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Preprint No. 44-2021

PRAHA 2021

THE TREE PACKING CONJECTURE FOR TREES OF ALMOST LINEAR MAXIMUM DEGREE

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ABSTRACT. We prove that there is $c > 0$ such that for all sufficiently large n, if $T_1, \ldots, T_n$ are any trees such that T_i has i vertices and maximum degree at most $cn/\log n$, then $\{T_1, \ldots, T_n\}$ packs into K_n. Our main result actually allows to replace the host graph K_n by an arbitrary quasirandom graph, and to generalize from trees to graphs of bounded degeneracy that are rich in bare paths, contain some odd degree vertices, and only satisfy much less stringent restrictions on their number of vertices.

Date: June 22, 2021.
(PA) Partially supported by EPSRC, grant number EP/P032125/1.
(JB) Partially supported by EPSRC, grant number EP/R00532X/1.
(JH) Supported by the Czech Science Foundation, grant number 18-01472Y.
(DP) Supported by the Czech Science Foundation, grant number 19-08740S.

Contents

1. Introduction

Let $G_1, \dots, G_t$ be a collection of graphs, and H be a graph. We say the family $\{G_1, \dots, G_t\}$ *packs into* H if there are edge-disjoint copies of $G_1, \dots, G_t$ in H. The packing is called *perfect* or *exact* if $\sum_{i\in[t]} e(G_i) = e(H)$, so that every edge of H is used exactly once. The study of (perfect) packings is one of the oldest topics in graph theory. Indeed, the problem of the existence of designs — one of the most fascinating questions of mathematics whose origins go back to the 19th century — can be phrased as a perfect (hyper)graph packing problem. This problem was solved only recently, first by Keevash [14], and independently by Glock, Lo, Kühn and Osthus [8].

In this paper, we concentrate on packings with larger graphs. The two most influential conjectures in this area concern the packing of trees, one of which is the following.

Conjecture 1 (Ringel's conjecture)**.** *For each $n \in \mathbb{N}$ and for tree T of order $n+1$, we have that $2n+1$ copies of T pack into the complete graph K_{2n+1}.*

Ringel's conjecture [24] was stated in 1968, and for a long time was only known to hold for very specific families of trees, such as stars, paths and similar trees. The first general result on this conjecture was proved in [4]. While the result in [4] has several further restrictions, the one we want to highlight here is that it is *approximate*, by which we mean that the total number of edges of the embedded trees must be at most $\binom{n}{2} - \Omega(n^2)$. For obtaining a corresponding *exact* result it remains to remove the $\Omega(n^2)$ term, which is hard. Indeed, this gap in difficulty between an approximate and an exact result is quite common in the area of packing and is best illustrated by the increase in difficulty needed to get from Rödl's proof [25] that approximate designs exist to the existence of designs [14, 8]. Joos, Kim, Kühn and Osthus [12] proved that Ringel's conjecture holds exactly for large bounded degree trees. Finally, Montgomery, Pokrovskiy and Sudakov [23], and later Keevash and Staden [18], proved that Ringel's conjecture holds for all sufficiently large trees.

The result of Joos, Kim, Kühn and Osthus is much more general than just Ringel's conjecture, and in particular their result allows the packing of any collection of bounded degree trees $\{T_1, \dots, T_{2n+1}\}$ each on $n+1$ vertices into K_{2n+1}. The results of [18, 23] do not allow for such an extension.

The second influential conjecture in the area is the following.

Conjecture 2 (Tree packing conjecture)**.** *For each $n \in \mathbb{N}$ and for each family of trees $(T_s)_{s\in[n]}$, $v(T_s) = s$, we have that $(T_s)_{s\in[n]}$ packs into the complete graph K_n.*

Gyárfás [10] formulated this conjecture in 1978, and again for a long time it was known only for specific path-like and star-like families. However even packing the few largest trees is already difficult: Balogh and Palmer [3] showed in 2013 that one can pack the largest about $n^{1/4}$ trees into K_{n+1}. Again, the first general approximate result is [4], and the theorem of Joos, Kim, Kühn and Osthus [12] mentioned above proves also that the tree packing conjecture holds for any family of bounded degree trees when n is sufficiently large.

Moving away from trees, Messuti, Rödl and Schacht [22] and Ferber, Lee and Mousset [5] considered approximate packings of bounded-degree graphs which were non-expanding in a suitable sense. Subsequently, Kim, Kühn, Osthus and Tyomkyn [19] proved a packing version of the blow-up lemma, which in particular allows for approximate packings of general bounded-degree graphs. In another direction, moving away from bounded-degree graphs, Ferber and Samotij [6] proved an approximate version of the tree packing conjecture for trees of maximum degree $O(n/\log n)$. We should remark that both these results apply in more generality than packing into K_n. Indeed, [19] in fact allows for packing into a Szemerédi partition, while [6] works also in sparse random graphs.

For packings into complete graphs, the following result generalises both [19, 6]. A graph is said to be *D-degenerate* if there is an ordering of its vertices such that each vertex has at most D neighbours preceding it in the order. Many interesting families of graphs are degenerate—for example trees are 1-degenerate and planar graphs are 5-degenerate.

Theorem 3 ([2, Theorem 2])**.** *For every $D \in \mathbb{N}$ and $\eta > 0$, there exists $n_0 \in \mathbb{N}$ and $c > 0$ so that for each $n > n_0$ the following holds. Suppose that $(G_s)_{s\in\mathcal{F}}$ is a family of D-degenerate graphs of orders at most n and maximum degrees at most $\frac{cn}{\log n}$, whose total number of edges is at most $\binom{n}{2} - \eta n^2$. Then $(G_s)_{s\in\mathcal{F}}$ packs into K_n.*

Building on this, in [1] a perfect packing result for degenerate graphs was obtained with the additional condition that many of the graphs are nonspanning and contain linearly many leaves (we state a generalisation of this result in Theorem 5). Here, a *leaf* in a graph is a vertex of degree 1. As observed in [1], this result implies that the tree packing conjecture holds for almost all families of trees. However, observe that a tree necessarily either contains many leaves or many short bare paths. Here, a subset U of vertices of a graph G induces a *bare path* if $G[U]$ is a path and for each vertex $u \in U$ we have $\deg_G(u) = 2$. Hence to prove the tree packing conjecture, at least for trees with maximum degree at most $cn/\log n$, it only remains to consider trees with many short bare paths. This paper resolves this case.

We remark that many results in this area allow for packing into more general graphs than K_n. One can also pack into quasirandom graphs, which we now define. Given a graph H with n vertices, its *density* is the number $e(H)/\binom{v(H)}{2}$. Given a bipartite graph K with parts of sizes a and b, its *bipartite density* is the number $e(K)/ab$. Suppose that $v \in V(H)$ and $S \subseteq V(H)$. The *neighbourhood* of v is denoted by $\mathsf{N}_H(v)$ and the *common neighbourhood* of S by $\mathsf{N}_H(S) = \bigcap_{v\in S} \mathsf{N}_H(v)$. Here, the convention is that $\mathsf{N}_H(\emptyset) = V(H)$. We write $\deg_H(v) := |\mathsf{N}_H(v)|$ and $\deg_H(S) := |\mathsf{N}_H(S)|$.

Definition 4 (quasirandom)**.** *Suppose that $L \in \mathbb{N}$ and $\gamma > 0$. Suppose that H is a graph with n vertices and with density p. We say that H is* (γ, L)-quasirandom *if for every set $S \subseteq V(H)$ of at most L vertices we have $|\mathsf{N}_H(S)| = (1 \pm \gamma)p^{|S|}n$.*

Theorem 3 allows more generally packings in quasirandom graphs. The same is true of the result of Joos, Kim, Kühn and Osthus [12], and Keevash and Staden [18] formulate and prove a version of Ringel's conjecture for quasirandom graphs. The following main result from [1] also handles quasirandom graphs.

Theorem 5 ([1, Theorem 2])**.** *For every $D \in \mathbb{N}$ and $d, \alpha > 0$, there exists $n_0, L \in \mathbb{N}$ and $c, \xi > 0$ so that for each $n > n_0$ the following holds. Suppose that H is a (ξ, L)-quasirandom graph of order n with at least dn^2 edges and that $(G_s)_{s\in\mathcal{F}}$ is a family of D-degenerate graphs of orders at most n, maximum degrees at most $\frac{cn}{\log n}$, and total number of edges at most $e(H)$. Suppose further, that there exists an index set $\mathcal{B} \subseteq \mathcal{F}$ such that*

- *for each $s \in \mathcal{B}$ we have $v(G_s) \le (1-\alpha)n$, and*
- *the total number of leaves in the family $(G_s)_{s\in\mathcal{B}}$ is at least αn^2.*

Then $(G_s)_{s\in\mathcal{F}}$ packs into H.

1.1. **Our results.** Our main result, Theorem 10, states that we can perfectly pack graphs from families we introduce in Definition 9 into a quasirandom graph. This definition is technical and tailored to the maximal possible generality allowed by our methods. However, combined with Theorem 5, it allows us to prove that the tree packing conjecture holds for trees with maximum degree $O(n/\log n)$.

