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A PURELY INFINITE CUNTZ-LIKE BANACH $*$-ALGEBRA WITH NO PURELY INFINITE ULTRAPOWERS

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Abstract. We continue our investigation, from [10], of the ring-theoretic infiniteness properties of ultrapowers of Banach algebras, studying in this paper the notion of being purely infinite. It is well known that a C^*-algebra is purely infinite if and only if any of its ultrapower is. We find examples of Banach algebras, as algebras of operators on Banach spaces, which do have purely infinite ultrapowers. Our main contribution is the construction of a "Cuntz-like" Banach $*$-algebra which is purely infinite, but does not have purely infinite ultrapowers. Our proof of being purely infinite is combinatorial, but direct, and so differs from the proof for the Cuntz algebra. We use an indirect method (and not directly computing norm estimates) to show that this algebra does not have purely infinite ultrapowers.

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1. Introduction and preliminaries

1.1. **Introduction.** We continue our study of infiniteness properties of Banach algebras, and how these interact with reduced products, in the continuous model theory sense, which we initiated in [10]. Recall that an idempotent p in an algebra $\mathcal{A}$ is *infinite* if it is (algebraically Murray–von Neumann) equivalent to a proper sub-idempotent of itself. One prominent property which we did not study in [10] is that of being *purely infinite*, which for simple rings could be defined by saying that every left ideal contains an infinite idempotent. We discuss this notion, and the literature surrounding it, in Section 1.2 below. This definition is equivalent, for a unital Banach algebra, to $\mathcal{A}$ not being $\mathbb{C}$, and that for $a \in \mathcal{A}$ non-zero there are $b, c \in \mathcal{A}$ with $bac = 1$. This generalises the definition for C^*-algebras.

As a purely infinite Banach algebra must be simple, the asymptotic sequence algebra of A is never purely infinite, see Lemma 2.7 below. We thus focus on ultrapowers in this paper. As in [10], and perhaps not surprisingly from the perspective of continuous model theory, we find that an ultrapower $(\mathcal{A})_{\mathcal{U}}$ is purely infinite if and only if it satisfies a "metric" form of the definition, where we have some sort of norm control. From this perspective, it is unsurprising to find that the fact that purely infinite C^*-algebras have purely infinite ultrapowers follows from such norm control always being available.

In [10] we found examples of Banach algebras which did, and did not, have suitable forms of norm control. Our major tool was to look at weighted semigroup algebras,

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where the weight allowed us to vary the norm control which we obtained. Surprisingly, in this paper we have no need to consider weights. Thus our examples are somewhat more "natural", and indeed, in showing that our principle example does not have purely infinite ultrapowers, we proceed in a somewhat indirect way, and avoid directly computing norms.

The structure of the paper is as follows. In the remainder of the introduction, we provide a more detailed introduction to purely infinite algebras, and recall the ultrapower construction. In Section 2 we define a suitable "quantified" definition of being purely infinite, and show that this does indeed capture when ultrapowers are purely infinite. We show quickly how this gives that purely infinite C^*-algebras have purely infinite ultrapowers.

In Section 2.2, we provide natural examples of Banach algebras which do have purely infinite ultrapowers. These are built as algebras of operators on suitable Banach spaces. Finally, we show that if a Banach algebra $\mathcal{A}$ does have purely infinite ultrapowers, then it behaves a little like a C^*-algebra, in the sense that continuous unital homomorphisms out of $\mathcal{A}$ must be bounded below. We use this property to show that our main example does not have purely infinite ultrapowers.

In Section 3 we present our main construction. As in [10], we use the *Cuntz semigroup* Cu_2, which is a semigroup with zero element, modelled on the relations of the Cuntz algebra $\mathcal{O}_2$. We study the semigroup algebra $\mathcal{A} = \ell^1(\mathrm{Cu}_2 \setminus \{\Diamond\}, \#)$, where we replace the semigroup zero by the algebra 0. We recall some of the combinatorics of this semigroup. There are two natural idempotents in this algebra, and we quotient by the relation that these idempotents sum to 1, say leading to the algebra $\mathcal{A}/\mathcal{J}$. By a delicate combinatorial argument, we show that the resulting Banach algebra is purely-infinite: for any non-zero $a \in \mathcal{A}/\mathcal{J}$ we find $f \in \mathcal{A}$ which maps to a, and $g, h \in \mathcal{A}$ with $g\#f\#h = 1$, see Theorem 3.16. To show that $\mathcal{A}/\mathcal{J}$ does not have purely infinite ultrapowers, we construct a faithful, but not bounded below, representation on the Banach space ℓ^1.

The Banach algebra $\mathcal{A}$ has been previously studied in [8], but in relation to being properly infinite (and further we studied a "weighted" version of this algebra in [10]). The underlying algebra, given by generators and relations, but without the ℓ^1-norm completion, has a much longer history, as noticed by Phillips in [22]; compare Remark 3.5 below. Indeed, Phillips makes a careful study of (in particular) the algebra $\mathcal{O}_2^1$, which, in our language, is the *closure* of the image of $\mathcal{A}/\mathcal{J}$ in $\mathcal{B}(\ell^1)$. As we consider in Remark 3.29, given the lack of "permanence" properties for purely infinite Banach algebras, there appears to be no logical connections between our results and those of Phillips. In particular, Phillips shows that $\mathcal{O}_2^1$ is purely infinite, but we have been unable to decide if $\mathcal{O}_2^1$ has purely infinite ultrapowers, or not.

Unless stated otherwise, we will use the same notation and terminology as in [10].

1.2. **Purely infinite algebras.** Let $\mathcal{A}$ be an algebra. We say that two idempotents $p, q \in \mathcal{A}$ are *algebraically Murray–von Neumann equivalent* or simply *equivalent* (in notation, $p \sim q$) if there exist $a, b \in \mathcal{A}$ such that $p = ab$ and $q = ba$. Note that $\sim$ is an equivalence relation on the set of idempotents of $\mathcal{A}$. We say that the idempotents $p, q \in \mathcal{A}$ are *orthogonal* (in notation, $p \perp q$) if $pq = 0 = qp$. An idempotent p is *infinite* if $p = q + r$ for orthogonal idempotents $q, r \in \mathcal{A}$ with $p \sim q$ and $r \neq 0$.

If $\mathcal{A}$ is additionally a $*$-algebra, then a self-adjoint idempotent is called a *projection*. Here one often takes a different notion of equivalence, which for C^*-algebras is well-known to give the same definitions; compare [10, Section 2].

The notion of a C^*-algebra being *purely infinite* is well-known, and has many equivalent definitions, mostly studied for simple algebras, but also in the non-simple case, [18]. Purely infinite C^*-algebras appear prominently in the classification programme for C^*-algebras, [21], in particular in the guise of the *Kirchberg algebras*. It is common to take as a definition that a C^*-algebra is purely infinite if every hereditary subalgebra contains an infinite projection.

In a more general direction, the notion of a simple ring being purely infinite was studied in [2], where it is taken as definition that a simple ring R is purely infinite if every right (or equivalently, left) ideal of R contains an infinite idempotent. Consideration of what it means for a non-simple ring to be purely infinite is given in [3].

Common to both definitions (in the simple case) is the following equivalence; for C^*-algebras see for example [9, Theorem V.5.5] while for rings see [2, Theorem 1.6].

Definition 1.1. *A complex unital algebra $\mathcal{A}$ is* purely infinite *if it is not a division algebra and for every $a \in \mathcal{A}$ non-zero there exist $b, c \in \mathcal{A}$ such that $1_{\mathcal{A}} = bac$.*

In this paper, we shall work only with this definition. Note that by the Gel'fand–Mazur Theorem a complex unital normed algebra is a division algebra if and only if it is isomorphic to the field of complex numbers $\mathbb{C}$.

We finish the section with the following. We recall that a unital algebra $\mathcal{A}$ is *properly infinite* if there exist idempotents $p, q \in \mathcal{A}$ with $p \sim 1_{\mathcal{A}}$, $q \sim 1_{\mathcal{A}}$ and $p \perp q$.

Lemma 1.2. *Let $\mathcal{A}$ be a purely infinite algebra. Then $\mathcal{A}$ is*

(1) *simple; and*
(2) *properly infinite.*

Proof. We first show that $\mathcal{A}$ is simple. Let $\mathcal{J}$ be a non-zero, two-sided ideal in $\mathcal{A}$ and pick $a \in \mathcal{J}$ non-zero. There exist $b, c \in \mathcal{A}$ such that $1_{\mathcal{A}} = bac$, hence $1_{\mathcal{A}} \in \mathcal{J}$. Thus $\mathcal{J} = \mathcal{A}$.

We now show that $\mathcal{A}$ is properly infinite. Recall that $\mathcal{A}$ is not a division algebra, hence we can find a non-zero, non-invertible element, say $a \in \mathcal{A}$. Let $b, c \in \mathcal{A}$ be such that $1_{\mathcal{A}} = bac$. We define $p := cba$ and $r := acb$, it is clear that $p, r \in \mathcal{A}$ are idempotents with $p \sim 1_{\mathcal{A}} \sim r$. However p and r need not be orthogonal. Nevertheless, either $p \neq 1_{\mathcal{A}}$ or $r \neq 1_{\mathcal{A}}$ (or both), otherwise a were invertible with inverse cb which is not possible. Without loss of generality we may assume $p \neq 1_{\mathcal{A}}$. Let $s := 1_{\mathcal{A}} - p$, then $s \in \mathcal{A}$ is a non-zero idempotent with $s \perp p$. We can find some $x, y \in \mathcal{A}$ such that $1_{\mathcal{A}} = xsy$. Define $q := syxs$, then $q^2 = syxssyxs = syxsyxs = syxs = q$, and clearly $q \sim 1_{\mathcal{A}}$. Now $p \perp q$ as $p \perp s$. □

1.3. **Ultrapowers of Banach algebras.** The main objects of study in this paper are the following. Let $\mathcal{A}$ be a Banach algebra and let $\ell^\infty(\mathcal{A})$ be the Banach space of all bounded sequences (a_n) in $\mathcal{A}$, turned into a Banach algebra with pointwise operations. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $c_{\mathcal{U}}(\mathcal{A})$ be the closed, two-sided ideal of $\ell^\infty(\mathcal{A})$ formed of sequences (a_n) with $\lim_{n\to\mathcal{U}} \|a_n\| = 0$. The quotient

$$(\mathcal{A})_{\mathcal{U}} = \ell^\infty(\mathcal{A})/c_{\mathcal{U}}(\mathcal{A}) \tag{1.1}$$

is the *ultrapower*, see [14].

We shall denote by a capital letter A, and so forth, an element $A = (a_n) \in \ell^\infty(\mathcal{A})$. Let $\pi_\mathcal{U} : \ell^\infty(\mathcal{A}_n) \to (\mathcal{A}_n)_\mathcal{U}$ be the quotient map; then

$$\|\pi_\mathcal{U}(A)\| = \lim_{n \to \mathcal{U}} \|a_n\|. \tag{1.2}$$

In particular, given any $a \in (\mathcal{A})_\mathcal{U}$ we can always find $A = (a_n) \in \ell^\infty(\mathcal{A})$ with $\pi(A) = a$ and $\|A\| = \sup_n \|a_n\| = \|a\|$. We always assume that our ultrafilters are non-principal, which on a countable indexing set, is equivalent to being *countably-incomplete* (see [14, Section 1]).

2. Norm control

In [10] we "quantified" Dedekind-finiteness, pure infiniteness and stable rank one, in order to characterise when an ultrapower $(\mathcal{A})_\mathcal{U}$ has these ring-theoretic properties of the underlying Banach algebra $\mathcal{A}$. We follow our previous approach in the present paper.

Definition 2.1. *Let $\mathcal{A}$ be a unital Banach algebra. For $a \in \mathcal{A} \setminus \{0\}$ define*

$$C_{pi}^{\mathcal{A}}(a) = \inf\{\|b\|\|c\| : b, c \in \mathcal{A},\, bac = 1\}$$

with $C_{pi}^{\mathcal{A}}(a) = \infty$ if there are no $b, c \in \mathcal{A}$ with $bac = 1$.

Then a unital Banach algebra $\mathcal{A}$ is purely infinite exactly when $C_{\mathrm{pi}}^{\mathcal{A}}(a) < \infty$ for each $a \in \mathcal{A} \setminus \{0\}$. Note that if $a \in \mathcal{A}$ is such that $C_{\mathrm{pi}}^{\mathcal{A}}(a) < \infty$ then $1/\|a\| \leqslant C_{\mathrm{pi}}^{\mathcal{A}}(a)$.

By homogeneity, we have

$$C_{\mathrm{pi}}^{\mathcal{A}}(za) = |z|^{-1} C_{\mathrm{pi}}^{\mathcal{A}}(a) \qquad (a \in \mathcal{A} \setminus \{0\},\ z \in \mathbb{C} \setminus \{0\}). \tag{2.1}$$

Thus it is enough to study the unit sphere of $\mathcal{A}$.

Usually, we will drop the superscript on $C_{\mathrm{pi}}^{\mathcal{A}}(a)$ and simply write $C_{\mathrm{pi}}(a)$, whenever it is clear from the context which Banach algebra the element a is taken from.

Proposition 2.2. *Let $\mathcal{U}$ be a countably-incomplete ultrafilter. Then for a unital Banach algebra $\mathcal{A}$ the following are equivalent.*

1. *$(\mathcal{A})_\mathcal{U}$ is purely infinite;*
2. *There is $K > 0$ such that $C_{pi}(a) < K$ for each $a \in \mathcal{A}$ with $\|a\| = 1$.*

Proof. As usual, we may suppose that $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.

