta<\alpha<1$ and $\lambda\in(0,\infty)$. If $\partial\Omega$ is connected then we shall look for a solution of the problem in the form of a single layer potential $(E^\lambda_\Omega\mathbf{\Phi},Q_\Omega\mathbf{\Phi})$ with $\mathbf{\Phi}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$. Unfortunately, it is not possible for Ω with disconnected boundary because

$$\int_C \mathbf{n}^\Omega\cdot E^\lambda_\Omega\mathbf{\Phi}\,\mathrm{d}\sigma=0$$

for each component C of $\partial\Omega$. Therefore we shall look for a solution in the form of a modified single layer potential. Let $G(1),\dots,G(k)$ be all bounded components of $\mathbb{R}^m\setminus\overline{\Omega}$. Fix open balls $B(j)$ such that $\overline{B(j)}\subset G(j)$. According to [6, Lemma 1.5.19] there exists $\mathbf{\Theta}\in\mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$ such that $\mathbf{\Theta}\cdot\mathbf{n}^\Omega>0$ on $\partial\Omega$. If $\mathbf{\Phi}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, we define the modified Brinkman single layer potential with density $\mathbf{\Phi}$ by

$$\hat{E}^\lambda_\Omega\mathbf{\Phi}:=E^\lambda_\Omega\mathbf{\Phi}+\sum_{j=1}^k\left(\int_{\partial G(j)}\mathbf{\Theta}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma\right)D^\lambda_{B(j)}\mathbf{n}^{B(j)},$$

$$\hat{Q}^\lambda_\Omega\mathbf{\Phi}:=Q_\Omega\mathbf{\Phi}+\sum_{j=1}^k\left(\int_{\partial G(j)}\mathbf{\Theta}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma\right)\Pi^\lambda_{B(j)}\mathbf{n}^{B(j)}.$$

(If $\partial\Omega$ is connected then $\hat{E}^\lambda_\Omega\mathbf{\Phi}=E^\lambda_\Omega\mathbf{\Phi}$ and $\hat{Q}^\lambda_\Omega\mathbf{\Phi}=Q_\Omega\mathbf{\Phi}$.)

Theorem 5.1. *Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0<\beta<\alpha<1$ and $\lambda\in(0,\infty)$. Let $h\in\mathcal{C}^\beta(\partial\Omega)$ with $h\geq 0$.*

(1) For $\mathbf{\Phi}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$ define

$$\tau^\lambda_{\Omega,h}\mathbf{\Phi}:=h\hat{E}^\lambda_\Omega\mathbf{\Phi}+\frac{1}{2}\mathbf{\Phi}-K'_{\Omega,\lambda}\mathbf{\Phi}$$

$$+\sum_{j=1}^k\left(\int_{\partial G(j)}\mathbf{\Theta}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma\right)T(D^\lambda_{B(j)}\mathbf{n}^{B(j)},\Pi^\lambda_{B(j)}\mathbf{n}^{B(j)})\mathbf{n}^\Omega.$$

Then $\tau^\lambda_{\Omega,h}$ is an isomorphism on $\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$.

(2) Suppose that $\mathbf{g}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, $\mathbf{f}\equiv 0$, $\Psi\equiv 0$. Put

$$\mathbf{\Phi}:=\left(\tau^\lambda_{\Omega,h}\right)^{-1}\mathbf{g},\qquad \mathbf{v}:=\hat{E}^\lambda_\Omega\mathbf{\Phi},\qquad p:=\hat{Q}^\lambda_\Omega\mathbf{\Phi}. \tag{5.1}$$

Then $(\mathbf{v}, p)$ *is a unique solution of the Robin problem* (1.3), (1.2) *in the space* $\left[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)\right]\times\left[\mathcal{C}^{\beta}(\overline{\Omega})\cap\mathcal{C}^1(\Omega)\right]$. *Moreover,*

$$\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+\|p\|_{\mathcal{C}^{\beta}(\overline{\Omega})}\le C\|\mathbf{g}\|_{\mathcal{C}^{\beta}(\partial\Omega)} \tag{5.2}$$

where C *does not depend on* $\mathbf{g}$.