Theorem 6. *There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that for each $n > n_0$ any family of trees $(T_s)_{s\in[n]}$ with $v(T_s) = s$ and $\Delta(T_s) \le \frac{cn}{\log n}$ packs into K_n.*

Similarly, we obtain an analogue of Ringel's conjecture for trees with degrees bounded by $O(n/\log n)$, where different trees are allowed.

Theorem 7. *There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that for each $n > n_0$ any family of trees $(T_s)_{s\in[n]}$ with $v(T_s) = n+1$ and $\Delta(T_s) \le \frac{cn}{\log n}$ packs into K_{2n+1}.*

In fact, Theorem 6 and Theorem 7 both follow immediately from the following more general packing result for trees, which we deduce in Section 2.

Theorem 8. *For each $\delta, d > 0$ there exist $c, \xi > 0$ and $n_0, L \in \mathbb{N}$ such that for each $n > n_0$ and any (ξ, L)-quasirandom graph H with n vertices and at least dn^2 edges the following holds. Any family of trees $(T_s)_{s\in[N]}$ satisfying*

(*a*) $\sum_{s\in[N]} e(T_s) \le e(H)$,

(*b*) $\Delta(T_s) \le \frac{cn}{\log n}$ *for all* $s \in [N]$,

(*c*) $\delta n \le v(T_s) \le (1-\delta)n$ *for all* $1 \le s \le (\frac{1}{2}+\delta)n$ *and* $v(T_s) \le n$ *for all* $(\frac{1}{2}+\delta)n < s \le N$,

packs into H.

We now introduce the class of graphs we can pack in our main result. The *length* of a path is the number of its edges.

Definition 9 (OurPackingClass)**.**
Given $n, m, D \in \mathbb{N}$ and $\alpha, \gamma > 0$, let $\mathsf{OurPackingClass}(n, m; \alpha, c, D)$ *be the set of all families*

$(G_s)_{s\in\mathcal{G}}$ *of graphs for which there are disjoint index sets* $\mathcal{K},\mathcal{J}\subseteq\mathcal{G}$ *with* $|\mathcal{J}|\geq\alpha n$*, and an odd number* $D_{\mathrm{odd}}\leq D$ *such that*

(*a*) *for each* $s\in\mathcal{G}$*, the graph* G_s *has* $v(G_s)\leq n$ *vertices, maximum degree* $\Delta(G_s)\leq\frac{cn}{\log n}$*, and is* D*-degenerate,*

(*b*) $\sum_{s\in\mathcal{G}}e(G_s)\leq m$*,*

(*c*) *for each* $s\in\mathcal{J}\cup\mathcal{K}$*, we have* $v(G_s)\leq(1-\alpha)n$*,*

(*d*) *for each* $s\in\mathcal{J}$*, the graph* G_s *contains a family* $\mathsf{BasicPaths}_s$ *of* αn *vertex-disjoint bare paths of length* 11*,*

(*e*) *for each* $s\in\mathcal{K}$ *there is a non-empty independent set* $\mathsf{OddVert}_s$ *of vertices of* G_s *whose degree is* D_{odd} *with* $|\mathsf{OddVert}_s|\leq\frac{cn}{\log n}$*, and such that* $\sum_{s\in\mathcal{K}}|\mathsf{OddVert}_s|\geq(1+\alpha)n$*.*

To summarise, Definition 9 allows families of graphs that

- by (*a*) may be spanning with respect to the host graph H the said family is to be embedded into if $n=v(H)$, have degeneracy bounded by a constant and maximum degrees bounded by $O(n/\log n)$,
- by (*b*) may have the same number of edges as the host graph H if $m=e(H)$,
- by (*c*) and (*d*) contain a reasonably sized collection of non-spanning graphs, each of which contains linearly many constant-length bare paths, and
- by (*c*) and(*e*) contain a collection of non-spanning graphs, each of which contains at least one vertex of odd and not too large degree (which we will use to correct parities), not too many of which are in any one graph and which total slightly more than n.

Our main result states that any family of graphs from this class can be perfectly packed into any dense and sufficiently quasirandom graph.

Theorem 10 (main result)**.** *For every* $D\in\mathbb{N}$ *and* $\delta,d>0$*, there exist* $n_0,L\in\mathbb{N}$ *and* $c,\xi>0$ *so that for each* $n>n_0$ *the following holds. Suppose that* H *is a* (ξ,L)*-quasirandom graph of order* n *with at least* dn^2 *edges. Then each family of graphs from* $\mathsf{OurPackingClass}(n,e(H);\delta,c,D)$ *packs into* H*.*

1.2. **Optimality.** Let us now briefly discuss the optimality of Theorem 10. Firstly, we cannot relax the maximum degree condition in Definition 9(*a*). Indeed, Komlós, Sárközy and Szemerédi show in [20] that if we connect a disjoint union of $\frac{\log n}{C}$ many stars of orders $\frac{Cn}{\log n}$ by a path going through the centres of these stars, then we get a tree which asymptotically almost surely does not appear in the random graph $\mathbb{G}(n,p)$ (here, $p\in(0,1)$ is arbitrary, and C is sufficiently big, depending on p). Of course, $\mathbb{G}(n,p)$ asymptotically almost surely satisfies our quasirandomness condition. This example shows that not only packing but already finding a single graph is impossible. This also shows the optimality of the maximum degree condition in Theorem 8.

Secondly, some of the graphs to be packed must be nonspanning, even if we only pack trees. So, condition (*c*) in Definition 9 cannot be omitted. Indeed, in Section 9.1 of [4], a family of bounded-degree trees of orders n and total number of edges $\binom{n}{2}$ is given that does not pack into K_n. Moving away from bounded degree graphs, suppose $\mathcal{G}$ is the following family of graphs. For any sufficiently small $c > 0$, we put $\frac{1}{3}n$ stars with $\frac{cn}{\log n}$ leaves into $\mathcal{G}$. We let the remaining graphs in $\mathcal{G}$ be long paths, such that in total we have $\binom{n}{2}$ edges. Suppose now that there is a packing of $\mathcal{G}$ into K_n. Let H be the subgraph of edges used by the stars. Consider the at least $\frac{2}{3}n$ vertices to which no star centre is embedded. These vertices form an independent set in H and are in total adjacent to at most $\frac{cn^2}{3\log n}$ edges, so in particular one must have degree at most $\frac{cn}{2\log n}$ in H. Thus in order to have a perfect packing, $\mathcal{G}$ must contain at least $\frac{n-1}{2} - \frac{cn}{4\log n}$ paths. If the average length of these paths is ℓ, then we have $\binom{n}{2} = \frac{cn^2}{3\log n} + \ell\big(\frac{n-1}{2} - \frac{cn}{4\log n}\big)$ from which we conclude $\ell \leq n - \frac{cn}{6\log n}$. In particular $\Omega(n/\log n)$ of the paths must be $\Omega(n/\log n)$-far from spanning. This shows that our requirement on the number of non-spanning graphs with many bare paths is sharp up to a log factor even for packing into K_n (and the same argument works in any other regular graph). For packing into a quasirandom H, our conditions permit one vertex of H to have $\Omega(n)$ more neighbours than the average degree. Letting $\mathcal{G}$ be a family of long paths, a similar argument tells us that the average length of the long paths is $n - \Omega(n)$, so that in particular $\Omega(n)$ of the graphs in $\mathcal{G}$ must be $\Omega(n)$ far from spanning, so that in this more general setting our conditions are sharp up to the value of δ.

1.3. **Organisation.** The remainder of this paper is structured as follows. In Section 2 we deduce Theorem 8 from our main result. In Section 3 we provide a sketch of the proof of our main theorem, Theorem 10. In Section 4 we provide a decomposition result that follows from the deep results on the existence of designs by Keevash [15, 16] and that we shall use in the final stage of our packing. In Section 5 we collect some probabilistic tools and facts about certain quasirandom properties that we use in our proof. In Section 6 we provide the main lemmas covering the different stages (Stage A–Stage G) of our proof and show how they imply Theorem 10. In Section 7 we provide an auxiliary non-perfect packing result for anchored paths that we need in Stages D and E of our proof. In Sections 8–14 we give the proofs of the lemmas for Stages A–G. We finish the paper with some concluding remarks in Section 15.

2. Deducing the tree packing results from the main theorem

In this section, we deduce Theorem 8 from our main result, Theorem 10, and Theorem 5.

Proof of Theorem 8. Given δ and d, let $\delta' = \delta/1000$. We choose $c, \xi > 0$ sufficiently small and n_0, L sufficiently large for both Theorem 10 with input $D = 1$, δ' and d, and Theorem 5 with input $D = 1$, d and $\alpha = (\delta')^2$.

Given H and trees $T_1, \dots, T_N$ satisfying the conditions of Theorem 8, we distinguish two cases. First, suppose that among the indices $\{1, \dots, (\frac{1}{2} + \delta)n\}$ there is a subset $\mathcal{B}$ of size $\delta' n$, such that each tree T_i with $i \in \mathcal{B}$ contains at least $\delta' n$ leaves. Then we apply Theorem 5, with constants as above and this $\mathcal{B}$, and it returns a perfect packing of $(T_i)_{i\in[N]}$ into H.