$((1) \Rightarrow (2))$: We prove the statement by way of a contraposition. Assume (2) does not hold. Then in particular we can pick sequence (a_n) in $\mathcal{A}$ consisting of norm one elements such that $C_{\mathrm{pi}}(a_n) > n$ for each $n \in \mathbb{N}$. Let $A := (a_n)$ so $A \in \ell^\infty(\mathcal{A})$. Assume towards a contradiction that $(\mathcal{A})_\mathcal{U}$ is purely infinite. Thus we can find $B = (b_n)$, $C = (c_n) \in \ell^\infty(\mathcal{A})$ such that $\pi(1) = \pi(B)\pi(A)\pi(C)$, or equivalently, $\lim_{n\to\mathcal{U}} \|1 - b_n a_n c_n\| = 0$. Let $\mathcal{N} := \{n \in \mathbb{N} : \|1 - b_n a_n c_n\| < 1/2\}$, then $\mathcal{N} \in \mathcal{U}$. By the Carl Neumann series $x_n := b_n a_n c_n \in \mathrm{inv}(\mathcal{A})$ with $\|x_n^{-1}\| \leqslant 2$ for each $n \in \mathcal{N}$. As $1 = x_n^{-1} x_n = (x_n^{-1} b_n) a_n c_n$, we conclude that

$$n < C_{\mathrm{pi}}(a_n) \leqslant \|x_n^{-1} b_n\|\|c_n\| \leqslant \|x_n^{-1}\|\|b_n\|\|c_n\| \leqslant 2\|B\|\|C\| \qquad (n \in \mathcal{N}). \tag{2.2}$$

As $\mathcal{N} \in \mathcal{U}$ and thus $\mathcal{N}$ is infinite, this gives a contradiction.

$((2) \Rightarrow (1))$: Assume (2) holds. Let $A = (a_n) \in \ell^\infty(\mathcal{A})$ be such that $\pi(A) \neq 0$. This is equivalent to saying that $\lim_{n\to\mathcal{U}} \|a_n\| \neq 0$, hence there is $\delta > 0$ such that $\{n \in \mathbb{N}\colon \|a_n\| < \delta\} \notin \mathcal{U}$, that is, $\mathcal{M} := \{n \in \mathbb{N}\colon \|a_n\| \geqslant \delta\} \in \mathcal{U}$. Thus we may set $a'_n := a_n/\|a_n\|$ whenever $n \in \mathcal{M}$, and $a'_n := 0$ otherwise. Clearly $\|a'_n\| = 1$ for each $n \in \mathcal{M}$, hence by the assumption it follows that $C_{\mathrm{pi}}(a'_n) < K$ for each $n \in \mathcal{M}$. Thus for every $n \in \mathcal{M}$ we can find $b'_n, c'_n \in \mathcal{A}$ such that $b'_n a'_n c'_n = 1$ and $\|b'_n\|\|c'_n\| < K$. We set

$$b_n := \begin{cases} \sqrt{\dfrac{\|c'_n\|}{\|b'_n\|\|a_n\|}}\, b'_n & \text{if } n \in \mathcal{M}, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad c_n := \begin{cases} \sqrt{\dfrac{\|b'_n\|}{\|c'_n\|\|a_n\|}}\, c'_n & \text{if } n \in \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Hence $b_n a_n c_n = \|a_n\|^{-1} b'_n a_n c'_n = b'_n a'_n c'_n = 1$ for each $n \in \mathcal{M}$. It is also follows from the definitions that $\|b_n\| = \sqrt{\|b'_n\|\|c'_n\|/\|a_n\|} < \sqrt{K/\delta}$ and similarly $\|c_n\| < \sqrt{K/\delta}$, hence $B := (b_n)$, $C := (c_n) \in \ell^\infty(\mathcal{A})$.

Fix $\varepsilon > 0$. Then

$$\mathcal{M} = \{n \in \mathbb{N}\colon 1 = b_n a_n c_n\} \subseteq \{n \in \mathbb{N}\colon \|1 - b_n a_n c_n\| < \varepsilon\}, \tag{2.4}$$

hence from $\mathcal{M} \in \mathcal{U}$ we conclude $\{n \in \mathbb{N}\colon \|1 - b_n a_n c_n\| < \varepsilon\} \in \mathcal{U}$. Thus $\lim_{n\to\mathcal{U}} \|1 - b_n a_n c_n\| = 0$, which is equivalent to $\pi(B)\pi(A)\pi(C) = \pi(A)$. Thus $(\mathcal{A})_\mathcal{U}$ is purely infinite. $\square$

Remark 2.3. In view of the comment before Proposition 2.2, we may rewrite condition (2) as

(2') There is $K > 0$ such that $C_{\mathrm{pi}}(a) \leqslant K/\|a\|$ for each non-zero $a \in \mathcal{A}$.

Consequently $(\mathcal{A})_\mathcal{U}$ is purely infinite if and only if there exists a $K > 0$ such that

$$1/\|a\| \leqslant C_{\mathrm{pi}}(a) \leqslant K/\|a\| \qquad (a \in \mathcal{A} \setminus \{0\}). \tag{2.5}$$

Corollary 2.4. *Let $\mathcal{U}$ be a countably-incomplete ultrafilter, and let $\mathcal{A}$ be a Banach algebra such that $(\mathcal{A})_\mathcal{U}$ is purely infinite. Then $\mathcal{A}$ is purely infinite.*

Proof. By the assumption we can take some $K > 0$ which satisfies the conditions of Proposition 2.2. Let $a \in \mathcal{A}$ be non-zero. We set $a' := a/\|a\|$, then $C_{\mathrm{pi}}(a') < K$. Thus there exist $b', c' \in \mathcal{A}$ such that $b'a'c' = 1$. Now define $b := b'/\sqrt{\|a\|}$ and $c := c'/\sqrt{\|a\|}$, thus $bac = 1$ as required. $\square$

In fact, we can make a quantitative statement in this direction. Given a Banach algebra $\mathcal{A}$ and an ultrafilter $\mathcal{U}$, define

$$\iota_\mathcal{A}\colon \mathcal{A} \to (\mathcal{A})_\mathcal{U}; \quad a \mapsto \pi_\mathcal{U}((a)) \tag{2.6}$$

to be the "diagonal" isometric embedding.

Lemma 2.5. *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{U}$ is an ultrafilter, then*

$$C_{pi}^{(\mathcal{A})_\mathcal{U}} \circ \iota_\mathcal{A} = C_{pi}^{\mathcal{A}}. \tag{2.7}$$

Proof. ($\geqslant$): Let $a \in \mathcal{A}$, and put $A := (a) \in \ell^\infty(\mathcal{A})$. Assume $B = (b_n)$, $C = (c_n) \in \ell^\infty(\mathcal{A})$ are such that $\pi_\mathcal{U}(1) = \pi_\mathcal{U}(B)\pi_\mathcal{U}(A)\pi_\mathcal{U}(C)$, which is equivalent to $\lim_{n\to\mathcal{U}} \|1-b_nac_n\| = 0$. Let us fix $\varepsilon \in (0,1)$. Then

$$\mathcal{N}_\varepsilon := \{n \in \mathbb{N} \colon \|1 - b_nac_n\| < \varepsilon\} \in \mathcal{U}, \tag{2.8}$$

and by the Carl Neumann series $x_n := b_nac_n \in \operatorname{inv}(\mathcal{A})$ with $\|x_n^{-1}\| \leqslant (1-\varepsilon)^{-1}$ for each $n \in \mathcal{N}_\varepsilon$. Thus $1 = x_n^{-1}x_n = (x_n^{-1}b_n)ac_n$, and consequently

$$C_{\mathrm{pi}}^{\mathcal{A}}(a) \leqslant \|x_n^{-1}b_n\|\|c_n\| \leqslant \|x_n^{-1}\|\|b_n\|\|c_n\| < (1-\varepsilon)^{-1}\|b_n\|\|c_n\| \quad (n \in \mathcal{N}_\varepsilon). \tag{2.9}$$

Therefore $C_{\mathrm{pi}}^{\mathcal{A}}(a) \leqslant \lim_{n\to\mathcal{U}} \|b_n\|\|c_n\|(1-\varepsilon)^{-1} = \|\pi_\mathcal{U}(B)\|\|\pi_\mathcal{U}(C)\|(1-\varepsilon)^{-1}$, which holds for all $\varepsilon \in (0,1)$, hence $C_{\mathrm{pi}}^{\mathcal{A}}(a) \leqslant \|\pi_\mathcal{U}(B)\|\|\pi_\mathcal{U}(C)\|$. Consequently $C_{\mathrm{pi}}^{\mathcal{A}}(a) \leqslant C_{\mathrm{pi}}^{(\mathcal{A})_\mathcal{U}}(\pi_\mathcal{U}(A)) = C_{\mathrm{pi}}^{(\mathcal{A})_\mathcal{U}}(\iota_\mathcal{A}(a))$, as claimed.

($\leqslant$): Let $a \in \mathcal{A}$. Assume $b, c \in \mathcal{A}$ are such that $1 = bac$. Putting $A := (a), B := (b), C := (c) \in \ell^\infty(\mathcal{A})$, we clearly have $\pi_\mathcal{U}(1) = \pi_\mathcal{U}(B)\pi_\mathcal{U}(A)\pi_\mathcal{U}(C)$. Consequently

$$C_{\mathrm{pi}}^{(\mathcal{A})_\mathcal{U}}(\iota_\mathcal{A}(a)) = C_{\mathrm{pi}}^{(\mathcal{A})_\mathcal{U}}(\pi_\mathcal{U}(A)) \leqslant \|\pi_\mathcal{U}(B)\|\|\pi_\mathcal{U}(C)\| = \|b\|\|c\|, \tag{2.10}$$

and therefore $C_{\mathrm{pi}}^{(\mathcal{A})_\mathcal{U}}(\iota_\mathcal{A}(a)) \leqslant C_{\mathrm{pi}}^{\mathcal{A}}(a)$, as required. ☐

One might wonder whether the converse to Corollary 2.4 could be true. We will show that this is not the case: there is a purely infinite Banach $*$-algebra which does not have purely infinite ultrapowers (see Theorems 3.21 and 3.27).

As it is well known (see [13, Section 3.13.7]) the converse to Corollary 2.4 remains true for C^*-algebras, however. Here we demonstrate how this can easily be deduced from Proposition 2.2.

Lemma 2.6. *Let $\mathcal{A}$ be a purely infinite C^*-algebra. Then $C_{pi}(a) = 1$ for each $a \in \mathcal{A}$ with $\|a\| = 1$, consequently $(\mathcal{A})_\mathcal{U}$ is purely infinite for every countably-incomplete ultrafilter $\mathcal{U}$.*

Proof. Let $a \in \mathcal{A}$ be norm one. Let us fix $\varepsilon > 0$. Clearly $a^*a \in \mathcal{A}$ is positive, hence by [9, Theorem V.5.5] there is some $x \in \mathcal{A}$ such that $(xa^*)ax^* = x(a^*a)x^* = 1$ and $\|x\| < \|a^*a\|^{-1/2} + \varepsilon = 1 + \varepsilon$. Thus

$$C_{\mathrm{pi}}(a) \leqslant \|xa^*\|\|x^*\| \leqslant \|x\|^2\|a\| < (1+\varepsilon)^2, \tag{2.11}$$

and therefore $C_{\mathrm{pi}}(a) \leqslant 1$. The "consequently" part follows from Proposition 2.2. ☐

We note that [9, Theorem V.5.5] has an elementary (functional calculus) proof, passing by way of an equivalent definition of what purely infinite means for C^*-algebras, compare our discussion in Section 1.2.

2.1. **A word about the asymptotic sequence algebra.** Let $c_0(\mathcal{A})$ be the closed, two-sided ideal of $\ell^\infty(\mathcal{A})$ which consists of sequences (a_n) with $\lim_n \|a_n\| = 0$. In fact, when $\mathcal{A}$ is unital, $\ell^\infty(\mathcal{A}_n)$ is the *multiplier algebra* of $c_0(\mathcal{A})$ (compare [12, Section 13] for example). The *asymptotic sequence algebra* $\operatorname{Asy}(\mathcal{A})$ is the quotient algebra $\ell^\infty(\mathcal{A})/c_0(\mathcal{A})$.

As opposed to the previously studied properties in [10] such as stable rank one, Dedekind-finitess and proper infiniteness; the theory for the asymptotic sequence algebra and the ultrapower of a Banach algebra seems to bifurcate here.

Lemma 2.7. *Let $\mathcal{A}$ be a non-zero unital Banach algebra. Then* $\mathrm{Asy}(\mathcal{A})$ *is not simple and hence not purely infinite.*

Proof. Note that $\mathrm{Asy}(\mathcal{A})$ is simple if and only if $c_0(\mathcal{A})$ is a maximal two-sided ideal in $\ell^\infty(\mathcal{A})$. But this latter is not possible, as for example, the following shows. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $2\mathbb{N} \in \mathcal{U}$. Let $A := (a_n)$ be a sequence in $\mathcal{A}$ defined by $a_{2n} := 1_{\mathcal{A}}$ and $a_{2n-1} := 0_{\mathcal{A}}$ for each $n \in \mathbb{N}$. Clearly $A \in \ell^\infty(\mathcal{A})$ and in fact $A \in c_{\mathcal{U}}(\mathcal{A})$ by definition. On the other hand clearly $A \notin c_0(\mathcal{A})$. Consequently $c_0(\mathcal{A}) \subsetneq c_{\mathcal{U}}(\mathcal{A})$ which shows that $c_0(\mathcal{A})$ cannot be maximal. The last part follows from Lemma 1.2. □

2.2. **Examples of Banach algebras with purely infinite ultrapowers.** It is a good point to give an example of a class of non-C^*, Banach algebras with purely infinite ultrapowers. In what follows, $\mathcal{B}(X)$ and $\mathcal{K}(X)$ denote the algebra of bounded linear operators on a Banach space X and the set of compact operators on X, respectively. Clearly $\mathcal{B}(X)$ is a unital Banach algebra and $\mathcal{K}(X)$ is a closed, two-sided ideal in $\mathcal{B}(X)$.