Proof. $K'_{\Omega,\lambda}$ is a compact operator in $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$ by Theorem 4.8. The operator $\tilde{E}^{\lambda}_{\Omega}:\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)\to\mathcal{C}^{1,\beta}(\partial\Omega;\mathbb{R}^m)$ is bounded. (It is an easy consequence of Proposition 3.1.) Therefore $\tilde{E}^{\lambda}_{\Omega}$ is a compact operator on $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$ by [18, Theorem 7.4]. Define the operator H by $H\mathbf{w}:=h\mathbf{w}$. Then H is a bounded operator on $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$ by [23, Lemma 1.16.8]. So, $H\tilde{E}^{\lambda}_{\Omega}$ is a compact operator on $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$. Denote by I the identity operator. The operator $\tau^{\lambda}_{\Omega,h}-H\tilde{E}^{\lambda}_{\Omega}-\frac{1}{2}I+K'_{\Omega,\lambda}$ is finite-dimensional and therefore compact. Since the operator $\tau^{\lambda}_{\Omega,h}-\frac{1}{2}I$ is compact in $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$, the operator $\tau^{\lambda}_{\Omega,h}$ is Fredholm with index 0 in $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$. The operator $\tau^{\lambda}_{\Omega,h}$ is one to one by [24, Theorem 9.2]. Hence $\tau^{\lambda}_{\Omega,h}$ is an isomorphism on $\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$.

Suppose now that $\mathbf{g}\in\mathcal{C}^{\beta}(\partial\Omega;\mathbb{R}^m)$, $\mathbf{f}\equiv 0$, $\Psi\equiv 0$. Let $\mathbf{\Phi}$, $\mathbf{v}$ and p be given by (5.1). Since $E^{\lambda}_{ij},Q^{\lambda}_{i}\in\mathcal{C}^{\infty}(\mathbb{R}^m\setminus\{0\})$, we infer that $\mathbf{v}\in\mathcal{C}^{\infty}(\Omega;\mathbb{R}^m)$ and $p\in\mathcal{C}^{\infty}(\Omega)$. Moreover, (1.3) holds true. The invertibility of $\tau^{\lambda}_{\Omega,h}$ and Proposition 3.1 give that $\mathbf{v}\in\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$, $p\in\mathcal{C}^{\beta}(\overline{\Omega})$ and the estimate (5.2) holds with some constant C. The boundary condition (1.2) is satisfied by Lemma 4.3.

We now show the uniqueness of a classical solution of the problem (1.3), (1.2). Let $(\mathbf{v},p)\in\left[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)\right]\times\left[\mathcal{C}^{\beta}(\overline{\Omega})\cap\mathcal{C}^1(\Omega)\right]$ be a solution of the problem (1.3), (1.2) with $\mathbf{f}\equiv 0$, $\Psi\equiv 0$ and $\mathbf{g}\equiv 0$. If $x\in\partial\Omega$, $a>0$, denote the non-tangential approach region of opening a at the point x by

$$\Gamma_a(x):=\{y\in\Omega;|x-y|<(1+a)\mathrm{dist}(y,\partial\Omega)\}.$$

According to [35, Theorem 1.12] there is a sequence of domains Ω_j with boundaries of class C^{∞} such that the following assertions hold:

- $\overline{\Omega}_j\subset\Omega$.
- There are $a>0$ and homeomorphisms $\Lambda_j:\partial\Omega\to\partial\Omega_j$, such that $\Lambda_j(y)\in\Gamma_a(y)$ for each j and each $y\in\partial\Omega$ and $\sup\{|y-\Lambda_j(y)|;y\in\partial\Omega\}\to 0$ as $j\to\infty$.
- There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E\subset\partial\Omega$,

  $$\int_E\omega_j\,\mathrm{d}\sigma=\int_{\Lambda_j(E)}1\,\mathrm{d}\sigma,$$

  and so that $\omega_j\to 1$ point-wise a.e. and in every $L^s(\partial\Omega)$, $1\le s<\infty$.
- The normal vectors to Ω_j, $n(\Lambda_j(y))$, converge point-wise a.e. and in every $L^s(\partial\Omega)$, $1\le s<\infty$, to $n(y)$.