Second, suppose no such $\mathcal{B}$ exists. Then there must be a set $\mathcal{J}$ of indices $i \leq (\frac{1}{2}+\delta)n$ such that T_i has less than $\delta' n$ leaves, with $|\mathcal{J}| = \delta' n$. Given any $i \in \mathcal{J}$ such that T_i has less than $\delta' n$ leaves, observe that the sum of the degrees of T_i is $2v(T_i) - 2$. Since at most $\delta' n$ vertices have degree 1 and all other vertices have degree at least 2, we see that at most $\delta' n$ vertices have degree exceeding 2 in T_i. Now let $\tilde{E}$ be the set of edges in T_i that contain at least one vertex that is not of degree 2 in T_i. We have

$$|\tilde{E}| \leq \sum_{\substack{x\in V(T_i),\\ \deg(x)=1}} 1 + \sum_{\substack{x\in V(T_i),\\ \deg(x)>2}} d(x) = 2v(T_i) - 2 - \sum_{\substack{x\in V(T_i),\\ \deg(x)=2}} 2 \leq 2v(T_i) - 2 - 2\big(v(T_i) - 2\delta' n\big) \leq 4\delta' n.$$

Removing all the edges in $\tilde{E}$ from T_i, we obtain a graph F with at most $4\delta' n + 1 \leq 5\delta' n$ components. The total number of vertices in components with less than 50 vertices is at most $250\delta' n$, so the remaining at least $\delta n - 250\delta' n \geq 50\delta' n$ vertices all lie in components with at least 50 vertices. Note that each such component forms the vertices of a bare path in G_i and that there are at most $\delta' n$ such components. From each component of F with at least 50 vertices, we choose greedily pairwise vertex-disjoint paths of length 11 until we have a set $\mathsf{SpecPaths}_i$ of $\delta' n$ vertex-disjoint bare paths in T_i of length 11, which we can do because after greedily choosing such paths in the at most $\delta' n$ components at most $10\delta' n$ vertices are left over.

We let $\mathcal{K}$ be an arbitrary subset of $[(\frac{1}{2}+\delta)n] \setminus \mathcal{J}$ of size $(\frac{1}{2}+\frac{1}{2}\delta)n$. For each T_i with $i \in \mathcal{K}$, let $\mathsf{OddVert}_i$ be a set of two leaves in T_i. Set $D_{\mathrm{odd}} = 1$. Observe that $\sum_{i\in\mathcal{K}} |\mathsf{OddVert}_i| = (1+\delta)n \geq (1+\delta')n$ and hence $(T_i)_{i\in[N]}$ is in $\mathsf{OurPackingClass}(n, \binom{n}{2}; \delta', c, D)$. Therefore, we can apply Theorem 10 which returns a perfect packing of $(T_i)_{i\in[N]}$ into H. □

3. Outline of the proof of our main theorem

In this section, we give a rough sketch of our proof. We will give a much more detailed discussion in Section 6, together with precise statements of the lemmas we need along the way.

We need to embed the graphs $(G_s)_{s\in\mathcal{G}}$ into H; we can without loss of generality suppose this will be a perfect packing (i.e. (b) of Definition 9 holds with equality). We first embed all the graphs G_s with $s \in \mathcal{G}\setminus(\mathcal{J}\cup\mathcal{K})$, and most vertices of all the remaining graphs, using a randomised packing algorithm from [2]; this is the *bulk embedding*, Stage A.

What remains is the following: for the graphs G_s with $s \in \mathcal{K}$ we still need to embed (some of) the vertices $\mathsf{OddVert}_s$. For the graphs G_s with $s \in \mathcal{J}$ we still need to embed some bare paths contained in $\mathsf{BasicPaths}_s$, whose ends are already embedded (we say the paths are *anchored*).

We next complete the embedding of the graphs G_s with $s \in \mathcal{K}$. Since after this step, only bare paths from $(\mathsf{BasicPaths}_s)_{s\in\mathcal{J}}$ — that is vertices of degree 2 — will be left to pack, there are some obvious parity restrictions ahead. So, we use the odd degrees of $\mathsf{OddVert}_s$ (c.f. Definition 9(e)) to fulfil these. This is the *parity correction*, Stage B.

Now we need to embed the anchored bare paths of G_s with $s \in \mathcal{J}$. At each vertex v of H, we have to use one edge per path anchored at v, and in addition each time we choose to use v as an interior vertex we use two edges; this is why we needed to set the parity in the previous step. We split these graphs up into three parts, $\mathcal{J} = \mathcal{J}_0 \dot\cup \mathcal{J}_1 \dot\cup \mathcal{J}_2$, and perform the embedding of these bare paths over several stages, preparing for an application of the results of Keevash [15, 16] on the existence of designs in our final stage. We now first detail in what setup we want to apply the design results in order to pack (parts of) our bare paths, we then outline how we can apply the design results in this setup, and finally explain which preparations are performed for achieving this setup.

Before the final stage we will be left with a subgraph of H; let us call this subgraph H^* here. We will ensure that H^* has an even number of vertices, which come in *terminal pairs* $\{\boxminus_i, \boxplus_i\}$. Thus we have a partition $V(H^*) = V_\boxminus \dot\cup V_\boxplus$ into equal parts, which we call *sides*. We will also ensure that all that remains is to embed some paths with three edges from some graphs G_s with $s \in \mathcal{J}_0$ such that each of these paths is anchored to a terminal pair: that is, if $xyzw$ is such a 3-edge path in G_s, then there exists some i such that x is embedded to the vertex $\boxminus_i$ of H^* and w to $\boxplus_i$. When packing these paths in the final stage, we shall insist that y gets embedded to the same side as x, i.e. $V_\boxminus$, and z to the same side as w, i.e. $V_\boxplus$.

Let us now explain how we apply the design results. We represent the embedding of the path $xyzw$ into H^* as the embedding of a 4-vertex configuration called a *diamond* in an auxiliary coloured and partially directed graph called a *chest*, which we now describe. (For an illustration see also Figure 2.) The vertices of the chest come in two parts, the set $V = \{1, 2, \ldots, |V_\boxminus|\}$ and the set $U = \mathcal{J}_0$. We put coloured edges into V as follows. For each edge $\boxminus_i \boxminus_j \in E(H^*)$, we put a blue edge ij. For each edge $\boxplus_i \boxplus_j$, we put a red edge ij. Finally for each edge $\boxminus_i \boxplus_j$ we put a green arc directed from i to j. Thus a 3-vertex cycle ijk in the chest, in which ij is blue, jk is green and directed from j to k, and ki is red, represents a 3-edge path $\boxminus_i \boxminus_j \boxplus_k \boxplus_i$ in H^*. We put in addition edges from $s \in U$ to V as follows. For each $i \in V$ such that G_s has a path anchored on $\{\boxminus_i, \boxplus_i\}$ we put a grey edge is. If $j \in V$ is such that $\boxminus_j$ has not been used in the embedding of G_s before the final stage, we put a black edge js, and if $\boxplus_k$ has not been used in the embedding of G_s before the final stage, we put a purple edge ks. A *diamond* is then the 4-vertex configuration obtained by adding a vertex $s \in U$ to the three-vertex cycle ijk described above, with is grey, js black and ks purple. A copy of this configuration in the chest represents a 3-edge path $\boxminus_i \boxminus_j \boxplus_k \boxplus_i$ in H^*, with the additional information that we can use this copy to embed a 3-edge path of G_s anchored at $\{\boxminus_i, \boxplus_i\}$ and that $\boxminus_j$ and $\boxplus_k$ have not been used in the embedding of G_s. In other words, a diamond in the chest represents a valid way to embed one path of one graph G_s in H^*. See Figure 3 for an illustration of this translation from paths to diamonds.