Proposition 2.8. *Let X be c_0 or ℓ^p, where $1 \leqslant p < \infty$. Then $(\mathcal{B}(X)/\mathcal{K}(X))_{\mathcal{U}}$ is purely infinite if $\mathcal{U}$ is a countably-incomplete ultrafilter. More precisely, for every $a \in \mathcal{B}(X)/\mathcal{K}(X)$ with $\|a\| = 1$ there exist $b, c \in \mathcal{B}(X)/\mathcal{K}(X)$ such that $1 = bac$ and $\|b\|\|c\| = 1$.*

The proof of Proposition 2.8 relies on the following result of G. K. Ware, see [27, Lemma 3.3.6]. Note that it is a strict strengthening of [26, Lemma 2.1]; the proof works by extracting a suitable block basic sequence equivalent to the standard unit vector basis for X.

Lemma 2.9. *Let X be c_0 or ℓ^p, where $1 \leqslant p < \infty$. Then for each $A \in \mathcal{B}(X)$ a non-compact operator, there exist $B, C \in \mathcal{B}(X)$ such that*

$$I_X = BAC \quad \textit{and} \quad \|\pi(B)\|\|\pi(C)\| = 1/\|\pi(A)\|, \tag{2.12}$$

where $\pi\colon \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$ the quotient map.

Proof of Proposition 2.8. Let $A \in \mathcal{B}(X)$ be such that $\|\pi(A)\| = 1$. Hence by Lemma 2.9 there are $B, C \in \mathcal{B}(X)$ such that $I_X = BAC$ and $\|\pi(B)\|\|\pi(C)\| = 1$. This obviously proves the first part of the claim. In particular, $C_{\mathrm{pi}}(\pi(A)) = 1$ follows whenever $\|\pi(A)\| = 1$. Now Proposition 2.2 yields that $(\mathcal{B}(X)/\mathcal{K}(X))_{\mathcal{U}}$ is purely infinite, whenever $\mathcal{U}$ is a countably-incomplete ultrafilter. □

2.2.1. *The connection to a certain maximal ideal.* Let us introduce some terminology, commonly found in the literature, for the property we have been studying. For a unital algebra $\mathcal{A}$, and given $a \in \mathcal{A}$, we say that $1_{\mathcal{A}}$ *factors through* a, if there exist $b, c \in \mathcal{A}$ such that $1_{\mathcal{A}} = bac$.

In a unital algebra $\mathcal{A}$ we define the set

$$\mathcal{M}_{\mathcal{A}} := \{a \in \mathcal{A} \colon 1_{\mathcal{A}} \text{ does not factor through } a\}. \tag{2.13}$$

The following result is folklore and easy to see; we omit the proof.

Proposition 2.10. *Let $\mathcal{A}$ be a unital algebra.*

- *The set $\mathcal{M}_{\mathcal{A}}$ is closed under scalar multiplication, and under multiplying elements of it from the left and right by elements from $\mathcal{A}$. Thus it is the largest proper (and therefore unique maximal) two-sided ideal in $\mathcal{A}$ if and only if $\mathcal{M}_{\mathcal{A}}$ is closed under addition.*
- *If $\mathcal{M}_{\mathcal{A}}$ is closed under addition and $\mathcal{A}/\mathcal{M}_{\mathcal{A}}$ is not a division algebra, then $\mathcal{A}/\mathcal{M}_{\mathcal{A}}$ is purely infinite.*

Note that in the second bullet point the condition that $\mathcal{A}/\mathcal{M}_{\mathcal{A}}$ is not a division algebra cannot be omitted. Indeed, Kania and Laustsen showed in [17, Theorem 1.2] that with $X := C[0, \omega_1]$, the one-codimensional *Loy–Willis ideal* coincides with $\mathcal{M}_{\mathcal{B}(X)}$ and hence $\mathcal{B}(X)/\mathcal{M}_{\mathcal{B}(X)} \cong \mathbb{C}$.

When the unital Banach algebra $\mathcal{A}$ is $\mathcal{B}(X)$ for some "classical" Banach space X, it happens very often that $\mathcal{M}_{\mathcal{B}(X)}$ is the unique maximal ideal in $\mathcal{M}_{\mathcal{B}(X)}$. Here we give a few examples, a more comprehensive list can be found in [17, p. 4832].

Example 2.11. *If X is any of the Banach spaces below then $\mathcal{M}_{\mathcal{B}(X)}$ is closed under addition and hence it is the unique maximal ideal in $\mathcal{B}(X)$:*

- $X = c_0$ *or* $X = \ell^p$*, where* $1 \leqslant p < \infty$*, in this case* $\mathcal{M}_{\mathcal{B}(X)} = \mathcal{K}(X)$ *(see* [15]*);*
- $X = \ell^\infty$ *(see* [19, p. 253]*);*
- $X = L^p[0,1]$*, where* $1 \leqslant p < \infty$ *(see* [11, Theorem 1.3 and the text after]*);*
- $X = C[0,1]$ *(see the explanation in* [17, p. 4832]*).*

Remark 2.12. Let $\mathcal{A}$ be a unital algebra and let $\mathcal{J}$ be a two-sided ideal in $\mathcal{A}$. If $\mathcal{A}/\mathcal{J}$ is purely infinite, then $\mathcal{M}_{\mathcal{A}}$ is closed under addition if and only if $\mathcal{J} = \mathcal{M}_{\mathcal{A}}$. Indeed, $\mathcal{A}/\mathcal{J}$ is simple by Lemma 1.2, or equivalently, $\mathcal{J}$ is a maximal ideal. Hence if $\mathcal{M}_{\mathcal{A}}$ is closed under addition then it is the unique maximal ideal in $\mathcal{A}$ by Proposition 2.10, thus $\mathcal{J} = \mathcal{M}_{\mathcal{A}}$. The other direction is trivial.

It is certainly not true however that for a unital Banach algebra $\mathcal{A}$ and a closed, two-sided ideal $\mathcal{J}$ of $\mathcal{A}$ the quotient $\mathcal{A}/\mathcal{J}$ is purely infinite only if $\mathcal{M}_{\mathcal{A}}$ is closed under addition.

We shall show the above statement by way of a counter-example. In order to do this, let us recall the following piece of terminology. For Banach spaces X and Y the symbol $\overline{\mathcal{G}}_Y(X)$ denotes the closed, two-sided ideal of operators on X which approximately factor through Y.

Lemma 2.13. *Let $X := \ell^p \oplus \ell^q$, where $1 \leqslant p < q < \infty$. Then $M_{\mathcal{B}(X)}$ is not closed under addition while $\big(\mathcal{B}(X)/\overline{\mathcal{G}}_Y(X)\big)_{\mathcal{U}}$ is purely infinite, where Y is ℓ^p or ℓ^q and $\mathcal{U}$ is a countably-incomplete ultrafilter. More precisely, for every $a \in \mathcal{B}(X)/\overline{\mathcal{G}}_Y(X)$ with $\|a\| = 1$ there exist $b, c \in \mathcal{B}(X)/\overline{\mathcal{G}}_Y(X)$ such that $1 = bac$ and $\|b\|\|c\| = 1$.*

Proof. The first part of the claim is well known; see *e.g.* [24, Theorem 5.3.2]. Indeed, $\mathcal{B}(X)$ has exactly two maximal two-sided ideals, namely, $\overline{\mathcal{G}}_{\ell^p}(X)$ and $\overline{\mathcal{G}}_{\ell^q}(X)$. We will work with $Y = \ell^p$, the other case in entirely analogous.

Let us recall that by Pitt's Theorem [1, Theorem 2.1.4], we can describe $\mathcal{B}(X)$ and $\overline{\mathcal{G}}_{\ell^p}(X)$ as

$$\mathcal{B}(X) = \begin{bmatrix} \mathcal{B}(\ell^p) & \mathcal{B}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix} = \begin{bmatrix} \mathcal{B}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix},$$
$$\overline{\mathcal{G}}_{\ell^p}(X) = \begin{bmatrix} \mathcal{K}(\ell^p) & \mathcal{B}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix} = \begin{bmatrix} \mathcal{K}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\ \mathcal{B}(\ell^p, \ell^q) & \mathcal{B}(\ell^q) \end{bmatrix}.$$

Consequently,

$$\mathcal{B}(X)/\overline{\mathcal{G}}_{\ell^p}(X) \cong \mathcal{B}(\ell^p)/\mathcal{K}(\ell^p),$$

where the isomorphism is clearly isometric. Hence the result follows from Propositions 2.8 and 2.2. □

2.3. **Further permanence properties.** The result below appears to be a very handy tool when showing that a certain Banach algebra cannot have purely infinite ultrapowers. Indeed, it is one of the key ideas in the proof of Theorem 3.27.

Proposition 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ unital Banach algebras, and let $\psi\colon \mathcal{A} \to \mathcal{B}$ be a continuous, unital algebra homomorphism. Assume that $(\mathcal{A})_{\mathcal{U}}$ is purely infinite for some countably-incomplete ultrafilter $\mathcal{U}$. Then the following hold:*

(a) *The map ψ is bounded below.*
(b) *If ψ has dense range then it is in fact an isomorphism, hence $\mathcal{B}$ is purely infinite.*

Proof. We first recall that in view of Proposition 2.2 there is $K > 0$ such that $C_{\mathrm{pi}}(a) < K$ for each $a \in \mathcal{A}$ with $\|a\| = 1$. To prove (a), we assume towards a contradiction that ψ is not bounded below. Then there exists some $a \in \mathcal{A}$ with $\|a\| = 1$ such that $\|\psi(a)\| < 1/(2K\|\psi\|^2)$. We can pick $b, c \in \mathcal{A}$ with $1_{\mathcal{A}} = bac$ and $\|b\|\|c\| < K$. Consequently $\psi(1_{\mathcal{A}}) = \psi(bac) = \psi(b)\psi(a)\psi(c)$, thus

$$1 = \|1_{\mathcal{B}}\| = \|\psi(1_{\mathcal{A}})\| \leqslant \|\psi\|^2\|b\|\|c\|\|\psi(a)\| < \|\psi\|^2 K\|\psi(a)\| < 1/2, \tag{2.14}$$

a contradiction. Therefore ψ must be bounded below.

Now (b) follows easily. Indeed, ψ is bounded below, hence if it is assumed to have dense range, then ψ is in fact an isomorphism. Thus $\mathcal{B}$ must be purely infinite. □

3. A "Banach- analogue" of the Cuntz algebra

In this section we construct a purely infinite, infinite-dimensional Banach $*$-algebra which does not have a (non-trivial) purely infinite ultrapower.

3.1. Preliminaries.

3.1.1. *Involutive semigroups with zero elements, and the Banach $*$-algebra $\ell^1(S \setminus \{\Diamond\})$.* We recall that a semigroup S is *involutive* if there is a map $s \mapsto s^*$, $S \to S$ with the property $(s^*)^* = s$ and $(st)^* = t^*s^*$ for each $s, t \in S$.

We say that S is a *monoid with a zero element* if S is a monoid with at least two elements and there exists a $\Diamond \in S$ such that $\Diamond s = \Diamond = s\Diamond$ for all $s \in S$. If such a $\Diamond \in S$ exists then it is necessarily unique. As we assume that S has more than one element, we have $\Diamond$ is different from the multiplicative identity $e \in S$. Note that if S is additionally involutive, then necessarily $\Diamond^* = \Diamond$.

Let us briefly recall that it is possible to endow the Banach space $\ell^1(S \setminus \{\Diamond\})$ with a unital Banach algebra structure; see [8] and [10] for details; compare also [16]. This is accomplished by identifying $\ell^1(S \setminus \{\Diamond\})$ with the quotient algebra $\ell^1(S)/\mathbb{C}\delta_\Diamond$, where $\ell^1(S)$ is endowed with the convolution product. This allows us to define a product # on $\ell^1(S \setminus \{\Diamond\})$ which satisfies

$$\delta_s \# \delta_t = \begin{cases} \delta_{st} & \text{if } st \neq \Diamond \\ 0 & \text{if } st = \Diamond \end{cases} \qquad (s, t \in S \setminus \{\Diamond\})\,. \tag{3.1}$$

In particular it follows from equation (3.1) that $(\ell^1(S \setminus \{\Diamond\}), \#)$ is a unital Banach algebra with δ_e being the unit, and such that $\|\delta_e\| = 1$.

If in addition S is involutive, then the formula

$$f^*(s) := \overline{f(s^*)} \qquad (f \in \ell^1(S \setminus \{\Diamond\}),\ s \in S \setminus \{\Diamond\}) \tag{3.2}$$

defines an isometric involution on $\ell^1(S \setminus \{\Diamond\})$. Hence $\ell^1(S \setminus \{\Diamond\})$ is a Banach $*$-algebra.

3.1.2. *The Cuntz semigroup Cu_2.* In the following Cu_2 denotes the second Cuntz semigroup (see also [25, Definition 2.2, p. 141]; this is also occasionally called the "polycyclic monoid" in the literature, [6]). (We warn the reader that "Cuntz semigroup" now also means something unrelated in C^*-algebra theory.) That is, Cu_2 is an involutive semigroup with multiplicative identity e and zero element $\Diamond$, and generators s_1, s_2, s_1^*, s_2^* subject to the relations $s_1^* s_1 = e = s_2^* s_2$ and $s_1^* s_2 = \Diamond = s_2^* s_1$. In notation, Cu_2 is

$$\langle s_1, s_2, s_1^*, s_2^* \colon s_1^* s_1 = e = s_2^* s_2,\ s_1^* s_2 = \Diamond = s_2^* s_1 \rangle. \tag{3.3}$$

We now mostly follow the notation of [8, Section 3.3].