Lebesgue lemma yields

$$0=\int_{\partial\Omega}\mathbf{v}\cdot[T(\mathbf{v},p)\mathbf{n}^{\Omega}+h\mathbf{v}]\,\mathrm{d}\sigma=\int_{\partial\Omega}h|\mathbf{v}|^2\,\mathrm{d}\sigma+\lim_{j\to\infty}\int_{\partial\Omega_j}\mathbf{v}\cdot T(\mathbf{v},p)\mathbf{n}\,\mathrm{d}\sigma.$$

According to the Green formula (compare [33, p. 14] or [24, §3])

$$0=\int_{\partial\Omega}h|\mathbf{v}|^2\,\mathrm{d}\sigma+\lim_{j\to\infty}\int_{\Omega_j}(2|\hat{\nabla}\mathbf{v}|^2+\lambda|\mathbf{v}|^2)\,\mathrm{d}x$$

$$= \int_{\partial\Omega} h|\mathbf{v}|^2 \, \mathrm{d}\sigma + \int_{\Omega} (2|\hat{\nabla}\mathbf{v}|^2 + \lambda|\mathbf{v}|^2) \, \mathrm{d}x.$$

Therefore $\mathbf{v} \equiv 0$. Since $0 = -\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p = \nabla p$, there exists a constant c such that $p \equiv c$. But $0 = T(\mathbf{v}, p)\mathbf{n}^\Omega + h\mathbf{v} = -c\mathbf{n}^\Omega$ on $\partial\Omega$. This forces that $c = 0$ and thus $p \equiv 0$. □

Now we are able to solve the Robin problem for the non-homogeneous system (1.3).

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0 < \beta < \alpha < 1$ and $\lambda \in (0, \infty)$. Let $h \in \mathcal{C}^\beta(\partial\Omega)$ with $h \geq 0$. Then there exists a constant C such that the following holds: If $\mathbf{f} \in \mathcal{C}^\beta(\overline{\Omega}; \mathbb{R}^m)$, $\psi \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ and $\mathbf{g} \in \mathcal{C}^\beta(\partial\Omega; \mathbb{R}^m)$, then there exists a unique solution $(\mathbf{v}, p)$ of the Robin problem (1.3), (1.2) in the space $\left[\mathcal{C}^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap \mathcal{C}^2(\Omega; \mathbb{R}^m)\right] \times \left[\mathcal{C}^\beta(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)\right]$. Moreover,*

$$\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|p\|_{\mathcal{C}^\beta(\overline{\Omega})} \leq C\left(\|\mathbf{g}\|_{\mathcal{C}^\beta(\partial\Omega)} + \|\mathbf{f}\|_{\mathcal{C}^\beta(\overline{\Omega})} + \|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\right). \tag{5.3}$$

Proof. The uniqueness follows from Theorem 5.1.

Choose a bounded domain ω with smooth boundary such that $\overline{\Omega} \subset \omega$. According to [5, Lemma 6.37] there exists $\tilde{\psi} \in \mathcal{C}^{1,\beta}(\overline{\omega})$ with compact support in ω such that $\tilde{\psi} = \psi$ in Ω and

$$\|\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\omega)} \leq C_1 \|\psi\|_{\mathcal{C}^{1,\beta}(\Omega)}$$

where the constant C_1 depends only on Ω, ω and β. Denote

$$\mathcal{C}_0^{3,\alpha}(\overline{\omega}) := \{\varphi \in \mathcal{C}^{3,\alpha}(\overline{\omega}); \varphi = 0 \text{ on } \partial\omega\}.$$

Then the Laplace operator is a continuous injective operator from $\mathcal{C}_0^{3,\beta}(\overline{\omega})$ onto $\mathcal{C}^{1,\beta}(\overline{\omega})$. (See [19, Lemma 3.10].) Therefore it is an isomorphism. So, there exists a solution $\varphi \in \mathcal{C}^{3,\beta}(\overline{\omega})$ of

$$\Delta\varphi = \tilde{\psi} \quad \text{in } \omega, \qquad \varphi = 0 \quad \text{on } \partial\omega.$$