We should at this point note that the chest can and will have multiple edges between its vertices, but we will ensure that $\boxminus_i \boxplus_i$ is not an edge of H^*, meaning that the chest has no loops. Suppose now that we have a collection of edge-disjoint diamonds in a chest, that is, we do not use any one coloured edge or arc in two different diamonds. We remark that it may still be the case that one red edge ij is used in one diamond while a parallel blue edge ij is used in another. This condition of edge-disjointness of the diamonds translated back to H^* means the following: the

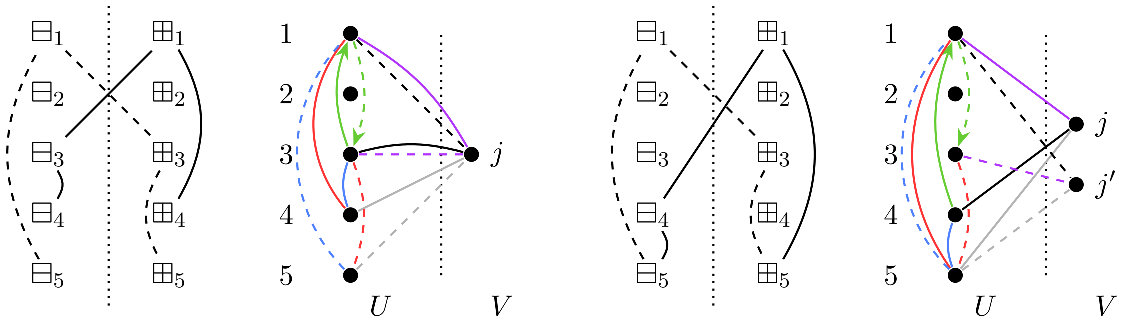

FIGURE 1. The two left-hand pictures show an example of two paths from the same graph G_j embedded in H^* to the paths $\boxminus_4 \boxminus_3 \boxplus_1 \boxplus_4$ and $\boxminus_5 \boxminus_1 \boxplus_3 \boxplus_5$ and the corresponding diamonds in the chest. The two right-hand pictures show an example of two paths from different graphs G_j and $G_{j'}$ embedded in H^* to the paths $\boxminus_5 \boxminus_4 \boxplus_1 \boxplus_5$ and $\boxminus_5 \boxminus_1 \boxplus_3 \boxplus_5$, respectively, and the corresponding diamonds in the chest. The solid and dashed lines in this picture are only used to distinguish the two paths/diamonds.

embeddings of 3-edge paths from various graphs G_s with $s \in \mathcal{J}_0$ encoded by the diamonds do not use any one edge of H^* twice, and any two paths from any one G_s get embedded to disjoint sets of vertices. In other words, a collection of edge-disjoint diamonds represents a way to extend the packing we have before the final stage to a larger packing. In particular, we will ensure the number of red, blue, green and grey edges are all the same. A collection of edge-disjoint diamonds which use all of them—which we call a *diamond core-decomposition*—then represents an extension of the packing before the final stage to the desired perfect packing. Note that our diamond core-decompositions will not use all the black or purple edges, since the graphs G_s with $s \in \mathcal{J}$ are not spanning. A diamond core-decomposition is precisely a generalised design, whose existence is proved by Keevash [15, 16] under certain conditions on the chest (see Section 4 for more details). This final stage of our packing will be Stage G, *designs completion*.

The preparation we need to do before this is then simply to ensure that we end up with the above setting, and that the Keevash conditions for the existence of the required generalised design are met. In Stage C, *partite reduction*, we split the vertices of H into two equal sides and pair them up into terminal pairs randomly. If $v(H)$ is odd, we have one leftover vertex $\boxdot$. We embed a few paths from the graphs G_s with $s \in \mathcal{J}$, including all of those anchored at $\boxdot$, in order to use up all the edges of H leaving $\boxdot$ and any edges of H between terminal pairs.

In Stage D, *connecting to terminal pairs*, we embed some vertices from the graphs G_s with $s \in \mathcal{J}_0$. Prior to this stage, the bare paths in this graph which we need to embed all have 11 edges, and they can be anchored anywhere in H (except $\boxdot$), not necessarily to terminal pairs. In this stage, we embed the first four and last four vertices of each path in order that what remains is to embed the middle 3 edges between some terminal pair $\{\boxminus_i, \boxplus_i\}$ which we choose randomly.

As mentioned above, for our designs completion Stage G, we need the same numbers of red, blue and green edges in the chest. That means we need the same number of edges of H within

$V_{\boxminus}$, within $V_{\boxplus}$ and crossing between $V_{\boxminus}$ and $V_{\boxplus}$. After Stage D, the left-over graph H has a fairly uniform density, and in particular there are about twice as many edges crossing from $V_{\boxminus}$ to $V_{\boxplus}$ as within $V_{\boxminus}$. In Stage E, *density correction*, we now make sure that we obtain precisely the correct number of edges within $V_{\boxminus}$ and $V_{\boxplus}$ and crossing. We will embed further paths in the following Stage F, but the number of edges from each set we use is fixed and taken into account. We do this density correction by completing the embedding of the graphs G_s with $s \in \mathcal{J}_2$.

In Stage F, *degree correction*, we ensure that some conditions of the following form hold. For any given $\boxminus_i \in V_{\boxminus}$, the number of edges of H going from $\boxminus_i$ to $V_{\boxminus}$ is equal to the number of edges going from $\boxminus_i$ to $V_{\boxplus}$ plus the number of $s \in \mathcal{J}_0$ which have a path anchored at $\boxminus_i$. Observe that if we embed a diamond using i, there is a blue edge at i and either a green arc leaving i or a grey edge at i, so that a diamond core-decomposition can only exist if the above equality holds. Following the language of Keevash [15, 16] we call these *divisibility conditions*. All these conditions are very close to holding already after Stage E, so only tiny corrections are necessary. We perform the degree correction by completing the embedding of the graphs G_s with $s \in \mathcal{J}_1$.

We have $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2$, so after Stage F, what remains is precisely the embedding of three-edge paths from graphs G_s with $s \in \mathcal{J}_0$ described above as Stage G, *designs completion*. For applying Keevash's designs result in this stage we need one further condition: The chest has to have certain quasirandomness properties (which we define precisely in Section 4). In Stages A, D and E (which is where we embed a significant number of edges) we use randomised algorithms to perform our embedding, and with high probability these algorithms give the required quasirandom output. In Stages B, C and F the number of edges we embed is tiny compared to $e(H)$. We simply ensure that we do not embed to any one vertex too often in these stages, and this is enough to ensure that quasirandomness cannot be seriously affected. We should note that while we eventually need quasirandomness to apply the Keevash machinery, we will also use it often in our analysis of the preparatory stages; it is for instance the quasirandomness that guarantees our divisibility conditions are all close to correct after Stage E.

Let us close this proof outline by describing the bulk embedding, Stage A, in some more detail. This part of our proof is an extension of the proof of Theorem 3 given in [2], which is built on the following idea. Enumerate the graphs as $G_1, \ldots, G_{|\mathcal{G}|}$, with the graphs G_s for $s \in \mathcal{J} \cup \mathcal{K}$ last. We embed the graphs one after another, edge-disjointly, in this order, except for the vertices in $\mathsf{OddVert}_s$ and $\mathsf{BasicPaths}_s$.

For each G_s, we pack the first $(1-\delta)n$ vertices in the D-degeneracy order as follows. We embed the first vertex of G_s uniformly to $V(H)$. Thereafter, when we need to embed vertex i, we look at the already embedded neighbours $x_1, \ldots, x_r$ of i. Suppose these are embedded to $v_1, \ldots, v_r$ in H. Then we need to embed i to a vertex which is adjacent to all of $v_1, \ldots, v_r$ in H and which we did not previously use in embedding G_s. We pick such a vertex v uniformly at random, embed i there, and delete all edges vv_j with $j \in [r]$ from H.

For the graphs G_s with $s \in \mathcal{G} \setminus (\mathcal{J} \cup \mathcal{K})$, we still need to embed the remaining δn vertices. This requires a separate argument which is not that relevant for this discussion.

To show that this random process succeeds, we need to show that after each G_s is embedded, the remaining graph H is still quasirandom. In turn, during the embedding of G_s, we need to argue that the first i vertices are embedded to roughly an i/n fraction of the common neighbourhood of any D or fewer vertices of H. This is called the *diet condition*, and together with the quasirandomness of H in particular it tells us that there will always be a large set of vertices to which we can embed vertex $i+1$ of G_s.

The above analysis is detailed in [2]. However to make Stages B–G work, we need quite a few additional properties of this packing, which we prove in this paper (in Section 8). In addition, the randomised algorithm which we use in Stages D and E is related to the procedure described above, but requires an entirely new analysis (which is given in Section 7).

We remark that another approach extending [2] was used in [1] to prove Theorem 5. However, the structure that is put aside and packed later there are the leaves in $(G_s)_{s\in\mathcal{B}}$. The subsequent perfect packing of these leaves is much easier than that of our systems of paths.

4. Designs

The main purpose of this section is to formulate a general decomposition result for directed coloured partite multigraphs (Theorem 21), which is a special case of the deep results on the existence of designs by Keevash [15, 16], and apply it obtain a specific decomposition result of certain partially directed coloured partite multigraphs (Proposition 14) that we shall use to complete the packing in our proof of Theorem 10. Here, a *partially directed multigraph* consists of a vertex set, a set (or a collection of sets) of directed edges and a set (or a collection of sets) of undirected edges, with multi-edges allowed but loops not.

We first introduce some basic notions. For a directed graph G and $v \in V(G)$ we define $\mathsf{N}_G^{out}(v)$ and $\mathsf{N}_G^{in}(v)$ as the out-neighbourhood and the in-neighbourhood of v, respectively. As before $\mathsf{N}_G^{out}(S) = \bigcap_{v\in S} \mathsf{N}_G^{out}(v)$ and $\mathsf{N}_G^{in}(S) = \bigcap_{v\in S} \mathsf{N}_G^{in}(v)$. We write $\deg_G^{out}(v) = |\mathsf{N}_G^{out}(v)|$ and $\deg_G^{in}(v) = |\mathsf{N}_G^{in}(v)|$ for the corresponding degrees. We shall be considering partially directed multigraphs $\mathcal{M}$ with a collection of different edge sets $E_1, E_2, \dots, E_k$, where E_i either consists only of directed edges or only of undirected edges. We also write $\mathsf{N}_{E_i}^{out}(v)$, $\mathsf{N}_{E_i}^{in}(v)$, $\deg_{E_i}^{out}(v)$, $\deg_{E_i}^{out}(v)$ in the former case, and $\mathsf{N}_{E_i}(v)$, $\deg_{E_i}(v)$ in the latter case for the neighbourhoods and degrees in the sub(di)graph of $\mathcal{M}$ with edge set E_i.