Definition 3.1. *We set*

$$\mathbf{I}_n := \{(i_1, i_2, \ldots, i_n) \colon i_1, i_2, \ldots, i_n \in \{1, 2\}\} \qquad (n \in \mathbb{N}), \tag{3.4}$$

and $\mathbf{I}_0 := \{\emptyset\}$*. Let* $\mathbf{I} := \bigcup_{n \in \mathbb{N}_0} \mathbf{I}_n$*, and* $\mathbf{L} := \prod_{n \in \mathbb{N}} \{1, 2\}$*.*

Let $\mathbf{n} = (n_i) \in \mathbf{L}$*, we then set*

$$\begin{aligned} &\mathbf{n}_0 := \emptyset, \\ &\mathbf{n}_l := (n_1, n_2, \ldots n_l) \in \mathbf{I}_l \qquad (l \in \mathbb{N}). \end{aligned} \tag{3.5}$$

If $\mathbf{i}, \mathbf{j} \in \mathbf{I}$*, then we define* $\mathbf{ij} \in \mathbf{I}$ *by concatenation*

$$\mathbf{ij} := \begin{cases} \mathbf{i} & \textit{if } \mathbf{j} = \emptyset, \\ \mathbf{j} & \textit{if } \mathbf{i} = \emptyset, \\ (i_1, i_2, \ldots, i_m, j_1, j_2, \ldots j_n) & \textit{if } \mathbf{i} = (i_1, i_2, \ldots, i_m) \textit{ and } \mathbf{i} = (j_1, j_2, \ldots, j_n). \end{cases} \tag{3.6}$$

For each $\mathbf{i} \in \mathbf{I}$ *we define* $s_{\mathbf{i}} \in Cu_2 \setminus \{\Diamond\}$ *by*

$$s_{\mathbf{i}} := \begin{cases} e & \textit{if } \mathbf{i} = \emptyset, \\ s_{i_1} s_{i_2} \cdots s_{i_n} & \textit{if } \mathbf{i} = (i_1, i_2, \ldots i_n) \in \mathbf{I} \setminus \{\emptyset\}. \end{cases} \tag{3.7}$$

We clearly have $s_{\mathbf{i}} s_{\mathbf{j}} = s_{\mathbf{ij}}$ and $s_{\mathbf{ij}}^* := (s_{\mathbf{ij}})^* = (s_{\mathbf{i}} s_{\mathbf{j}})^* = s_{\mathbf{j}}^* s_{\mathbf{i}}^*$.

3.2. **Basic combinatorics of Cu_2.** The following result is perhaps the single most important tool for our purposes. As stated below, it can be found in [10, Lemma 3.7], where it is attributed to Cuntz (see [7, Lemmas 1.2 and 1.3]).

Lemma 3.2. (1) *For every* $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ *we have*

$$s_{\mathbf{i}}^* s_{\mathbf{j}} = \begin{cases} s_{\mathbf{k}}^* & \text{if } \mathbf{i} = \mathbf{jk} \text{ for some } \mathbf{k} \in \mathbf{I}, \\ s_{\mathbf{k}} & \text{if } \mathbf{j} = \mathbf{ik} \text{ for some } \mathbf{k} \in \mathbf{I}, \\ \Diamond & \text{otherwise.} \end{cases} \tag{3.8}$$

(2) *For every* $t \in Cu_2 \setminus \{\Diamond\}$ *there exist unique* $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ *such that* $t = s_{\mathbf{i}} s_{\mathbf{j}}^*$.

Remark 3.3. Let $t \in \mathrm{Cu}_2 \setminus \{\Diamond\}$. By Lemma 3.2 (2) there exist unique $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ such that $t = s_{\mathbf{i}} s_{\mathbf{j}}^*$. Let $\alpha, \beta \in \mathbb{N}_0$ be the unique numbers such that $\mathbf{i} \in \mathbf{I}_\alpha$ and $\mathbf{j} \in \mathbf{I}_\beta$.

Thus we may define the *length* of t as

$$\mathrm{length}(t) := \alpha + \beta. \tag{3.9}$$

In fact, Lemma 3.2 (2) features so frequently in our arguments that we shall mostly use it implicitly without referring to it.

A very important corollary of the above is the lemma below, which we will use numerous times throughout the rest of the paper.

Lemma 3.4. *Let* $\mathbf{i}, \mathbf{j}, \mathbf{m}, \mathbf{n} \in \mathbf{I}$. *Then*

$$s_{\mathbf{i}}^* s_{\mathbf{m}} s_{\mathbf{n}}^* s_{\mathbf{j}} = \begin{cases} s_{\mathbf{qp}}^* & \text{if } \mathbf{i} = \mathbf{mp} \text{ and } \mathbf{n} = \mathbf{jq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{r}}^* & \text{if } \mathbf{i} = \mathbf{mqr} \text{ and } \mathbf{j} = \mathbf{nq} \text{ for some } \mathbf{r}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{p}} s_{\mathbf{q}}^* & \text{if } \mathbf{m} = \mathbf{ip} \text{ and } \mathbf{n} = \mathbf{jq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{pq}} & \text{if } \mathbf{m} = \mathbf{ip} \text{ and } \mathbf{j} = \mathbf{nq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{r}} & \text{if } \mathbf{i} = \mathbf{mp} \text{ and } \mathbf{j} = \mathbf{npr} \text{ for some } \mathbf{p}, \mathbf{r} \in \mathbf{I}, \\ \Diamond & \text{otherwise.} \end{cases} \tag{3.10}$$

Consequently, $s_{\mathbf{i}}^* s_{\mathbf{m}} s_{\mathbf{n}}^* s_{\mathbf{j}} = e$ *if and only if* $\mathbf{i} = \mathbf{mk}$ *and* $\mathbf{j} = \mathbf{nk}$ *for some* $\mathbf{k} \in \mathbf{I}$.

Proof. Applying Lemma 3.2 to $s_{\mathbf{i}}^* s_{\mathbf{m}}$ and $s_{\mathbf{n}}^* s_{\mathbf{j}}$, we immediately obtain that

$$s_{\mathbf{i}}^* s_{\mathbf{m}} s_{\mathbf{n}}^* s_{\mathbf{j}} = \begin{cases} s_{\mathbf{p}}^* s_{\mathbf{q}}^* = s_{\mathbf{qp}}^* & \text{if } \mathbf{i} = \mathbf{mp} \text{ and } \mathbf{n} = \mathbf{jq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{p}}^* s_{\mathbf{q}} & \text{if } \mathbf{i} = \mathbf{mp} \text{ and } \mathbf{j} = \mathbf{nq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{p}} s_{\mathbf{q}}^* & \text{if } \mathbf{m} = \mathbf{ip} \text{ and } \mathbf{n} = \mathbf{jq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ s_{\mathbf{p}} s_{\mathbf{q}} = s_{\mathbf{pq}} & \text{if } \mathbf{m} = \mathbf{ip} \text{ and } \mathbf{j} = \mathbf{nq} \text{ for some } \mathbf{p}, \mathbf{q} \in \mathbf{I}, \\ \Diamond & \text{otherwise.} \end{cases} \tag{3.11}$$

Once more we apply Lemma 3.2 to $s_{\mathbf{p}}^* s_{\mathbf{q}}$, which yields the desired formula (3.10).

The "consequently" part follows from inspecting the cases in the above formula and from observing that

- $s_{\mathbf{qp}}^* = e$ or $s_{\mathbf{pq}} = e$ or $s_{\mathbf{p}} s_{\mathbf{q}}^* = e$ if and only if $\mathbf{p} = \emptyset = \mathbf{q}$ if and only if $\mathbf{i} = \mathbf{m}$ and $\mathbf{j} = \mathbf{n}$,
- $s_{\mathbf{r}}^* = e$ if and only if $\mathbf{r} = \emptyset$ if and only if $\mathbf{i} = \mathbf{mq}$ and $\mathbf{j} = \mathbf{nq}$,
- $s_{\mathbf{r}} = e$ if and only if $\mathbf{r} = \emptyset$ if and only if $\mathbf{i} = \mathbf{mp}$ and $\mathbf{j} = \mathbf{np}$.

□

3.3. **A purely infinite quotient of** $(\ell^1(\mathbf{Cu}_2 \setminus \{\Diamond\}), \#)$**.** From now on we let $\mathcal{A} := (\ell^1(\mathrm{Cu}_2 \setminus \{\Diamond\}), \#)$. In this section we show that $\mathcal{A}$ has a quotient $\mathcal{A}/\mathcal{J}$ which is a purely infinite Banach $*$-algebra.

Remark 3.5. Suppose we start instead with the group ring $\mathbb{C}[\mathrm{Cu}_2]$, which is just the algebra of finitely supported elements of $\ell^1(\mathrm{Cu}_2)$, and similarly quotient by the span of $\delta_\Diamond$. As observed in [22, Section 1], the algebra $\mathbb{C}[\mathrm{Cu}_2]/\mathbb{C}\delta_\Diamond$ was studied, with a different presentation, by Cohn in [5, Section 5], and is sometimes called the *Cohn algebra* C_2.

We could hence view $\mathcal{A}$ as being a Banach algebra completion of C_2. To our knowledge, this algebra has not been studied from this perspective; for example, it is not mentioned in [8]. We make remarks about links, or lack thereof, with Phillips's work in [22] below, Remark 3.29.

Let us observe first that in view of Lemma 3.2, we may write

$$f = \sum_{t \in \mathrm{Cu}_2 \setminus \{\Diamond\}} f(t)\delta_t = \sum_{\mathbf{i},\mathbf{j} \in \mathbf{I}} f(s_\mathbf{i} s_\mathbf{j}^*)\delta_{s_\mathbf{i} s_\mathbf{j}^*} \qquad (f \in \mathcal{A}). \tag{3.12}$$

3.3.1. *Purely infinite elements of* $\mathcal{A}$. Our goal in this section is to find a useful sufficient condition which guarantees that an element $f \in \mathcal{A}$ is purely infinite, in other words, that there exist $g, h \in \mathcal{A}$ with $g\#f\#h = \delta_e$.

Definition 3.6.

- *Let* $v \in Cu_2 \setminus \{\Diamond\}$*, and let* $\mathbf{i}, \mathbf{j}$ *be the unique elements in* $\mathbf{I}$ *with* $v = s_\mathbf{i} s_\mathbf{j}^*$.
  - *Suppose* $\mathbf{n} \in \mathbf{L}$*. We define*

$$v_l^\mathbf{n} := s_{\mathbf{i}\mathbf{n}_l} s_{\mathbf{j}\mathbf{n}_l}^* = s_\mathbf{i} s_{(n_1,\ldots,n_l)} s_{(n_1,\ldots,n_l)}^* s_\mathbf{j}^* \qquad (l \in \mathbb{N}_0). \tag{3.13}$$

  - *Suppose* $\mathbf{n} \in \mathbf{I}$*. There is a unique* $\alpha \in \mathbb{N}_0$ *satisfying* $\mathbf{n} \in \mathbf{I}_\alpha$*; hence* $\mathbf{n} = (n_1, n_2, \ldots, n_\alpha)$*, where* $n_i \in \{1, 2\}$ *whenever* $1 \leqslant i \leqslant \alpha$*. We define* $v_l^\mathbf{n}$ *as in* (3.13) *provided* $l \in \mathbb{N}_0$ *is such that* $l \leqslant \alpha$*. Otherwise* $v_l^\mathbf{n}$ *is undefined.*

  We have in particular $v_0^\mathbf{n} = s_{\mathbf{i}\mathbf{n}_0} s_{\mathbf{j}\mathbf{n}_0}^* = s_\mathbf{i} s_\mathbf{j}^* = v$*, and that* $e_l^\mathbf{n} = s_{(n_1,\ldots,n_l)} s_{(n_1,\ldots,n_l)}^*$.
- *We say that* $f \in \mathcal{A}$ has zero sums at $v = s_\mathbf{i} s_\mathbf{j}^* \in \mathrm{Cu}_2 \setminus \{\Diamond\}$ *if*

$$\sum_{l \in \mathbb{N}_0} f(v_l^\mathbf{n}) = \sum_{l \in \mathbb{N}_0} f(s_{\mathbf{i}\mathbf{n}_l} s_{\mathbf{j}\mathbf{n}_l}^*) = 0 \qquad (\mathbf{n} \in \mathbf{L}). \tag{3.14}$$

Notice that as f is an ℓ^1 element, the sum in (3.14) is absolutely convergent.