Moreover.

$$\|\varphi\|_{\mathcal{C}^{3,\beta}(\overline{\omega})} \leq C_2 \|\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\omega)}$$

where C_2 depends only on ω and β. Put $\mathbf{w} := \nabla\varphi$. Then $\mathbf{w} \in \mathcal{C}^{2,\beta}(\overline{\Omega})$, $\nabla \cdot \mathbf{w} = \Delta\varphi = \psi$ in Ω and

$$\|\mathbf{w}\|_{\mathcal{C}^{2,\beta}(\overline{\Omega})} \leq mC_1C_2\|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}.$$

According to [25, Theorem 2.2] there exists a unique solution

$$(\mathbf{u}, q) \in \left[\mathcal{C}^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap \mathcal{C}^2(\Omega; \mathbb{R}^m)\right] \times \left[\mathcal{C}^\beta(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)\right]$$

of the problem

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla q = \mathbf{f} + \Delta\mathbf{w} - \lambda\mathbf{w}, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega,$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega q \, \mathrm{d}x = 0.$$

Moreover,

$$\|\mathbf{u}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|q\|_{\mathcal{C}^\beta(\overline{\Omega})} \leq C_3 \|\mathbf{f} + \Delta\mathbf{w} - \lambda\mathbf{w}\|_{\mathcal{C}^\beta(\overline{\Omega})},$$

where C_3 depends only on Ω, β and λ. According to Theorem 5.1 there exists a unique solution

$$(\tilde{\mathbf{u}}, \tilde{q}) \in \left[\mathcal{C}^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap \mathcal{C}^2(\Omega; \mathbb{R}^m)\right] \times \left[\mathcal{C}^\beta(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)\right]$$

of the problem

$$-\Delta\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} + \nabla\tilde{q} = 0, \quad \nabla\cdot\tilde{\mathbf{u}} = 0 \qquad \text{in } \Omega,$$
$$T(\tilde{\mathbf{u}},\tilde{q})\mathbf{n}^\Omega + h\tilde{\mathbf{u}} = \mathbf{g} - T(\mathbf{u}+\mathbf{w},q)\mathbf{n}^\Omega - h(\mathbf{u}+\mathbf{w}).$$

Moreover,

$$\|\tilde{\mathbf{u}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|\tilde{q}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C_4\|\mathbf{g} - T(\mathbf{u}+\mathbf{w},q)\mathbf{n}^\Omega - h(\mathbf{u}+\mathbf{w})\|_{\mathcal{C}^{\beta}(\partial\Omega)},$$

where C_4 depends only on Ω, β, λ and h. Put $\mathbf{v} = \mathbf{w}+\mathbf{u}+\tilde{\mathbf{u}}$, $p = q+\tilde{q}$. Then $(\mathbf{v},p)$ is a solution of (1.3), (1.2). Moreover, the estimate (5.3) holds with C depending only on Ω, β, λ and h. □

6. Darcy-Forchheimer-Brinkman system

Theorem 6.1. *Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$, $0<\beta<\alpha<1$ and $\lambda\in(0,\infty)$. Let $h\in\mathcal{C}^\beta(\partial\Omega)$ with $h\ge 0$ and $a,b\in\mathcal{C}^\beta(\overline{\Omega})$. Then there exist $\delta,\epsilon,C\in(0,\infty)$ such that the following holds: If $\mathbf{g}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, $\mathbf{F}\in\mathcal{C}^\beta(\overline{\Omega};\mathbb{R}^m)$, $\psi\in\mathcal{C}^{1,\beta}(\overline{\Omega})$ and*

$$\|\mathbf{g}\|_{\mathcal{C}^{\beta}(\partial\Omega)} + \|\mathbf{F}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} + \|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} < \delta, \tag{6.1}$$

then there exists a unique solution $(\mathbf{v},p)\in[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)]\times[\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)]$ of the Robin problem for the Darcy-Forchheimer-Brinkman system (1.1), (1.2) such that

$$\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} < \epsilon. \tag{6.2}$$