We now define the setup in which we want to apply the decomposition results mentioned above. The partially directed multigraph which we want to decompose, and which we shall call *chest*, has the following form. See the left hand side of Figure 2 for an illustration of this setup.

Definition 11 (chest)**.** *We have disjoint sets V and U of vertices and the following collection of edge sets. The edge set $\vec{E}_1$ contains directed edges in V, the edge sets E_2 and E_3 contain edges in V, and E_4, E_5, E_6 contain edges between V and U. Parallel edges or loops are not allowed within any of these edge sets, though antiparallel edges are allowed in the set of directed edges $\vec{E}_1$, and different edge sets may have edges in common. If V, U, $\vec{E}_1$, $E_2, \dots, E_6$ satisfy these properties, we say that $\mathcal{M} = (V\dot\cup U; \vec{E}_1, E_2, E_3, E_4, E_5, E_6)$ is a* chest.

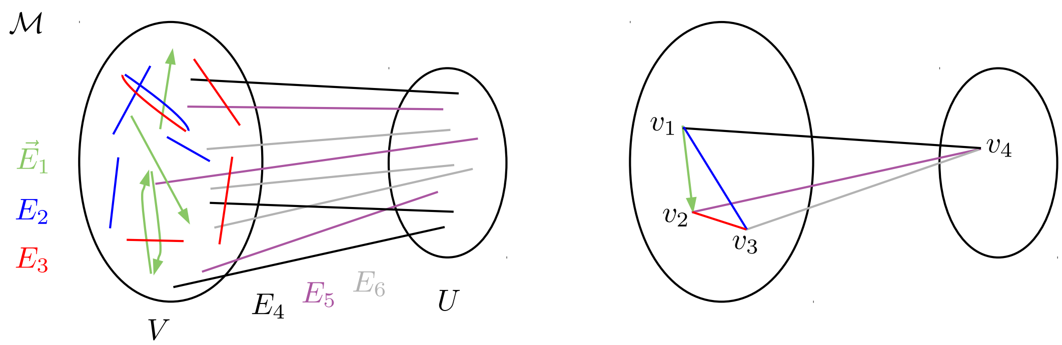

FIGURE 2. A chest (on the left) and a diamond in it (on the right).

In order to be able to apply the decomposition results we need our chest to have certain quasirandomness properties.

Definition 12 (quasirandom chest)**.** *Define* $d_1 := \frac{|\vec{E}_1|}{|V|^2}$, $d_i := \frac{|E_i|}{\binom{|V|}{2}}$ *for* $i = 2, 3$, *and* $d_i := \frac{|E_i|}{|V||U|}$ *for* $i = 4, 5, 6$. *We say that a chest* $\mathcal{M}$ *is* (γ, L)-quasirandom *if for every choice of* $S_1, S'_1, S_2, S_3, S'_4, S'_5, S'_6 \subseteq V$ *and* $S_4, S_5, S_6 \subseteq U$ *of mutually disjoint sets of total size at most* L *we have that*

$$\left| V \cap \mathsf{N}^{out}_{\vec{E}_1}(S_1) \cap \mathsf{N}^{in}_{\vec{E}_1}(S'_1) \cap \bigcap_{i=2}^{6} \mathsf{N}_{E_i}(S_i) \right| = (1 \pm \gamma) \cdot d_1^{|S_1|+|S'_1|} \cdot \prod_{i=2}^{6} d_i^{|S_i|} \cdot |V| \quad \textit{and}$$

$$\left| U \cap \bigcap_{i=4}^{6} \mathsf{N}_{E_i}(S'_i) \right| = (1 \pm \gamma) \cdot \prod_{i=4}^{6} d_i^{|S'_i|} \cdot |U| \, .$$

We next define the partially directed graph into which we want to decompose a quasirandom chest.

Definition 13 (diamond)**.** *A* diamond *in* $\mathcal{M}$ *is a graph* $(\{v_i\}_{i=1}^4, \{e_i\}_{i=1}^6)$ *with* $e_1 \in \vec{E}_1$ *and* $e_i \in E_i$ *for* $i = 2, \dots, 6$, *and so that* $e_1 = (v_1, v_2)$, $e_2 = \{v_3, v_1\}$, $e_3 = \{v_2, v_3\}$, $e_4 = \{v_1, v_4\}$, $e_5 = \{v_2, v_4\}$, *and* $e_6 = \{v_3, v_4\}$.

See the right hand side of Figure 2 for an illustration of a diamond. A collection $\mathcal{D}$ of diamonds in a chest $\mathcal{M}$ is a *diamond core-decomposition* of $\mathcal{M}$ if each edge of $\vec{E}_1 \dot\cup E_2 \dot\cup E_3 \dot\cup E_6$ is used by exactly one diamond of $\mathcal{D}$, and each edge of $E_4 \dot\cup E_5$ is used by at most one diamond of $\mathcal{D}$.

It turns out that, in order to complete the packing of our trees in the proof of Theorem 10, we need a diamond core-decomposition of a quasirandom chest (this part of our proof is encapsulated in Lemma 38). The following proposition provides conditions under which we can obtain such a diamond core-decomposition.

Proposition 14 (diamond core-decomposition)**.** *For every* $d, \sigma > 0$ *there exists* $L, n_0 \in \mathbb{N}$ *and* $\gamma > 0$ *with the following property. Let* $\mathcal{M} = (V \dot\cup U; \vec{E}_1, E_2, E_3, E_4, E_5, E_6)$ *be a* (γ, L)*-quasirandom chest with* $|U \cup V| = n > n_0$ *and* $\min\{|U|, |V|\} \geq \sigma n$, *such that* $d_1 = \frac{|\vec{E}_1|}{|V|^2} > d$, *and*

$d_i = \frac{|E_i|}{|V||U|} > d$ *for* $i = 4, 5, 6$. *Suppose that* $|\vec{E}_1| = |E_2| = |E_3| = |E_6|$. *Suppose further that for any vertex* $v \in V$ *we have*

(*i*) $\deg_{E_2}(v) = \deg^{out}_{E_1}(v) + \deg_{E_6}(v)$,

(*ii*) $\deg_{E_3}(v) = \deg^{in}_{E_1}(v) + \deg_{E_6}(v)$,

(*iii*) $\deg_{E_4}(v) \geq \deg^{out}_{E_1}(v) + 4d^{-1}\gamma|U|$,

(*iv*) $\deg_{E_5}(v) \geq \deg^{in}_{E_1}(v) + 4d^{-1}\gamma|U|$,

and for any vertex $u \in U$ *we have*

(*v*) $\deg_{E_4}(u) \geq \deg_{E_6}(u) + 4d^{-1}\gamma|V|$

(*vi*) $\deg_{E_5}(u) \geq \deg_{E_6}(u) + 4d^{-1}\gamma|V|$.

Then $\mathcal{M}$ *has a diamond core-decomposition.*

The remainder of this subsection is dedicated to the deduction of Proposition 14 from [16, Theorem 19]. This theorem is very general and relies on heavy terminology specific to [16]. So, as an intermediate step, we present in Theorem 21 a decomposition result which is fairly general yet relatively easy to state. In order to state this result we need a number of definitions.

Theorem 21 allows us to decompose multi-digraphs into a family of digraphs in a coloured and partite setting. For an integer D, a $[D]$*-edge-coloured digraph* is a digraph whose edges are assigned colours from the set $[D]$. We remark that in this theorem the rôles of H and G are interchanged compared to the setting in the remainder of the paper: We decompose a digraph G into graphs H from a family $\mathcal{H}$. The reason for this change is that this will make it easier for us to explain how Theorem 21 follows from [16, Theorem 19].

Definition 15 (decomposition)**.** *Let* $\mathcal{H}$ *be a family of* $[D]$*-edge-coloured digraphs on* $[q]$*, and let* G *be a* $[D]$*-edge-coloured digraph on* $[n]$*. We say that* G *has an* $\mathcal{H}$-decomposition *if the edges of* G *can be partitioned into copies of digraphs from* $\mathcal{H}$ *that preserve the colouring.*

We can only decompose into certain types of edge-coloured digraphs, which we call simple canonical digraphs.