Lemma 3.7. *Let* $\mathbf{n} \in \mathbf{I} \cup \mathbf{L}$ *and* $f \in \mathcal{A}$*. Then*

$$\delta_{s_{\mathbf{n}_l}^*} \# f \# \delta_{s_{\mathbf{n}_l}} = \left(\sum_{k=0}^{l} f(e_k^\mathbf{n}) \right) \delta_e + \sum_{\substack{\mathbf{i},\mathbf{j} \in \mathbf{I} \\ s_{\mathbf{n}_l}^* s_\mathbf{i} s_\mathbf{j}^* s_{\mathbf{n}_l} \notin \{e, \Diamond\}}} f(s_\mathbf{i} s_\mathbf{j}^*) \delta_{s_{\mathbf{n}_l}^* s_\mathbf{i} s_\mathbf{j}^* s_{\mathbf{n}_l}} \qquad (l \in \mathbb{N}_0). \tag{3.15}$$

Proof. Let us fix an $l \in \mathbb{N}_0$. We first note that by Lemma 3.4

$$\begin{aligned} \{s_\mathbf{i} s_\mathbf{j}^* \colon \mathbf{i}, \mathbf{j} \in \mathbf{I},\ s_{\mathbf{n}_l}^* s_\mathbf{i} s_\mathbf{j}^* s_{\mathbf{n}_l} = e\} &= \{s_\mathbf{i} s_\mathbf{j}^* \colon \mathbf{i}, \mathbf{j} \in \mathbf{I},\ \mathbf{n}_l = \mathbf{i}\mathbf{p},\ \mathbf{n}_l = \mathbf{j}\mathbf{p} \text{ for some } \mathbf{p} \in \mathbf{I}\} \\ &= \{s_\mathbf{i} s_\mathbf{i}^* \colon \mathbf{i} = \mathbf{n}_k \text{ for some } 0 \leqslant k \leqslant l\} \\ &= \{e_k^\mathbf{n} \colon 0 \leqslant k \leqslant l\}. \end{aligned}$$

Consequently

$$\begin{aligned}\delta_{s^*_{\mathbf{n}_l}}\#f\#\delta_{s_{\mathbf{n}_l}} &= \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I}\\ s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}\neq\Diamond}} f(s_{\mathbf{i}}s^*_{\mathbf{j}})\delta_{s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}}\\ &= \left(\sum_{k=0}^{l} f(e^{\mathbf{n}}_k)\right)\delta_e + \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I}\\ s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}\notin\{e,\Diamond\}}} f(s_{\mathbf{i}}s^*_{\mathbf{j}})\delta_{s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}},\end{aligned}$$

as claimed. □

Proposition 3.8. *Let $f \in \mathcal{A}$ be such that it does not have zero sums at the multiplicative unit $e \in Cu_2$. Then there exist $g, h \in \mathcal{A}$ with $g\#f\#h = \delta_e$.*

Proof. By the assumption there is an $\mathbf{n} = (n_1, n_2, \ldots, n_k, \ldots) \in \mathbf{L}$ such that $\sum_{k\in\mathbb{N}_0} f(e^{\mathbf{n}}_k) \neq 0$. Let us set $z_N := \sum_{k=0}^{N} f(e^{\mathbf{n}}_k)$ for each $N \in \mathbb{N}_0$. As $f \in \mathcal{A}$, the sequence (z_N) converges to some non-zero element in $\mathbb{C}$, therefore there is an $\varepsilon > 0$ and $N' \in \mathbb{N}_0$ such that $|z_n| \geqslant 2\varepsilon$ for each $n \geqslant N'$. From Lemma 3.7 we see that

$$\delta_{s^*_{\mathbf{n}_l}}\#f\#\delta_{s_{\mathbf{n}_l}} = z_l\delta_e + \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I}\\ s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}\notin\{e,\Diamond\}}} f(s_{\mathbf{i}}s^*_{\mathbf{j}})\delta_{s^*_{\mathbf{n}_l}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_l}} \qquad (l \in \mathbb{N}_0). \tag{3.16}$$

Let us take an $f' \in \mathcal{A}$ with finite support such that $\|f - f'\| < \varepsilon$. We can hence pick some $M \in \mathbb{N}$ such that $M \geqslant N'$ and $M \geqslant \max\{\mathrm{length}(t)\colon t \in \mathrm{supp}(f')\}$. In particular, $f'(t) = 0$ whenever $t \in \mathrm{Cu}_2 \setminus \{\Diamond\}$ is such that $\mathrm{length}(t) > M$.

To ease notation, we put $z := z_M$.

Claim 3.9. *There is a $\mathbf{p} \in \mathbf{I}$ such that*

$$\delta_{s^*_{\mathbf{p}}}\#\delta_{s^*_{\mathbf{n}_M}}\#f'\#\delta_{s_{\mathbf{n}_M}}\#\delta_{s_{\mathbf{p}}} = \delta_{s^*_{\mathbf{n}_M\mathbf{p}}}\#f'\#\delta_{s_{\mathbf{n}_M\mathbf{p}}} = z\delta_e. \tag{3.17}$$

Proof of Claim 3.9. By Lemma 3.7 we have

$$\delta_{s^*_{\mathbf{n}_M}}\#f'\#\delta_{s_{\mathbf{n}_M}} = z\delta_e + \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I}\\ s^*_{\mathbf{n}_M}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_M}\notin\{e,\Diamond\}}} f'(s_{\mathbf{i}}s^*_{\mathbf{j}})\delta_{s^*_{\mathbf{n}_M}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_M}} =: z\delta_e + h_0. \tag{3.18}$$

If $h_0 = 0$ then we are done. Otherwise, $H_0 := \mathrm{supp}(h_0) \neq \emptyset$. We claim that there is an $i \in \{1, 2\}$ such that

$$|\{s^*_i t s_i \colon t \in H_0,\ s^*_i t s_i \neq \Diamond\}| < |H_0|. \tag{3.19}$$

To show this, observe that as $\mathrm{supp}(f')$, and hence also H_0, is finite, it is enough to see that $s^*_1 t s_1 = \Diamond$ or $s^*_2 t s_2 = \Diamond$ for some $t \in H_0$. This readily follows however from $\{e\} \neq H_0$ (which clearly holds as $e \notin H_0$).

For this choice of i, applying Lemma 3.7 again we see that

$$\begin{aligned}\delta_{s^*_{\mathbf{n}_M i}}\#f'\#\delta_{s_{\mathbf{n}_M i}} &= \delta_{s_i^*}\#(z\delta_e+h_0)\#\delta_{s_i} = z\delta_e + \delta_{s_i^*}\#h_0\#\delta_{s_i}\\ &= (z+h_0(e)+h_0(e_1^i))\delta_e + \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathbf{I}\\ s_i^*s_{\mathbf{i}}s_{\mathbf{j}}^*s_i\notin\{e,\Diamond\}}} h_0(s_{\mathbf{i}}s_{\mathbf{j}}^*)\delta_{s_i^*s_{\mathbf{i}}s_{\mathbf{j}}^*s_i}\\ &=: (z+h_0(e)+h_0(e_1^i))\delta_e + h_1. \end{aligned}\tag{3.20}$$

Note that $\operatorname{supp}(h_1)\subseteq\{s_i^*ts_i\colon t\in H_0,\ s_i^*ts_i\neq\Diamond\}$.

On the one hand $h_0(e)=0$. On the other hand $s^*_{\mathbf{n}_M}s_{\mathbf{i}}s^*_{\mathbf{j}}s_{\mathbf{n}_M}=s_is_i^*$ if and only if $\mathbf{i}=\mathbf{n}_Mi$ and $\mathbf{j}=\mathbf{n}_Mi$ by Lemma 3.4, hence

$$h_0(e_1^i)=h_0(s_is_i^*)=f'(s_{\mathbf{n}_Mi}s^*_{\mathbf{n}_Mi})=0.\tag{3.21}$$

The last equality follows because $\operatorname{length}(s_{\mathbf{n}_Mi}s^*_{\mathbf{n}_Mi})=2(M+1)$, and f' vanishes on elements of Cu_2 of length at least $M+1$. Consequently,

$$\delta_{s^*_{\mathbf{n}_M i}}\#f'\#\delta_{s_{\mathbf{n}_M i}} = z\delta_e+h_1,\tag{3.22}$$

where $H_1:=\operatorname{supp}(h_1)$ is such that $|H_1|<|H_0|$.

Let us fix some $k_0>|\operatorname{supp}(f')|$. Continuing recursively, we obtain $i_1,i_2,\dots,i_{k_0}\in\{1,2\}$ and finitely supported functions $(h_k)_{k=1}^{k_0}$ in $\mathcal{A}$ with $H_k:=\operatorname{supp}(h_k)$ such that

$$\delta_{s^*_{\mathbf{n}_M(i_1,\dots,i_k)}}\#f'\#\delta_{s_{\mathbf{n}_M(i_1,\dots,i_k)}} = z\delta_e+h_k \qquad (1\leqslant k\leqslant k_0),\tag{3.23}$$

$$|H_0|>|H_1|>\ldots>|H_{k_0}|.\tag{3.24}$$

As $\operatorname{supp}(f')$ is finite, we must have that $H_{k_0}=\emptyset$ or equivalently $h_{k_0}=0$. Thus setting $\mathbf{p}:=(i_1,\dots,i_{k_0})\in\mathbf{I}$ yields the claim. □

From the claim we obtain

$$\begin{aligned}\|\delta_e - z^{-1}\delta_{s^*_{\mathbf{n}_M\mathbf{p}}}\#f\#\delta_{s_{\mathbf{n}_M\mathbf{p}}}\| &= |z|^{-1}\|\delta_{s^*_{\mathbf{n}_M\mathbf{p}}}\#(f'-f)\#\delta_{s_{\mathbf{n}_M\mathbf{p}}}\|\\ &\leqslant |z|^{-1}\|f-f'\|<1/2,\end{aligned}\tag{3.25}$$

thus the Carl Neumann series implies $u:=z^{-1}\delta_{s^*_{\mathbf{n}_M\mathbf{p}}}\#f\#\delta_{s_{\mathbf{n}_M\mathbf{p}}}\in\operatorname{inv}(\mathcal{A})$. Hence setting $g:=u^{-1}\#z^{-1}\delta_{s^*_{\mathbf{n}_M\mathbf{p}}}$ and $h:=\delta_{s_{\mathbf{n}_M\mathbf{p}}}$ concludes the proof. □

3.3.2. *The description of purely infinite elements in terms of the ideal $\mathcal{J}$.* In the following, let $\mathcal{J}$ denote the closed, two-sided ideal in $\mathcal{A}$ generated by the element

$$f_0:=\delta_e-\delta_{s_1s_1^*}-\delta_{s_2s_2^*}.\tag{3.26}$$

Clearly f_0 is an projection in $\mathcal{A}$, in other words, $f_0^2=f_0$ and $f_0^*=f_0$. We immediately see from the formula (3.12) and Lemma 3.2 (2) that

$$\mathcal{J}=\overline{\operatorname{span}}\{g\#f_0\#h\colon g,h\in\mathcal{A}\}=\overline{\operatorname{span}}\{\delta_{s_{\mathbf{i}}s^*_{\mathbf{k}}}\#f_0\#\delta_{s_{\mathbf{l}}s^*_{\mathbf{j}}}\colon \mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\in\mathbf{I}\}.\tag{3.27}$$

Corollary 3.10. *$\mathcal{A}/\mathcal{J}$ is a Banach $*$-algebra.*

Proof. We already saw that $\mathcal{A}$ is a Banach $*$-algebra, hence it is enough to show that the closed, two-sided ideal $\mathcal{J}$ is also a $*$-ideal. This however readily follows from (3.27) and $f_0^*=f_0$. □

Remark 3.11. Continuing Remark 3.5, in the Cohn Algebra $C_2 \cong \mathbb{C}[\mathrm{Cu}_2]/\mathbb{C}\delta_\lozenge$ we could also consider the ideal, say J_2, generated by f_0. Then C_2/J_2 is seen to be isomorphic to the *Leavitt algebra* L_2, see [22, Section 1], which was first considered (over the field with 2 elements) in [20].

Again, $\mathcal{A}/\mathcal{J}$ is a Banach algebraic completion of L_2, which again seems not to have been considered in the literature before. Compare with Remark 3.29 below.

Let us introduce some new terminology which will render the technical proofs in this section significantly more transparent.

Definition 3.12.

(i) *An element* $t = s_{\mathbf{i}} s_{\mathbf{j}}^* \in \mathit{Cu}_2 \setminus \{\lozenge\}$ *is* symmetric *if* $\mathbf{i} = \mathbf{j}$.

(ii) *We say that* $t \in \mathit{Cu}_2 \setminus \{\lozenge\}$ *is a* symmetric extension of $r = s_{\mathbf{m}} s_{\mathbf{n}}^* \in \mathit{Cu}_2 \setminus \{\lozenge\}$ *if there exists a symmetric* $u \in \mathit{Cu}_2 \setminus \{\lozenge\}$ *with* $t = s_{\mathbf{m}} u s_{\mathbf{n}}^*$. *If in addition* $u \neq e$ *then we say that* t *is a* proper symmetric extension of r.

(iii) *For some* $t \in \mathit{Cu}_2 \setminus \{\lozenge\}$ *the set of symmetric extensions of* t *is denoted by* S_t.

(iv) *An element of* $\mathit{Cu}_2 \setminus \{\lozenge\}$ has no symmetry *if it is not the proper symmetric extension of any element in* $\mathit{Cu}_2 \setminus \{\lozenge\}$.

The following are immediate from the definition.

Remark 3.13.

- An element $t \in \mathrm{Cu}_2 \setminus \{\lozenge\}$ is a symmetric extension of $r = s_{\mathbf{m}} s_{\mathbf{n}}^* \in \mathrm{Cu}_2 \setminus \{\lozenge\}$ if and only if there exists $\mathbf{i} \in \mathbf{I}$ with $t = s_{\mathbf{mi}} s_{\mathbf{ni}}^*$. Also, t is a proper extension of r if and only if $\mathbf{i} \neq \emptyset$.
- An element $t \in \mathrm{Cu}_2 \setminus \{\lozenge\}$ has no symmetry if and only if whenever $\mathbf{m}, \mathbf{n}, \mathbf{i} \in \mathbf{I}$ are such that $t = s_{\mathbf{mi}} s_{\mathbf{ni}}^*$ then $\mathbf{i} = \emptyset$.

Lemma 3.14. *The set*

$$\{S_v \colon v \in \mathit{Cu}_2 \setminus \{\lozenge\} \textit{ has no symmetry}\} \tag{3.28}$$

forms a partition of $\mathit{Cu}_2 \setminus \{\lozenge\}$.