Moreover,

$$\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|p\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C\left(\|\mathbf{g}\|_{\mathcal{C}^{\beta}(\partial\Omega)} + \|\mathbf{F}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} + \|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\right). \tag{6.3}$$

If $\tilde{\mathbf{g}}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, $\tilde{\mathbf{F}}\in\mathcal{C}^\beta(\overline{\Omega};\mathbb{R}^m)$, $\tilde{\psi}\in\mathcal{C}^{1,\beta}(\overline{\Omega})$, $\tilde{\mathbf{v}}\in\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)$ and $\tilde{p}\in\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)$,

$$\nabla\tilde{p} - \Delta\tilde{\mathbf{v}} + a|\tilde{\mathbf{v}}|\tilde{\mathbf{v}} + \lambda\tilde{\mathbf{v}} + b(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}} = \tilde{\mathbf{F}}, \quad \nabla\cdot\tilde{\mathbf{v}} = \tilde{\psi} \quad \text{in } \Omega, \tag{6.4a}$$

$$T(\tilde{\mathbf{v}},\tilde{p})\mathbf{n}^\Omega + h\tilde{\mathbf{v}} = \tilde{\mathbf{g}} \qquad \text{on } \partial\Omega \tag{6.4b}$$

and $\|\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} < \epsilon$, then

$$\|\mathbf{v}-\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|p-\tilde{p}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C\left(\|\mathbf{g}-\tilde{\mathbf{g}}\|_{\mathcal{C}^{\beta}(\partial\Omega)} + \|\mathbf{F}-\tilde{\mathbf{F}}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} + \|\psi-\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\right).$$

Proof. For $\mathbf{u}\in\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ define

$$D_{ab}\mathbf{u} := a|\mathbf{u}|\mathbf{u} + b(\mathbf{u}\cdot\nabla)\mathbf{u}.$$

According to [25, Lemma 3.1 and Lemma 3.2] there exists a constant C_1 such that

$$\|D_{ab}\mathbf{u}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C_1\|\mathbf{u}\|^2_{\mathcal{C}^{1,\beta}(\overline{\Omega})}, \tag{6.5}$$

$$\|D_{ab}\mathbf{v} - D_{ab}\mathbf{u}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C_1\|\mathbf{v}-\mathbf{u}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\left[\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|\mathbf{u}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\right]. \tag{6.6}$$

For each $\mathbf{g}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, $\mathbf{f}\in\mathcal{C}^\beta(\overline{\Omega};\mathbb{R}^m)$ and $\psi\in\mathcal{C}^{1,\beta}(\overline{\Omega})$ there exists a unique solution $(\mathbf{v},p)\in[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)]\times[\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)]$ of the Robin problem (1.3), (1.2). Moreover,

$$\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} + \|p\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C_2\left(\|\mathbf{g}\|_{\mathcal{C}^{\beta}(\partial\Omega)} + \|\mathbf{f}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} + \|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\right) \tag{6.7}$$

where C_2 depends only on Ω, β and h. (See Theorem 5.2.)

Remark that $(\mathbf{v},p)$ is a solution of (1.1) if $(\mathbf{v},p)$ is a solution of (1.3) with $\mathbf{f}=\mathbf{F}-D_{ab}\mathbf{v}$. Put

$$\epsilon := \frac{1}{4(C_1+1)(C_2+1)}, \quad \delta := \frac{\epsilon}{2(C_2+1)}.$$

If $(\mathbf{v},p),(\tilde{\mathbf{v}},\tilde{p}) \in [\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)]\times[\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)]$ are solutions of (1.1), (1.2) and (6.4) with (6.2) and $\|\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}<\epsilon$, then

$$\|\mathbf{v}-\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+\|p-\tilde{p}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le C_2\big(\|\mathbf{g}-\tilde{\mathbf{g}}\|_{\mathcal{C}^{\beta}(\partial\Omega)}+\|\mathbf{F}-\tilde{\mathbf{F}}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}$$

$$+\|\psi-\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+\|D_{ab}\mathbf{v}-D_{ab}\tilde{\mathbf{v}}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}\big)\le C_2\big(\|\mathbf{g}-\tilde{\mathbf{g}}\|_{\mathcal{C}^{\beta}(\partial\Omega)}$$

$$+\|\mathbf{F}-\tilde{\mathbf{F}}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}+\|\psi-\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+2\epsilon C_1\|\mathbf{v}-\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\big).$$