Definition 16 (simple canonical digraphs)**.** *Let* $D, q \in \mathbb{N}$*, and let* $\mathcal{P} = \{P_1, \ldots, P_t\}$ *be a partition of* $[q]$*. Let* $\mathcal{H}$ *be a family of* $[D]$*-edge-coloured digraphs on* $[q]$*. Further, for each colour* $d \in [D]$ *assume we are given a pair* $(i, j) \in [t]^2$*, which we call* colour location of d*. For a colour* $d \in [D]$ *with colour location* (i, j) *such that* $i = j$ *we assume further that* d *is specified as being an* oriented colour *or an* unoriented colour*.*

In this case, we say that the family $\mathcal{H}$ *is* simple $\mathcal{P}$-canonical *if the following hold for every* $H \in \mathcal{H}$ *and every colour* $d \in [D]$ *and its colour location* (i, j)*.*

- *There is no loop in* H*, there are no parallel edges in* H*, and there are no anti-parallel edges of different colours in* H*.*
- *All the edges of* H *of colour* d *start in* P_i *and end in* P_j*.*

- *If $i = j$ and d is an unoriented colour, then all the edges of H of colour d come in pairs that form directed 2-cycles.*
- *If $i = j$ and d is an oriented colour, then H contains no directed 2-cycles in colour d.*

As illustration, let us explain how this definition is used in our application. Recall that in our setup we decompose into diamonds on vertex set $[4]$ (replacing v_i in the definition of a diamond with i) containing directed edges and undirected edges and with vertices $1, 2, 3$ mapped to V and vertex 4 mapped to U, as well as leftover edges from E_4 and E_5. Translating this to the setting of simple canonical digraphs, we would choose $\mathcal{P} = \{P_1, P_2\}$ with $P_1 = \{1, 2, 3\}$ and $P_2 = \{4\}$ and $\mathcal{H} = \{H_1, H_2, H_3\}$ containing the following digraphs. The digraph H_1 is a directed version of the diamond: We give the 6 edges of the diamond 6 different colours, and then replace each undirected edge $(i, 4)$ of colour d with an edge directed towards 4 of colour d (this choice of direction is arbitrary), and each other undirected edge (within P_1) of colour d with two antiparallel edges of colour d. The digraphs H_2 and H_3 can be used for the leftover edges from E_4 and E_5: H_2 only contains the edge $(1, 4)$ in the same colour as in H_1 and H_3 only contains the edge $(2, 4)$ in the same colour as in H_1.

We next define the types of digraphs that can be decomposed with the help of Theorem 21.

Definition 17 (general multi-digraph)**.** *Let $\mathcal{P}' = \{P'_1, \dots, P'_t\}$ be a partition of $[n]$. We say that a $[D]$-edge-coloured multi-digraph G on $[n]$ with partition $\mathcal{P}'$ is* general *if G has no loop, but multi-edges, parallel and anti-parallel, of the same or different colours are allowed, as long as for any colour $d \in [D]$ between any two parts P'_i and P'_j with $i \neq j$, either all edges of colour d are directed towards P'_i or all edges of colour d are directed towards P'_j.*

Further, the digraphs we want to decompose need to satisfy a number of conditions, which we define next, and which allow for a partite setting. We start with the divisibility conditions. For a $[D]$-edge coloured (multi-)digraph H, a vertex v of H and a colour $d \in [D]$, we write H_d for the sub-(multi-)digraph of H containing exactly those edges of colour d. We also write $\deg^{out}_{H,d}(v)$ for $\deg^{out}_{H_d}(v)$ and $\deg^{in}_{H,d}(v)$ for $\deg^{in}_{H_d}(v)$, and analogously for $\mathsf{N}^{out}_{H,d}(v)$, $\mathsf{N}^{in}_{H,d}(v)$, $\mathsf{N}^{out}_{H,d}(S)$, and $\mathsf{N}^{out}_{H,d}(S)$, where $S \subseteq V(H)$.

Definition 18 (divisibility)**.** *Let $\mathcal{P} = \{P_1, \dots, P_t\}$ be a partition of $[q]$ and $\mathcal{P}' = \{P'_1, \dots, P'_t\}$ be a partition of $[n]$. Let G be a $[D]$-edge-coloured general multi-digraph on $[n]$. Let $\mathcal{H}$ be a family of $[D]$-edge-coloured simple $\mathcal{P}$-canonical digraphs on $[q]$. We say that $(G, \mathcal{P}')$ is $(\mathcal{H}, \mathcal{P})$-*divisible *if the following hold.*

0-divisibility**:** *For $d \in [D]$, and $H \in \mathcal{H}$, let $c_{d,H}$ denote the number of edges in H coloured d. There are integers $(m_H)_{H \in \mathcal{H}}$ such that for each $d \in [D]$ the number of edges of G in colour d equals $\sum_{H \in \mathcal{H}} m_H \cdot c_{d,H}$.*

1-divisibility**:** *For any $i \in [t]$ and any vertex $v \in P'_i$, there exist integers $(m_{H,x})_{H\in\mathcal{H},x\in P_i}$ such that for each $d \in [D]$ we have*

$$\deg^{out}_{G,d}(v) = \sum_{H\in\mathcal{H},x\in P_i} m_{H,x} \cdot \deg^{out}_{H,d}(x) \quad \text{and} \quad \deg^{in}_{G,d}(v) = \sum_{H\in\mathcal{H},x\in P_i} m_{H,x} \cdot \deg^{in}_{H,d}(x) \; .$$

2-divisibility**:** *For each $d \in [D]$ with colour location (i, j) (with respect to $\mathcal{H}$), all edges in G of colour d start in P'_i and end in P'_j. Further, if $i = j$ and colour d is unoriented then all the edges of G of colour d come in pairs that form directed 2-cycles.*

Observe that 2-divisibility mandates that in G edges of colour d only run between a unique pair (P'_i, P'_j) of parts (with possibly $i = j$). Since $\mathcal{H}$ is canonical also in any $H \in \mathcal{H}$ we can have edges of this colour d only between the corresponding pair (P_i, P_j)

The next condition requires that copies of the digraphs H into which we seek to decompose the digraph G are distributed regularly on edges of G.

Definition 19 (regularity)**.** *Let $\mathcal{H}$ be a family of $[D]$-edge coloured digraphs on $[q]$ and let G be a general $[D]$-edge-coloured multi-digraph on $[n]$. Let $c, \omega > 0$ be reals. We say that G is* $(\mathcal{H}, c, \omega)$-regular *if we can assign a weight $w_{H'} \in [\omega \cdot n^{2-q}, \omega^{-1} \cdot n^{2-q}]$ to each coloured copy $H' \subseteq G$ of any $H \in \mathcal{H}$, such that for every edge $e \in E(G)$ we have*

$$\sum_{H\in\mathcal{H}} \sum_{H'\subseteq G: e\in E(H'), H'\sim H} w_{H'} = (1 \pm c) \, ,$$

where we denote the fact that H' is a coloured copy of H by $H' \sim H$.

In this definition the normalisation of the weights is n^{2-q} because we fix an edge in G and sum over copies of H in G which contain this fixed edge. We also remark that we do not need to see the partitions of H and G to formulate this condition. Never the less we will sometimes say that $(G, \mathcal{P}')$ is $(\mathcal{H}, c, \omega)$-regular if G is $(\mathcal{H}, c, \omega)$-regular.

The following final condition considers any vertex x in any $H \in \mathcal{H}$ and requires that whenever we choose for each vertex y in H other than x a set A_y of at most h vertices in the part of G where y should be embedded then the common neighbourhood of these sets A_y in the part where x should be embedded is of linear size, where for this common neighbourhood we take edges directed as mandated by corresponding edges in H.

Definition 20 (vertex-extendability)**.** *Let $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$ and let $\mathcal{P}' = (P'_1, \dots, P'_t)$ be a partition of $[n]$. Let $\mathcal{H}$ be a simple $\mathcal{P}$-canonical family of $[D]$-edge-coloured digraphs on $[q]$. Let G be a general $[D]$-edge-coloured multi-digraph on $[n]$.*

Let $H \in \mathcal{H}$, let $x \in [q]$ be any vertex of H, and let P_i be the part of $\mathcal{P}$ containing x. Let y be any other vertex of H, that is, $y \in [q] \setminus \{x\}$. If (x, y) or (y, x) is an edge of H, then let d be its colour. (Recall that since $\mathcal{H}$ is canonical, if (x, y) and (y, x) are both edges, then they have the *same unoriented colour.) For any set $A \subseteq [n]$ of vertices in G, we define*

$$\mathsf{N}_G^{(y,x,H)}(A) = \begin{cases} \mathsf{N}_{G,d}^{out}(A) \cap P'_i & \text{if only } (y,x) \in E(H)\,, \\ \mathsf{N}_{G,d}^{in}(A) \cap P'_i & \text{if only } (x,y) \in E(H)\,, \\ \mathsf{N}_{G,d}^{out}(A) \cap \mathsf{N}_{G,d}^{in}(A) \cap P'_i & \text{if both } (y,x),(x,y) \in E(H)\,, \\ P'_i & \text{otherwise}\,. \end{cases}$$

*Let h be an integer and $\omega > 0$. We say that $(G, \mathcal{P}')$ is $(\mathcal{H}, \mathcal{P}, \omega, h)$-*vertex-extendable *if the following holds for every $H \in \mathcal{H}$ and every $x \in [q]$. For every choice of pairwise disjoint sets $\{A_y\}_{y\in[q]\setminus\{x\}}$ of size $|A_y| \leq h$ with $A_y \subseteq P'_j$ whenever $y \in P_j$, the set $A_x := \bigcap_{y\in[q]\setminus\{x\}} \mathsf{N}_G^{(y,x,H)}(A_y)$ has size $|A_x| \geq \omega n$.*

In our setting, regularity and vertex-extendability both follow from the quasirandomness properties of the chest. We are now ready to state the decomposition result that implies Proposition 14.