Proof. Let $t \in \mathrm{Cu}_2 \setminus \{\lozenge\}$ be arbitrary. There exist unique $\mathbf{p}, \mathbf{q} \in \mathbf{I}$ such that $t = s_{\mathbf{p}} s_{\mathbf{q}}^*$. Let $\alpha \in \mathbb{N}_0$ be maximal with respect to the property that there is an $\mathbf{i} \in \mathbf{I}_\alpha$ with $\mathbf{p} = \mathbf{mi}$ and $\mathbf{q} = \mathbf{ni}$ for some $\mathbf{m}, \mathbf{n} \in \mathbf{I}$. Then $t = s_{\mathbf{p}} s_{\mathbf{q}}^* = s_{\mathbf{mi}} s_{\mathbf{ni}}^* = s_{\mathbf{m}} (s_{\mathbf{i}} s_{\mathbf{i}}^*) s_{\mathbf{n}}^*$ shows that t is the symmetric extension of $v := s_{\mathbf{m}} s_{\mathbf{n}}^*$. Observe that v has no symmetry. For assume towards a contradiction it has, then there exists $\mathbf{k} \in \mathbf{I} \setminus \mathbf{I}_0$ such that $v = s_{\mathbf{ak}} s_{\mathbf{bk}}^*$ for some $\mathbf{a}, \mathbf{b} \in \mathbf{I}$. Therefore $\mathbf{m} = \mathbf{ak}$ and $\mathbf{n} = \mathbf{bk}$ must hold, consequently $t = s_{\mathbf{mi}} s_{\mathbf{ni}}^* = s_{\mathbf{aki}} s_{\mathbf{bki}}^*$. This contradicts the maximality of α.

Let $v, w \in \mathrm{Cu}_2 \setminus \{\lozenge\}$ be without symmetry. Assume there is some $t \in S_v \cap S_w$. Let $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ be unique with $v = s_{\mathbf{i}} s_{\mathbf{j}}^*$, then $t = s_{\mathbf{ik}} s_{\mathbf{jk}}^*$ for some $\mathbf{k} \in \mathbf{I}$. Similarly, let $\mathbf{p}, \mathbf{q} \in \mathbf{I}$ be unique with $w = s_{\mathbf{p}} s_{\mathbf{q}}^*$, then $t = s_{\mathbf{pl}} s_{\mathbf{ql}}^*$ for some $\mathbf{l} \in \mathbf{I}$. As $s_{\mathbf{ik}} s_{\mathbf{jk}}^* = t = s_{\mathbf{pl}} s_{\mathbf{ql}}^*$, it follows that $\mathbf{ik} = \mathbf{pl}$ and $\mathbf{jk} = \mathbf{ql}$. We want to show that $v = w$, equivalently $\mathbf{i} = \mathbf{p}$ and $\mathbf{j} = \mathbf{q}$. Let $\alpha, \beta \in \mathbb{N}_0$ be the unique numbers such that $\mathbf{k} \in \mathbf{I}_\alpha$ and $\mathbf{l} \in \mathbf{I}_\beta$. Note that it is enough to show that $\alpha = \beta$. Assume towards a contradiction that, say, $\alpha < \beta$. Then there are $\mathbf{m} \in \mathbf{I}_{\beta-\alpha}$ and $\mathbf{n} \in \mathbf{I}_\alpha$ with $\mathbf{l} = \mathbf{mn}$. Thus $\mathbf{ik} = \mathbf{pmn}$, hence from $\mathbf{k}, \mathbf{n} \in \mathbf{I}_\alpha$ we obtain $\mathbf{i} = \mathbf{pm}$. Similarly, we get $\mathbf{j} = \mathbf{qm}$. But then $v = s_{\mathbf{i}} s_{\mathbf{j}}^* = s_{\mathbf{pm}} s_{\mathbf{qm}}^*$, which by $\beta - \alpha > 0$ contradicts that v has no symmetry. $\square$

Proposition 3.15. *Suppose $f \in \mathcal{A}$ is such that it has zero sums at some $v \in Cu_2 \setminus \{\Diamond\}$. Define $h := \sum_{t\in S_v} f(t)\delta_t \in \mathcal{A}$. Then $h \in \mathcal{J}$.*

Proof. Let $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ be such that $v = s_{\mathbf{i}}s_{\mathbf{j}}^*$. Then $S_v = \{s_{\mathbf{ik}}s_{\mathbf{jk}}^* \colon \mathbf{k} \in \mathbf{I}\}$ and hence $h = \sum_{\mathbf{k}\in\mathbf{I}} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*}$. From $f_0 \in \mathcal{J}$ we immediately get

$$\delta_{s_{\mathbf{im}}s_{\mathbf{jm}}^*} - \delta_{s_{\mathbf{im}1}s_{\mathbf{jm}1}^*} - \delta_{s_{\mathbf{im}2}s_{\mathbf{jm}2}^*} = \delta_{s_{\mathbf{im}}}\#f_0\#\delta_{s_{\mathbf{jm}}^*} \in \mathcal{J} \qquad (\mathbf{m} \in \mathbf{I}). \tag{3.29}$$

In particular, setting $\mathbf{m} := \emptyset$ in (3.29) yields

$$\delta_v - \delta_{s_{\mathbf{i}1}s_{\mathbf{j}1}^*} - \delta_{s_{\mathbf{i}2}s_{\mathbf{j}2}^*} \in \mathcal{J}. \tag{3.30}$$

Hence from

$$h = f(v)\delta_v + f(s_{\mathbf{i}1}s_{\mathbf{j}1}^*)\delta_{s_{\mathbf{i}1}s_{\mathbf{j}1}^*} + f(s_{\mathbf{i}2}s_{\mathbf{j}2}^*)\delta_{s_{\mathbf{i}2}s_{\mathbf{j}2}^*} + \sum_{\mathbf{k}\in\mathbf{I}\setminus(\mathbf{I}_0\cup\mathbf{I}_1)} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*} \tag{3.31}$$

and (3.30) we see that

$$\begin{aligned} &\big(f(v) + f(s_{\mathbf{i}1}s_{\mathbf{j}1}^*)\big)\delta_{s_{\mathbf{i}1}s_{\mathbf{j}1}^*} + \big(f(v) + f(s_{\mathbf{i}2}s_{\mathbf{j}2}^*)\big)\delta_{s_{\mathbf{i}2}s_{\mathbf{j}2}^*} \\ &\qquad + \sum_{\mathbf{k}\in\mathbf{I}\setminus(\mathbf{I}_0\cup\mathbf{I}_1)} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*} - h \in \mathcal{J} \\ \iff &\sum_{\mathbf{k}\in\mathbf{I}_1} \big(f(v) + f(s_{\mathbf{i}}e_1^{\mathbf{k}}s_{\mathbf{j}}^*)\big)\delta_{s_{\mathbf{i}}e_1^{\mathbf{k}}s_{\mathbf{j}}^*} \\ &\qquad + \sum_{\mathbf{k}\in\mathbf{I}\setminus(\mathbf{I}_0\cup\mathbf{I}_1)} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*} - h \in \mathcal{J}. \end{aligned} \tag{3.32}$$

Continuing inductively, we obtain

$$\sum_{\mathbf{k}\in\mathbf{I}_n} \left(f(v) + \sum_{l=1}^{n} f(s_{\mathbf{i}}e_l^{\mathbf{k}}s_{\mathbf{j}}^*)\right)\delta_{s_{\mathbf{i}}e_n^{\mathbf{k}}s_{\mathbf{j}}^*} + \sum_{\mathbf{k}\in\mathbf{I}\setminus\bigcup_{r=0}^{n}\mathbf{I}_r} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*} - h \in \mathcal{J} \qquad (n \in \mathbb{N}). \tag{3.33}$$

That f has zero sums at v is to say $f(v) = -\sum_{l\in\mathbb{N}} f(s_{\mathbf{i}}e_l^{\mathbf{k}}s_{\mathbf{j}}^*)$. Therefore (3.33) is equivalent to

$$-\sum_{\mathbf{k}\in\mathbf{I}_n} \left(\sum_{l>n} f(s_{\mathbf{i}}e_l^{\mathbf{k}}s_{\mathbf{j}}^*)\right)\delta_{s_{\mathbf{i}}e_n^{\mathbf{k}}s_{\mathbf{j}}^*} + \sum_{\mathbf{k}\in\mathbf{I}\setminus\bigcup_{r=0}^{n}\mathbf{I}_r} f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)\delta_{s_{\mathbf{ik}}s_{\mathbf{jk}}^*} - h \in \mathcal{J} \qquad (n \in \mathbb{N}). \tag{3.34}$$

Now $\sum_{t\in\mathrm{Cu}_2\setminus\{\Diamond\}} |f(t)| < \infty$ implies

$$\sum_{\mathbf{k}\in\mathbf{I}_n} \left|\sum_{l>n} f(s_{\mathbf{i}}e_l^{\mathbf{k}}s_{\mathbf{j}}^*)\right| \to 0 \quad \text{and} \quad \sum_{\mathbf{k}\in\mathbf{I}\setminus\bigcup_{r=0}^{n}\mathbf{I}_r} |f(s_{\mathbf{ik}}s_{\mathbf{jk}}^*)| \to 0 \qquad (n \to \infty). \tag{3.35}$$

As $\mathcal{J}$ is closed, we conclude from (3.35) and (3.34) that $h \in \mathcal{J}$. ☐

The main result of this section is the following.

Theorem 3.16. *Let $f \in \mathcal{A}$ be such that $f \notin \mathcal{J}$. Then there exist $g, h \in \mathcal{A}$ such that $g\#f\#h = \delta_e$.*

Proof. Assume first that there exists $v = s_{\mathbf{i}}s_{\mathbf{j}}^* \in Cu_2 \setminus \{\Diamond\}$ without symmetry such that f does not have zero sums at v. We *claim* that $\overline{f} := \delta_{s_{\mathbf{i}}^*}\#f\#\delta_{s_{\mathbf{j}}} \in \mathcal{A}$ does not have zero sums at e. To see this, let us fix an $\mathbf{n} \in \mathbf{L}$. Using Lemma 3.4 we see that

$$\begin{aligned}\sum_{l\in\mathbb{N}_0} \overline{f}(e_l^{\mathbf{n}}) &= \sum_{l\in\mathbb{N}_0} \overline{f}(s_{\mathbf{n}_l}s_{\mathbf{n}_l}^*) = \sum_{l\in\mathbb{N}_0} (\delta_{s_{\mathbf{i}}^*}\#f\#\delta_{s_{\mathbf{j}}})(s_{\mathbf{n}_l}s_{\mathbf{n}_l}^*) \\ &= \sum_{l\in\mathbb{N}_0}\left(\sum_{\mathbf{p},\mathbf{q}\in\mathbf{I}} f(s_{\mathbf{p}}s_{\mathbf{q}}^*)\delta_{s_{\mathbf{i}}^*s_{\mathbf{p}}s_{\mathbf{q}}^*s_{\mathbf{j}}}\right)(s_{\mathbf{n}_l}s_{\mathbf{n}_l}^*) = \sum_{l\in\mathbb{N}_0} f(s_{\mathbf{i}\mathbf{n}_l}s_{\mathbf{j}\mathbf{n}_l}^*) = \sum_{l\in\mathbb{N}_0} f(v_l^{\mathbf{n}}),\end{aligned} \tag{3.36}$$

hence the claim follows because f does not have zero sums at v. We can thus apply Proposition 3.8; there exist $\overline{g}, \overline{h} \in \mathcal{A}$ with $\delta_e = \overline{g}\#\overline{f}\#\overline{h}$. Consequently $\delta_e = (\overline{g}\#\delta_{s_{\mathbf{i}}^*})\#f\#(\delta_{s_{\mathbf{j}}}\#\overline{h})$.

Assume towards a contradiction that f has zero sums at every $v \in Cu_2 \setminus \{\Diamond\}$ without symmetry. We set $f_v := \sum_{t\in S_v} f(t)\delta_t$ for every $v \in \mathrm{Cu}_2 \setminus \{\Diamond\}$ without symmetry. As $\mathrm{Cu}_2 \setminus \{\Diamond\}$ is countable, the set of elements without symmetry may be enumerated as (v_n). In view of Lemma 3.14 the set $\{S_{v_n} : n \in \mathbb{N}\}$ consists of mutually disjoint sets, consequently

$$\left\| f - \sum_{n=1}^{N} f_{v_n} \right\| = \sum_{\substack{t\in\mathrm{Cu}_2\setminus\{\Diamond\} \\ t\notin\cup_{n=1}^N S_{v_n}}} |f(t)| \to 0 \qquad (N \to \infty). \tag{3.37}$$

The convergence of the right-hand side of (3.37) follows from Lemma 3.14; namely, that $\{S_{v_n} : n \in \mathbb{N}\}$ covers $\mathrm{Cu}_2 \setminus \{\Diamond\}$. This shows $f \in \overline{\mathrm{span}}\{f_{v_n} : n \in \mathbb{N}\}$. Proposition 3.15 yields however $f_{v_n} \in \mathcal{J}$ for each $n \in \mathbb{N}$. Thus $f \in \mathcal{J}$ must hold, a contradiction. ☐

From now on we let $\pi_{\mathcal{J}} \colon \mathcal{A} \to \mathcal{A}/\mathcal{J}$ denote the quotient map.

Corollary 3.17. *Let $a \in \mathcal{A}/\mathcal{J}$ be non-zero. Then there exist $b, c \in \mathcal{A}/\mathcal{J}$ such that $bac = 1_{\mathcal{A}/\mathcal{J}}$.*

Proof. Let $f \in \mathcal{A}$ be such that $a = \pi_{\mathcal{J}}(f)$. That a is non-zero is equivalent to $f \notin \mathcal{J}$. Hence by Theorem 3.16 there are $g, h \in \mathcal{A}$ such that $g\#f\#h = \delta_e$. Setting $b := \pi_{\mathcal{J}}(g)$ and $c := \pi_{\mathcal{J}}(h)$ finishes the proof. ☐

Note that in order to conclude that $\mathcal{A}/\mathcal{J}$ is purely infinite we still need to show that it is not isomorphic to $\mathbb{C}$. This will immediately follow from Proposition 3.19 in the next section.

3.4. **Representing $\mathcal{A}/\mathcal{J}$ in $\mathcal{B}(\ell^1)$.** In this section we show how to represent a quotient of $\mathcal{A}$ inside $\mathcal{B}(\ell^1(\mathbb{N}))$, the unital Banach algebra of bounded linear operators on $\ell^1(\mathbb{N})$. As an application, we will see that $\mathcal{A}/\mathcal{J}$ is infinite-dimensional, hence together with Corollary 3.17 we conclude that $\mathcal{A}/\mathcal{J}$ is purely infinite (see Theorem 3.21).