Since $2C_1C_2\epsilon<1/2$ we get subtracting $2\epsilon C_1C_2\|\mathbf{v}-\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}$ from the both sides

$$\|\mathbf{v}-\tilde{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+\|p-\tilde{p}\|_{\mathcal{C}^{\beta}(\overline{\Omega})} \le 2C_2\big(\|\mathbf{g}-\tilde{\mathbf{g}}\|_{\mathcal{C}^{\beta}(\partial\Omega)}+\|\mathbf{F}-\tilde{\mathbf{F}}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}+\|\psi-\tilde{\psi}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\big).$$

Therefore a solution of (1.1), (1.2) satisfying (6.2) is unique. Putting $\tilde{p}\equiv 0$, $\tilde{\mathbf{u}}\equiv 0$, $\tilde{\mathbf{F}}\equiv 0$, $\tilde{\psi}\equiv 0$ and $\tilde{\mathbf{g}}\equiv 0$, we obtain (6.3) with $C=2C_2$.

Denote $X:=\{\mathbf{v}\in\mathcal{C}^{1,\beta}(\overline{\Omega},\mathbb{R}^m);\|\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\le\epsilon\}$. Fix $\mathbf{g}\in\mathcal{C}^\beta(\partial\Omega;\mathbb{R}^m)$, $\mathbf{F}\in\mathcal{C}^\beta(\overline{\Omega};\mathbb{R}^m)$ and $\psi\in\mathcal{C}^{1,\beta}(\overline{\Omega})$ satisfying (6.1). For $\mathbf{v}\in X$ there exists a unique solution $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})\in[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)]\times[\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)]$ of the Robin problem (1.3), (1.2) with $\mathbf{f}=\mathbf{F}-D_{ab}\mathbf{v}$. Remember that $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})$ is a solution of (1.1) if and only if $\mathbf{u}^{\mathbf{v}}=\mathbf{v}$. According to (6.7)

$$\|\mathbf{u}^{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\le C_2\left[\|\mathbf{g}\|_{\mathcal{C}^{\beta}(\partial\Omega)}+\|\psi\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}+\|\mathbf{F}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}+\|D_{ab}\mathbf{v}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}\right].$$

By virtue of (6.1) and (6.5)

$$\|\mathbf{u}^{\mathbf{v}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\le C_2\delta+C_2C_1\epsilon^2.$$

As $C_2\delta+C_2C_1\epsilon^2<\epsilon$, we infer $\mathbf{u}^{\mathbf{v}}\in X$. If $\mathbf{w}\in X$ then

$$\|\mathbf{u}^{\mathbf{v}}-\mathbf{u}^{\mathbf{w}}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}\le C_2\|D_{ab}\mathbf{v}-D_{ab}\mathbf{w}\|_{\mathcal{C}^{\beta}(\overline{\Omega})}\le C_2C_1 2\epsilon\|\mathbf{w}-\mathbf{v}\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}$$

by (6.7) and (6.6). Since $C_2C_12\epsilon<1$, the Fixed point theorem ([4, Satz 1.24]) gives that there exists $\mathbf{v}\in X$ such that $\mathbf{u}^{\mathbf{v}}=\mathbf{v}$. So, $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})$ is a solution of (1.1), (1.2) in $[\mathcal{C}^{1,\beta}(\overline{\Omega};\mathbb{R}^m)\cap\mathcal{C}^2(\Omega;\mathbb{R}^m)]\times[\mathcal{C}^\beta(\overline{\Omega})\cap\mathcal{C}^1(\Omega)]$ satisfying (6.2). □

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8. Conflict of interest

The author declares no conflict of interest in this paper.

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