Theorem 21 (decomposition result)**.** *Given $q, D \in \mathbb{N}$ and $\sigma > 0$, there exist numbers $\omega_0 > 0$ and $n_0 \in \mathbb{N}$ such that with $q' = \max\{q, 8 + \lfloor \log_2(1/\sigma)\rfloor\}$, $h = 2^{50q'^3}$ and $\delta = 2^{-10^3 q'^5}$ the following holds for each $n > n_0$ and $\omega \in (n^{-\delta}, \omega_0)$.*

Let $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$ and $\mathcal{P}' = (P'_1, \dots, P'_t)$ be a partition of $[n]$ such that $|P'_i| \geq \sigma n$ for each $i \in [t]$. Let $\mathcal{H}$ be a simple $\mathcal{P}$-canonical family of $[D]$-edge-coloured digraphs on $[q]$, and let G be a general $[D]$-edge-coloured multi-digraph on $[n]$.

Suppose that $(G, \mathcal{P}')$ is $(\mathcal{H}, \mathcal{P})$-divisible, $(\mathcal{H}, \omega^{h^{20}}, \omega)$-regular and $(\mathcal{H}, \mathcal{P}, \sqrt[q'h]{\omega}, h)$-vertex-extendable. Then G has an $\mathcal{H}$-decomposition.

Theorem 21 is a special case of [16, Theorem 19]. In Appendix A, we explain the connections between the notion used in [16] and the one used here. We now use Theorem 21 to prove Proposition 14.

Proof of Proposition 14. We first choose the constants. Let $d, \sigma > 0$ be given. Set $q := 4$ and $D := 6$. Let ω_0 and n'_0 be the output parameters of Theorem 21 for input q, D and σ and set

$$q' := \max\{q, 8 + \lfloor \log_2(1/\sigma)\rfloor\}\,, \quad h := 2^{50q'^3} \quad \text{and} \quad \delta := 2^{-10^3 q'^5}\,,$$

as specified in Theorem 21. Further, set

$$L := h\,, \quad n_0 := \max\{n'_0, \omega^{-1/\delta}\}\,, \quad \omega := \min\left\{\left(\frac{\sigma d}{2}\right)^{5Lhq'}, \frac{\omega_0}{2}\right\}, \quad \text{and} \quad \gamma := \frac{\omega^{h^{20}}}{6} > 0\,.$$

Let $n > n_0$ and observe that $n^{-\delta} < \omega < \omega_0$ as required by Theorem 21.

Now let $\mathcal{M}$ be a (γ, L)-quasirandom chest with vertex set $[n] = U \dot\cup V$ satisfying the assumptions of Proposition 14. As Theorem 21 only decomposes digraphs and $\mathcal{M}$ has undirected edges, we need to transform $\mathcal{M}$ as follows. Replace every unoriented edge inside V by an oriented 2-cycle of the same colour and replace every unoriented edge between V and U by an oriented edge from V to U of the same colour. Abusing notation we call this transformed digraph $\mathcal{M}$

as well. Observe that $\mathcal{M}$ is a general multigraph. Naturally, we shall be using the partition $\mathcal{P}' = \{P'_1, P'_2\}$ with $P'_1 = U$ and $P'_2 = V$ for $\mathcal{M}$, hence $t := 2$ in our application of Theorem 21. By assumption we have $|P'_i| \geq \sigma n$ for $i = 1, 2$.

We next define the family $\mathcal{H}$ of simple canonical digraphs on $[q] = [4]$ we are decomposing $\mathcal{M}$ into. (This was motivated already after the definition of simple canonical digraphs.) Indeed, let $P_1 := \{1, 2, 3\}$ and $P_2 := \{4\}$. Let $\mathcal{H} := \{H_1, H_2, H_3\}$, where

- H_1 has an edge coloured by 1 directed from 1 to 2, a 2-cycle of colour 2 between 2 and 3, a 2-cycle of colour 3 between 3 and 1, an edge in colour 4 directed from 1 to 4, an edge in colour 5 directed from 2 to 4, and an edge in colour 6 directed from 3 to 4;
- H_2 consists of a single edge coloured by 4 directed from 1 to 4;
- H_3 consists of a single edge coloured by 5 directed from 2 to 4.

In other words, H_1 is a directed analogue of the diamond, with directions and colours consistent with directions and colours in $\mathcal{M}$. The digraphs H_2 and H_3 consist of a single edge and their purpose is that instead of using each edge $\vec{E}_1 \dot\cup E_2 \dot\cup E_3 \dot\cup E_6$ exactly once and each edge of $E_4 \dot\cup E_5$ at most once in the definition of the diamond core-decomposition, we now want to use each edge exactly once in a copy of a graph from $\mathcal{H}$ as in Definition 15.

By construction, an $\mathcal{H}$-decomposition of $\mathcal{M}$ gives the desired diamond core-decomposition. Theorem 21 provides us with such an $\mathcal{H}$-decomposition. Hence it remains to check that the conditions of this theorem are satisfied: We shall show that $(\mathcal{M}, \mathcal{P}')$ is $(\mathcal{H}, \mathcal{P})$-divisible, $(\mathcal{H}, 6\gamma, \omega)$-regular and $(\mathcal{H}, \mathcal{P}, \sqrt[q'h]{\omega}, h)$-vertex-extendable.

Divisibility: The 0-divisibility condition is implied by the fact that by assumption $|\vec{E}_1| = |E_2| = |E_3| = |E_4| \leq |E_5|, |E_6|$, where the inequality follows from (iii)–(vi) of Proposition 14. The 1-divisibility condition follows from (i)–(vi) of Proposition 14, the fact that each colour in the (original) diamond appears only once, and the fact that H_2 and H_3 are single edges. The 2-divisibility comes from the definition of the chest $\mathcal{M}$ and the corresponding digraphs H_1, H_2, H_3.

Regularity: For every copy H'_1 of H_1 in $\mathcal{M}$ and every edge H'_2 of colour 4 and every edge H'_3 of colour 5 in $\mathcal{M}$, we choose the weights

$$w_{H'_1} := \frac{1}{d_1^2 d_4 d_5 d_6 |V||U|}\,, \quad w_{H'_2} := \frac{d_4 - d_6}{d_4|V|^2}\,, \quad w_{H'_3} := \frac{d_5 - d_6}{d_5|V|^2}\,.$$

Observe that $w_{H'_1}, w_{H'_2}, w_{H'_3} \in [n^{-2}, (d^5\sigma^2)^{-1}n^{-2}]$ and we have $\omega \leq d^5\sigma^2 < 1$, so these weights are in the allowed range. By the (L, γ)-quasirandomness of the chest $\mathcal{M}$ we have that every edge e in $\mathcal{M}$ with colour $i \in \{1, 2, 3\}$ is contained in $(1 \pm \gamma)^5 d_1^2 |V| \cdot d_4 d_5 d_6 |U|$ copies of H_1 and 0 copies of H_2 and H_3. Hence,

$$\sum_{H \in \mathcal{H}} \sum_{H' \subseteq \mathcal{M}; e \in E(H'), H' \sim H} w_{H'} = (1 \pm \gamma)^5 d_1^2 |V| \cdot d_4 d_5 d_6 |U| \cdot \frac{1}{d_1^2 d_4 d_5 d_6 |V||U|} = (1 \pm 6\gamma)\,.$$

Every edge e in $\mathcal{M}$ with colour 6 is contained in $(1\pm\gamma)^5 d_1^3 d_4 d_5 |V|^2$ copies of H_1 and 0 copies of H_2 and H_3. Recall that $|\vec{E}_1| = d_1|V|^2$, $|E_6| = d_6|U||V|$, and $|\vec{E}_1| = |E_6|$ by assumption. Hence,

$$\sum_{H\in\mathcal{H}} \sum_{H'\subseteq\mathcal{M}; e\in E(H'), H'\sim H} w_{H'} = (1\pm\gamma)^5 d_1^3 d_4 d_5 |V|^2 \cdot \frac{1}{d_1^2 d_4 d_5 d_6 |V||U|}$$
$$= (1\pm 6\gamma)\frac{d_1|V|}{d_6|U|} = (1\pm 6\gamma)\frac{d_1|V|^2}{|E_6|} = (1\pm 6\gamma)\frac{|\vec{E}_1|}{|E_6|} = (1\pm 6\gamma)\ .$$

Next, every edge e in $\mathcal{M}$ of colour 4 is contained in $(1\pm\gamma)^5 d_1^3 d_5 d_6 |V|^2$ copies of H_1, in $(|V|-2)(|V|-3)$ copies of H_2 corresponding of the placement of the two isolated vertices $\{2,3\}$ in $\mathcal{M}$, and in 0 copies of H_3. Hence, using $|E_4| = d_4|U||V|$ and $d_1|V|^2 = |\vec{E}_1| = |E_6| = d_6|U||V|$, we get

$$\sum_{H\in\mathcal{H}} \sum_{H'\subseteq\mathcal{M}; e\in E(H'), H'\sim H} w_{H'} = (1\pm\gamma)^5 \frac{d_1^3 d_5 d_6 |V|^2}{d_1^2 d_4 d_5 d_6 |V||U|} + (1\pm\gamma)|V|^2\left(\frac{d_4-d_6}{d_4|V|^2}\right)$$
$$= (1\pm 6\gamma)\left(\frac{d_1|V|}{d_4|U|} + \frac{(d_4-d_6)|U|}{d_4|U|}\right) = (1\pm 6\gamma)\frac{d_1|V|^2 + d_4|U||V| - d_6|U||V|}{|E_4|}$$
$$= (1\pm 6\gamma)\frac{|E_4|}{|E_4|} = (1\pm 6\gamma)\ .$$

The calculation for an edge of colour 5 is analogous to the previous case.