To this end, we first define operators A_1, A_2, B_1, B_2 on $\ell^1 := \ell^1(\mathbb{N})$ by

$$(A_1x)(n) = x_{2n}, \qquad (A_2x)(n) = x_{2n-1} \qquad (x \in \ell^1), \tag{3.38}$$

and

$$(B_1x)(n) = \begin{cases} x_{n/2} & \text{if } n \in 2\mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \quad (B_2x)(n) = \begin{cases} x_{(n+1)/2} & \text{if } n \in 2\mathbb{N}-1, \\ 0 & \text{otherwise.} \end{cases} \qquad (x \in \ell^1). \tag{3.39}$$

It is immediate that $A_i, B_i, \in \mathcal{B}(\ell^1)$ with $\|A_i\| = 1 = \|B_i\|$ for $i \in \{1,2\}$. Moreover, the following relations hold:

$$A_1B_1 = I_{\ell^1} = A_2B_2, \quad A_1B_2 = 0 = A_2B_1, \quad B_1A_1 + B_2A_2 = I_{\ell^1}, \tag{3.40}$$

where I_{ℓ^1} denotes the identity operator on ℓ^1.

Remark 3.18. Let us note that the set $\{B_1^n \colon n \in \mathbb{N}\}$ is linearly independent in $\mathcal{B}(\ell^1)$. Indeed, suppose $(\alpha_n)_{n=1}^N$ is a finite family of scalars such that $\sum_{n=1}^N \alpha_n B_1^n = 0$. Let (e_n) be the standard unit vector basis of ℓ^1. We see that

$$0 = \sum_{n=1}^{N} \alpha_n B_1^n e_1 = \sum_{n=1}^{N} \alpha_n e_{2^n}, \tag{3.41}$$

hence $\alpha_n = 0$ must hold whenever $1 \leqslant n \leqslant N$.

Proposition 3.19. *There is a continuous, unital algebra homomorphism* $\Theta \colon \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1)$ *with*

$$\Theta(\pi_{\mathcal{J}}(\delta_{s_i^*})) = A_i \quad \textit{and} \quad \Theta(\pi_{\mathcal{J}}(\delta_{s_i})) = B_i \qquad (i \in \{1,2\}). \tag{3.42}$$

In particular $\mathcal{A}/\mathcal{J}$ *is infinite-dimensional and non-commutative.*

Proof. The operators $A_1, A_2, B_1, B_2 \in \mathcal{B}(\ell^1)$ are subject to the relations $A_1B_1 = I_{\ell^1} = A_2B_2$ and $A_2B_1 = 0 = A_1B_2$, hence there is a unique semigroup homomorphism

$$\phi \colon \mathrm{Cu}_2 \to \mathcal{B}(\ell^1) \tag{3.43}$$

which satisfies $\phi(s_1^*) = A_1$, $\phi(s_1) = B_1$, $\phi(s_2^*) = A_2$ and $\phi(s_2) = B_2$. Notice that in particular $\phi(e) = \phi(s_1^*s_1) = \phi(s_1^*)\phi(s_1) = A_1B_1 = I_{\ell^1}$ and $\phi(\Diamond) = \phi(s_1^*s_2) = \phi(s_1^*)\phi(s_2) = A_1B_2 = 0$. By Lemma 3.2 (2), and because the operators $A_1, A_2, B_1, B_2 \in \mathcal{B}(\ell^1)$ have norm one, we see that $\|\phi(t)\| \leqslant 1$ for every $t \in \mathrm{Cu}_2$.

It follows that there is a unique continuous algebra homomorphism

$$\theta \colon \mathcal{A} = (\ell^1(\mathrm{Cu}_2 \setminus \{\Diamond\}), \#) \to \mathcal{B}(\ell^1) \tag{3.44}$$

such that $\|\theta\| \leqslant 1$ and $\theta(\delta_t) = \phi(t)$ for all $t \in \mathrm{Cu}_2 \setminus \{\Diamond\}$.

In particular θ is unital as $\theta(\delta_e) = \phi(e) = I_{\ell^1}$. Moreover, from the relation $B_1A_1 + B_2A_2 = I_{\ell^1}$ we see

$$\theta(f_0) = \theta(\delta_e) - \theta(\delta_{s_1})\theta(\delta_{s_1^*}) - \theta(\delta_{s_2})\theta(\delta_{s_2^*}) = I_{\ell^1} - B_1A_1 - B_2A_2 = 0, \tag{3.45}$$

consequently $\mathcal{J} \subseteq \ker(\theta)$. Therefore there is unique continuous algebra homomorphism

$$\Theta \colon \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1) \tag{3.46}$$

with $\|\Theta\| \leqslant 1$ such that $\Theta \circ \pi_{\mathcal{J}} = \theta$, where $\pi_{\mathcal{J}} \colon \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is the quotient map. Clearly $\Theta(\pi_{\mathcal{J}}(\delta_t)) = \theta(\delta_t) = \phi(t)$ for each $t \in \mathrm{Cu}_2 \setminus \{\Diamond\}$. Consequently the required relations hold.

Let us show that $\mathcal{A}/\mathcal{J}$ is infinite-dimensional. We observe that

$$\{B_1^n \colon n \in \mathbb{N}\} = \{\Theta(\pi_{\mathcal{J}}(\delta_{s_1^n})) \colon n \in \mathbb{N}\} \subseteq \mathrm{Ran}(\Theta), \tag{3.47}$$

and hence $\mathrm{Ran}(\Theta)$ is infinite-dimensional by Remark 3.18. From this it readily follows that $\mathcal{A}/\mathcal{J}$ is infinite-dimensional too.

Finally, it is clear that $\mathcal{A}/\mathcal{J}$ is non-commutative. □

Remark 3.20. It is obvious that the continuous homomorphism $\theta \colon \mathcal{A} \to \mathcal{B}(\ell^1)$ in the proof above is not injective. We remark in passing however, that it is possible to find (even explicitly construct) a continuous, unital, faithful $*$-homomorphism $\mathcal{A} \to \mathcal{B}(\ell^2)$; see [8, Remark 3.16].

Theorem 3.21. *$\mathcal{A}/\mathcal{J}$ is an infinite-dimensional, purely infinite Banach $*$-algebra.*

Proof. This is immediate from Corollaries 3.10 and 3.17, and from Proposition 3.19. □

3.4.1. *A description of the annihilator $\mathcal{J}^{\perp}$.* Let us start by pushing the characterisation of $\mathcal{J}$ given by (3.27) a bit further:

Lemma 3.22. *The following holds:*

$$\mathcal{J} = \overline{\mathrm{span}}\{\delta_{s_{\mathbf{i}}} \# f_0 \# \delta_{s_{\mathbf{j}}^*} \colon \mathbf{i}, \mathbf{j} \in \mathbf{I}\}. \tag{3.48}$$

Proof. Let us fix $\mathbf{k} \in \mathbf{I} \setminus \mathbf{I}_0$. In view of Lemma 3.2 (1) we have either

- $s_{\mathbf{k}}^* s_1 = \Diamond$ and $s_{\mathbf{k}}^* s_2 = s_{\mathbf{p}}^*$, where $\mathbf{p} \in \mathbf{I}$ is such that $\mathbf{k} = 2\mathbf{p}$; or
- $s_{\mathbf{k}}^* s_2 = \Diamond$ and $s_{\mathbf{k}}^* s_1 = s_{\mathbf{q}}^*$, where $\mathbf{q} \in \mathbf{I}$ is such that $\mathbf{k} = 1\mathbf{q}$.

We may assume without loss of generality that the first bullet point holds. Consequently

$$\delta_{s_{\mathbf{k}}^*} \# f_0 = \delta_{s_{\mathbf{k}}^*} - \delta_{s_{\mathbf{k}}^*} \# \delta_{s_1 s_1^*} - \delta_{s_{\mathbf{k}}^*} \# \delta_{s_2 s_2^*} = \delta_{s_{\mathbf{k}}^*} - 0 - \delta_{s_{\mathbf{p}}^* s_2^*} = 0. \tag{3.49}$$

With an entirely analogous argument we can show $f_0 \# \delta_{s_{\mathbf{l}}} = 0$ for any $\mathbf{l} \in \mathbf{I} \setminus \mathbf{I}_0$.

Hence from (3.27) and the above we conclude

$$\mathcal{J} = \overline{\mathrm{span}}\{\delta_{s_{\mathbf{i}} s_{\mathbf{k}}^*} \# f_0 \# \delta_{s_{\mathbf{l}} s_{\mathbf{j}}^*} \colon \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{I}\} = \overline{\mathrm{span}}\{\delta_{s_{\mathbf{i}}} \# f_0 \# \delta_{s_{\mathbf{j}}^*} \colon \mathbf{i}, \mathbf{j} \in \mathbf{I}\}, \tag{3.50}$$

as required. □

Let us define the maps

$$\tau_k \colon \mathrm{Cu}_2 \setminus \{\Diamond\} \to \mathrm{Cu}_2 \setminus \{\Diamond\}; \quad s_{\mathbf{i}} s_{\mathbf{j}}^* \mapsto s_{\mathbf{i}k} s_{\mathbf{j}k}^* \qquad (k \in \{1, 2\}). \tag{3.51}$$

Lemma 3.2 (2) ensures that τ_k is in fact well-defined. By the very same result we actually find that τ_k is injective.

For both $k \in \{1, 2\}$, we can find "induced" bounded linear operators

$$T_k \colon \mathcal{A} \to \mathcal{A} \quad \text{with} \quad T_k(\delta_t) = \delta_{\tau_k(t)} \qquad (t \in \mathrm{Cu}_2 \setminus \{\Diamond\}). \tag{3.52}$$

From injectivity of τ_k it easily follows that T_k is an isometry. Let $T := T_1 + T_2$. Then T is a bounded linear operator on $\mathcal{A}$.

In the following, $\mathcal{A}^*$ denotes the (continuous) dual of $\mathcal{A}$, which we identify with $\ell^\infty(\mathrm{Cu}_2 \setminus \{\Diamond\})$ as a Banach space. Let $T^*\colon \mathcal{A}^* \to \mathcal{A}^*$ denote the adjoint of T. We observe that

$$\begin{aligned}(T^*\mu)(s_{\mathbf{i}}s_{\mathbf{j}}^*) &= \langle T^*\mu, \delta_{s_{\mathbf{i}}s_{\mathbf{j}}^*}\rangle = \langle \mu, T\delta_{s_{\mathbf{i}}s_{\mathbf{j}}^*}\rangle = \langle \mu, T_1\delta_{s_{\mathbf{i}}s_{\mathbf{j}}^*}\rangle + \langle \mu, T_2\delta_{s_{\mathbf{i}}s_{\mathbf{j}}^*}\rangle \\ &= \langle \mu, \delta_{\tau_1(s_{\mathbf{i}}s_{\mathbf{j}}^*)}\rangle + \langle \mu, \delta_{\tau_2(s_{\mathbf{i}}s_{\mathbf{j}}^*)}\rangle = \langle \mu, \delta_{s_{\mathbf{i}1}s_{\mathbf{j}1}^*}\rangle + \langle \mu, \delta_{s_{\mathbf{i}2}s_{\mathbf{j}2}^*}\rangle \\ &= \mu(s_{\mathbf{i}1}s_{\mathbf{j}1}^*) + \mu(s_{\mathbf{i}2}s_{\mathbf{j}2}^*) \qquad (\mathbf{i},\mathbf{j} \in \mathbf{I}).\end{aligned} \tag{3.53}$$

We recall that the *annihilator* of $\mathcal{J}$ is $\mathcal{J}^\perp := \{\mu \in \mathcal{A}^*\colon \langle f, \mu\rangle = 0 \text{ for all } f \in \mathcal{J}\}$. In what follows, $I_{\mathcal{A}}$ denotes the identity operator in $\mathcal{A}$.

Lemma 3.23. *The following hold:*

1. $\mathcal{J} = \overline{\mathrm{Ran}}(I_{\mathcal{A}} - T)$, *and*
2. $\mathcal{J}^\perp = \{\mu \in \mathcal{A}^*\colon T^*\mu = \mu\}$.

Proof. That $\mathcal{J} = \overline{\mathrm{Ran}}(I_{\mathcal{A}} - T)$ is immediate from Lemma 3.22 and (3.52).

We now prove (2). Let us fix $\mu \in \mathcal{A}^*$. Suppose first $T^*\mu = \mu$. Then $\langle f, \mu\rangle = \langle f, T^*\mu\rangle = \langle Tf, \mu\rangle$ or equivalently $\langle f - Tf, \mu\rangle = 0$ for every $f \in \mathcal{A}$. Hence by continuity, $\langle g, \mu\rangle = 0$ for every $g \in \overline{\mathrm{Ran}}(I_{\mathcal{A}} - T)$. By (1) this is equivalent to $\mu \in \mathcal{J}^\perp$.

In the other direction suppose $\mu \in \mathcal{J}^\perp$. By (1) we clearly have $f - Tf \in \overline{\mathrm{Ran}}(I_{\mathcal{A}} - T) = \mathcal{J}$, and hence $\langle f - Tf, \mu\rangle = 0$ or equivalently $\langle f, \mu\rangle = \langle Tf, \mu\rangle = \langle f, T^*\mu\rangle$ for each $f \in \mathcal{A}$. Thus $T^*\mu = \mu$. □

3.4.2. *$\mathcal{A}/\mathcal{J}$ does not have purely infinite ultrapowers.*

Proposition 3.24. *Let $F \subseteq \mathbf{I}$ be finite with $\emptyset \notin F$, and set $f := \sum_{\mathbf{i}\in F} \delta_{s_{\mathbf{i}}^*}$. Then $\|\pi_{\mathcal{J}}(f)\| = |F|$.*

Proof. Clearly $\|f\| = \sum_{\mathbf{i}\in F} \|\delta_{s_{\mathbf{i}}^*}\| = |F|$ and hence $\|\pi_{\mathcal{J}}(f)\| \leqslant |F|$. Thus it suffices to show $\|\pi_{\mathcal{J}}(f)\| \geqslant |F|$. This in turn follows if we can find $\xi \in (\mathcal{A}/\mathcal{J})^*$ satisfying $\|\xi\| = 1$ and $|\langle \pi_{\mathcal{J}}(f), \xi\rangle| \geqslant |F|$. Recall that $\pi_{\mathcal{J}}^*\colon (\mathcal{A}/\mathcal{J})^* \to \mathcal{A}^*$ is a linear isometry with range equal to $\mathcal{J}^\perp$. Hence it is sufficient to find $\mu \in \mathcal{J}^\perp$ satisfying $\|\mu\| = 1$ and $|\langle f, \mu\rangle| \geqslant |F|$.

We shall now define such a μ. To this end, let us consider the following property. Let $\alpha \in \mathbb{N}_0$ be fixed. We say that $t \in \mathrm{Cu}_2 \setminus \{\Diamond\}$ has property $(\alpha - \maltese)$ if

$$t = s_{\mathbf{i}}s_{\mathbf{kj}}^* \text{ for some } \mathbf{i},\mathbf{j} \in \mathbf{I}_\alpha, \text{ and } \mathbf{k} \in F. \qquad (\alpha - \maltese)$$

Now define $\mu\colon \mathrm{Cu}_2 \setminus \{\Diamond\} \to \mathbb{C}$ by setting

$$\mu(t) := \begin{cases} 2^{-\alpha} & \text{if } t \text{ has property } (\alpha - \maltese) \text{ for some } \alpha \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases} \qquad (t \in \mathrm{Cu}_2 \setminus \{\Diamond\}). \tag{3.54}$$

We need to check that μ is well-defined. Assume $\alpha, \beta \in \mathbb{N}_0$, $\mathbf{i},\mathbf{j} \in \mathbf{I}_\alpha$, $\mathbf{p},\mathbf{q} \in \mathbf{I}_\beta$ and $\mathbf{k},\mathbf{l} \in F$ are such that $s_{\mathbf{i}}s_{\mathbf{kj}}^* = s_{\mathbf{p}}s_{\mathbf{lq}}^*$. Then it follows from Lemma 3.2 (2) that $\mathbf{i} = \mathbf{p}$ and hence $\alpha = \beta$.

It is clear that μ is bounded with $\|\mu\| = 1$, hence $\mu \in \mathcal{A}^*$. We want to show that in fact $\mu \in \mathcal{J}^\perp$, which in view of Lemma 3.23 (2) is equivalent to the following claim.

Claim 3.25. $\mu = T^*\mu$.

Proof of Claim. *Assume first* $t \in Cu_2 \setminus \{\Diamond\}$ *has property* $(\alpha - ✠)$ *for some* $\alpha \in \mathbb{N}_0$. Then $t = s_{\mathbf{i}} s_{\mathbf{kj}}^*$ for some $\mathbf{i}, \mathbf{j} \in \mathbf{I}_\alpha$ and $\mathbf{k} \in F$. Notice that $s_{\mathbf{i}1} s_{\mathbf{kj}1}^*$ and $s_{\mathbf{i}2} s_{\mathbf{kj}2}^*$ have property $((\alpha+1) - ✠)$, hence by (3.53)

$$\begin{aligned}\mu(t) &= 2^{-\alpha} = 2^{-\alpha-1} + 2^{-\alpha-1} = \mu(s_{\mathbf{i}1} s_{\mathbf{kj}1}^*) + \mu(s_{\mathbf{i}2} s_{\mathbf{kj}2}^*) \\ &= (T^*\mu)(s_{\mathbf{i}} s_{\mathbf{kj}}^*) = (T^*\mu)(t).\end{aligned} \tag{3.55}$$

Assume now $t \in Cu_2 \setminus \{\Diamond\}$ *does not have property* $(\alpha - ✠)$ *for any* $\alpha \in \mathbb{N}_0$. By definition, $\mu(t) = 0$. By Lemma 3.2 (2) we can find unique $\alpha, \beta \in \mathbb{N}_0$ and $\mathbf{i} \in \mathbf{I}_\alpha$, $\mathbf{j} \in \mathbf{I}_\beta$ such that $t = s_{\mathbf{i}} s_{\mathbf{j}}^*$.

We observe that $s_{\mathbf{i}1} s_{\mathbf{j}1}^*$ does not have property $(\gamma - ✠)$ for any $\gamma \in \mathbb{N}_0$. For assume towards a contradiction that there exist $\gamma \in \mathbb{N}_0$, $\mathbf{p}, \mathbf{q} \in \mathbf{I}_\gamma$ and $\mathbf{l} \in F$ such that $s_{\mathbf{i}1} s_{\mathbf{j}1}^* = s_{\mathbf{p}} s_{\mathbf{lq}}^*$. Then $\mathbf{i}1 = \mathbf{p}$ and $\mathbf{j}1 = \mathbf{lq}$. In particular $\alpha + 1 = \gamma$ and $\mathbf{lq} \in \mathbf{I}_{\beta+1}$.

- *Suppose* $\alpha \geqslant \beta$. By the above $\mathbf{l} \in \mathbf{I}_{\beta+1-\gamma} = \mathbf{I}_{\beta-\alpha}$. Consequently $\mathbf{l} \in \mathbf{I}_0$ must hold, which is equivalent to saying $\mathbf{l} = \emptyset$. This contradicts $\emptyset \notin F$.
- *Suppose* $\alpha < \beta$. Then $\mathbf{j} = \mathbf{wu}$ for some $\mathbf{u} \in \mathbf{I}_\alpha$ and $\mathbf{w} \in \mathbf{I}_{\beta-\alpha}$ and therefore $t = s_{\mathbf{i}} s_{\mathbf{wu}}^*$. As t does not have property $(\alpha - ✠)$ it follows that $\mathbf{w} \notin F$. However $\mathbf{u}1 \in \mathbf{I}_{\alpha+1}$ and $\mathbf{q} \in \mathbf{I}_{\alpha+1}$, thus from $\mathbf{wu}1 = \mathbf{j}1 = \mathbf{lq}$ we conclude $\mathbf{w} = \mathbf{l} \in F$, a contradiction.

An analogous argument shows that $s_{\mathbf{i}2} s_{\mathbf{j}2}^*$ does not have property $(\gamma - ✠)$ either for any $\gamma \in \mathbb{N}_0$. From (3.53) we obtain $(T^*\mu)(t) = (T^*\mu)(s_{\mathbf{i}} s_{\mathbf{j}}^*) = \mu(s_{\mathbf{i}1} s_{\mathbf{j}1}^*) + \mu(s_{\mathbf{i}2} s_{\mathbf{j}2}^*) = 0$, as required. □

Lastly, from the definition of μ we see

$$|\langle f, \mu \rangle| = \left| \sum_{\mathbf{i} \in F} \langle \delta_{s_{\mathbf{i}}^*}, \mu \rangle \right| = \left| \sum_{\mathbf{i} \in F} \mu(s_{\mathbf{i}}^*) \right| = \sum_{\mathbf{i} \in F} 2^{-0} = |F|. \tag{3.56}$$

Hence the proposition is proved. □

Proposition 3.26. *Let* $\Theta \colon \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1)$ *be a continuous algebra homomorphism such that* $\Theta(\pi_{\mathcal{J}}(\delta_{s_i^*})) = A_i$ *for all* $i \in \{1, 2\}$. *Then* Θ *is injective but it is not bounded below.*

Proof. Let us consider the operator $S := A_1 + A_2 \in \mathcal{B}(\ell^1)$. It immediately follows from the definitions of A_1 and A_2 that $Se_{2k} = e_k$ and $Se_{2k-1} = e_k$ for all $k \in \mathbb{N}$. Thus,

$$\|Sx\| = \sum_{k=1}^{\infty} |x_{2k-1} + x_{2k}| \leqslant \sum_{k=1}^{\infty} (|x_{2k-1}| + |x_{2k}|) = \sum_{n=1}^{\infty} |x_n| = \|x\| \qquad \left(x \in \ell^1\right).$$

Thus $\|S\| \leqslant 1$. Let us fix an $N \in \mathbb{N}$. We see that $(\delta_{s_1^*} + \delta_{s_2^*})^N = \sum_{\mathbf{i} \in \mathbf{I}_N} \delta_{s_{\mathbf{i}}^*}$, where clearly $|\mathbf{I}_N| = 2^N$ and $\emptyset \notin \mathbf{I}_N$. Therefore by Proposition 3.24 we obtain $\|\pi_{\mathcal{J}}(\delta_{s_1^*} + \delta_{s_2^*})^N\| = 2^N$.

Hence Θ cannot be bounded below. For assume towards a contradiction that there is some $K > 0$ such that $K\|\pi_{\mathcal{J}}(f)\| \leqslant \|\Theta(\pi_{\mathcal{J}}(f))\|$ for every $f \in \mathcal{A}$. Therefore

$$\begin{aligned}K2^N &= K\|\pi_{\mathcal{J}}(\delta_{s_1^*} + \delta_{s_2^*})^N\| \leqslant \|\Theta(\pi_{\mathcal{J}}(\delta_{s_1^*} + \delta_{s_2^*})^N)\| \\ &\leqslant \|\Theta(\pi_{\mathcal{J}}(\delta_{s_1^*})) + \Theta(\pi_{\mathcal{J}}(\delta_{s_2^*}))\|^N = \|A_1 + A_2\|^N \\ &= \|S\|^N \leqslant 1,\end{aligned} \tag{3.57}$$

which is impossible as $N \in \mathbb{N}$ can be arbitrarily big.

Lastly, $\mathcal{A}/\mathcal{J}$ is purely infinite by Theorem 3.21, hence in particular it is simple by Lemma 1.2 (1). As Θ is a non-zero continuous algebra homomorphism, $\mathrm{Ker}(\Theta) = \{0\}$ must hold. □

Theorem 3.27. *The Banach $*$-algebra $(\mathcal{A}/\mathcal{J})_{\mathcal{U}}$ is not purely infinite for any countably-incomplete ultrafilter $\mathcal{U}$.*

Proof. By Proposition 3.19 there is a continuous, unital algebra homomorphism $\Theta\colon \mathcal{A}/\mathcal{J} \to \mathcal{B}(\ell^1)$ with $\Theta(\pi_{\mathcal{J}}(\delta_{s_i^*})) = A_i$ for all $i \in \{1,2\}$. Thus Θ is not bounded below by Proposition 3.26. Hence $(\mathcal{A}/\mathcal{J})_{\mathcal{U}}$ cannot be purely infinite for any countably-incomplete ultrafilter $\mathcal{U}$ by Proposition 2.14 (a). □

Remark 3.28. Even though $(\mathcal{A}/\mathcal{J})_{\mathcal{U}}$ is not purely infinite for any countably-incomplete ultrafilter $\mathcal{U}$, it is always properly infinite. Indeed, $\mathcal{A}/\mathcal{J}$ is purely infinite by Theorem 3.21 hence it is properly infinite by Lemma 1.2 (2). Now it follows from [10, Corollary 4.18] that $(\mathcal{A}/\mathcal{J})_{\mathcal{U}}$ is properly infinite for any ultrafilter $\mathcal{U}$.

Remark 3.29. In [22] Phillips considers certain representations of the Leavitt algebra L_2 (see Remark 3.11) on L^p spaces, in particular on L^1 spaces. Indeed, [22, Example 3.1] constructs a representation of L_2 on ℓ^1 which is essentially the same as the restriction of our Θ to L_2. Phillips explores generalisations of these representations, which are called *spatial*, see [22, Definition 7.4, Lemma 7.5]. It is shown in [22, Theorem 8.7] that all spatial representations give rise to isometrically isomorphic closures. This gives rise to the p-analogues of the Cuntz algebras, [22, Definition 8.8]; see also [4] for more recent study of these algebras. Thus the closure of the image of Θ, inside $\mathcal{B}(\ell^1)$, is isometric to $\mathcal{O}_2^1$, in the language of [22]. Our result of course shows that $\Theta : \mathcal{A}/\mathcal{J} \to \mathcal{O}_2^1$ is *not* an isomorphism, because it is not bounded below.

Phillips shows in [23] that, in particular, $\mathcal{O}_2^1$ is purely infinite (with the same definition as we use). The proof, however, is different to our proof that $\mathcal{A}/\mathcal{J}$ is purely infinite, and much more closely parallels the C^*-algebraic proof that $\mathcal{O}_2$ is purely infinite. A close examination of the proof shows that it does not work for $\mathcal{A}/\mathcal{J}$, as various necessary norm estimates are different (in the sense of not even being equivalent up to a constant) for $\mathcal{A}/\mathcal{J}$.

It is not obvious to us that the proof in [23] provides an estimate for how $C_{\mathrm{pi}}^{\mathcal{O}_2^1}$ behaves, and hence if $\mathcal{O}_2^1$ has purely infinite ultrapowers. Furthermore, given a lack of nice "permanence" properties for purely infinite Banach algebras, it seems that knowing $\mathcal{O}_2^1$ is purely infinite is no direct help in showing that $\mathcal{A}/\mathcal{J}$ is purely infinite, or vice versa. We remark that similar questions around "permanence properties" are raised at the end of [3].

We end the paper with a question of interest to Banach algebraists. Motivated by [23], we ask if $\mathcal{A}/\mathcal{J}$ is an *amenable* Banach algebra? Phillips shows that $\mathcal{O}_2^1$ is amenable, but his techniques do not appear applicable to $\mathcal{A}/\mathcal{J}$ due to differing (again, incomparable) norm estimates. However, if $\mathcal{A}/\mathcal{J}$ were amenable, this would immediately give a new proof that $\mathcal{O}_2^1$ is amenable.

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