Extendability: We provide the details for checking vertex-extendability only for $H = H_1$ and $x = 1 \in V(H_1)$. The other cases are analogous. We choose any collection of pairwise disjoint sets $\{A_i\}_{i\in[4]\setminus\{1\}}$ of size $|A_i| \le h = L$ with $A_i \subseteq V$ if $i = 2,3$ and $A_i \subseteq U$ if $i = 4$. Using the notation from the definition of vertex-extendability, we have $\mathsf{N}_{\mathcal{M}}^{(2,1,H_1)}(A_2) = \mathsf{N}_{\mathcal{M},1}^{in}(A_2) = \mathsf{N}_{\vec{E}_1}^{in}(A_2)$. Similarly $\mathsf{N}_{\mathcal{M}}^{(3,1,H_1)}(A_3) = \mathsf{N}_{\mathcal{M},2}^{in}(A_3) \cap \mathsf{N}_{\mathcal{M},2}^{out}(A_3) = \mathsf{N}_{E_2}(A_3)$, where we use that (undirected) edges in E_2 were replaced by oriented 2-cycles. Analogously, $\mathsf{N}_{\mathcal{M}}^{(4,1,H_1)}(A_4) = \mathsf{N}_{E_4}(A_4)$. Accordingly, we want to lower-bound the size of

$$A_1 := \bigcap_{y\in[4]\setminus\{1\}} \mathsf{N}_{\mathcal{M}}^{(y,1,H_1)}(A_y) = \mathsf{N}_{\vec{E}_1}^{in}(A_2) \cap \mathsf{N}_{E_2}(A_3) \cap \mathsf{N}_{E_4}(A_4)\,.$$

By the definition of (L,γ)-quasirandomness for the chest $\mathcal{M}$, setting $S_1' := A_2$, $S_2 := A_3$, $S_4 := A_4$ and $S_1 = S_3 = S_4' = S_5' = S_6' = S_5 = S_6 = \emptyset$, and using that $\omega \le (\frac{\sigma d}{2})^{5Lhq'}$, we get that

$$|A_1| \ge (1-\gamma) d_1^L d_2^L d_4^L |V| > \frac{1}{2} d^{3L}\sigma n \ge \sqrt[q'h]{\omega}\, n\ ,$$

as required for $(\mathcal{H}, \sqrt[q'h]{\omega}, h)$-vertex-extendability. □

5. Probabilistic tools and quasirandomness

In this section we collect a number of other tools that we shall need for the proof of Theorem 10. We start with some standard probabilistic results concerning properties of the hypergeometric distribution, McDiarmid's inequality, and Freedman's inequality. We then turn to the discussion of various quasirandomness properties.

5.1. **Probabilistic tools.** Let us quickly recall the hypergeometric distribution.

Definition 22. *Suppose that X is a set of order n, and $Y \subseteq X$ has ℓ elements. Let Z be a uniformly random subset of X of size h. Then $|Y \cap Z|$ has* hypergeometric distribution *with parameters (n, ℓ, h).*

It is a well known fact that the hypergeometric distribution is at least as concentrated as the binomial distribution with the same mean. In particular, we have the following bounds which apply to the binomial, and hence also hypergeometric, distributions.

Fact 23 (Corollary 2.4, Theorems 2.8 and 2.10, and (2.9) in [11])**.** *Given integers (n, ℓ, h), the hypergeometric distribution with parameters (n, ℓ, h) has mean $\frac{\ell h}{n}$. Suppose that N is a random variable whose distribution is either hypergeometric with parameters (n, ℓ, h), or binomial with parameters (n, p). Then, for each $c \in (0, \frac{3}{2})$, we have*

$$\mathbf{P}\left[|N - \mathbf{E}[N]| > c\mathbf{E}[N]\right] < 2\exp\left(-\frac{c^2}{3} \cdot \mathbf{E}[N]\right) .$$

We also have

$$\mathbf{P}\left[N - \mathbf{E}[N] > s\right] < \exp(-s)$$

whenever $s \geq 6\mathbf{E}[N]$.

Putting these two bounds together, we get the following observation, which we will use rather often. We have $N = \mathbf{E}[N] \pm n^{0.9}$ with probability at least $1 - \exp(-n^{0.65})$ for all sufficiently large n. To see that this is true, observe that if $\mathbf{E}[N] \leq n^{0.89}$ then we only need to consider the possibility $N \geq \mathbf{E}[N] + n^{0.9}$; since $n^{0.9} \geq 6\mathbf{E}[N]$ the second bound applies and gives us the desired result. If on the other hand $n^{0.89} \leq \mathbf{E}[N] \leq n$, then we use the first bound with $c = n^{-0.1}$.

Suppose that $\Omega = \prod_{i=1}^{k} \Omega_i$ is a product probability space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be *C-Lipschitz* if for each $\omega_1 \in \Omega_1, \ldots, \omega_k \in \Omega_k$, for each $i \in [k]$ and each $\omega_i' \in \Omega_i$ we have $|f(\omega_1, \ldots, \omega_i, \ldots, \omega_k) - f(\omega_1, \ldots, \omega_i', \ldots, \omega_k)| \leq C$. McDiarmid's Inequality states that Lipschitz functions are concentrated around their expectation.

Lemma 24 (McDiarmid's Inequality, [21])**.** *Let $f : \prod_{i=1}^{k} \Omega_i \to \mathbb{R}$ be a C-Lipschitz function. Then for each $t > 0$ we have*

$$\mathbf{P}\Big[|f - \mathbf{E}[f]| > t\Big] \leq 2\exp\left(-\frac{2t^2}{C^2 k}\right) .$$

Let Ω be a finite probability space. A *filtration* $\mathcal{F}_0$, $\mathcal{F}_1, \ldots$, $\mathcal{F}_n$ is a sequence of partitions of Ω such that $\mathcal{F}_i$ refines $\mathcal{F}_{i-1}$ for all $i \in [n]$. Note that in this finite setting, a function $f : \Omega \to \mathbb{R}$ is *$\mathcal{F}_i$-measurable* if f is constant on each part of $\mathcal{F}_i$. Further, for any random variable $Y : \Omega \to \mathbb{R}$

the *conditional expectation* $\mathbf{E}(Y|\mathcal{F}_i)\colon \Omega \to \mathbb{R}$ and the *conditional variance* $\mathbf{Var}(Y|\mathcal{F}_i)\colon \Omega \to \mathbb{R}$ of Y with respect to $\mathcal{F}_i$ are defined by

$$\begin{aligned} \mathbf{E}(Y|\mathcal{F})(x) &= \mathbf{E}(Y|X), \\ \mathbf{Var}(Y|\mathcal{F})(x) &= \mathbf{Var}(Y|X), \end{aligned} \qquad \text{where } X \in \mathcal{F} \text{ is such that } x \in X\,.$$

Suppose that we have an algorithm which proceeds in m rounds using a new source of randomness Ω_i in each round i. Then the probability space underlying the run of the algorithm is $\prod_{i=1}^m \Omega_i$. By *history up to time t* we mean a set of the form $\{\omega_1\} \times \cdots \times \{\omega_t\} \times \Omega_{t+1} \times \cdots \Omega_m$, where $\omega_i \in \Omega_i$. We shall use the symbol $\mathscr{H}_t$ to denote any particular history of such a form. By a *history ensemble up to time t* we mean any union of histories up to time t; we shall use the symbol $\mathscr{L}$ to denote any one such. Observe that there are natural filtrations associated to such a probability space: given times $t_1 < t_2 < \dots$ we let $\mathcal{F}_{t_i}$ denote the partition of Ω into the histories up to time t_i.

The following inequality, a corollary of Freedman's inequality [7] derived in [1], will be our main concentration tool for analysing random processes. The event $\mathcal{E}$ will generally be an assertion that various good properties are maintained up to some stage in a random process, and in particular it is important that $\mathcal{E}$ need not be measurable with respect to any elements of the given filtration.

Corollary 25 ([1, Corollary 7])**.** *Let Ω be a finite probability space, and $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$ be a filtration. Suppose that we have $R > 0$, and for each $1 \le i \le n$ we have an $\mathcal{F}_i$-measurable non-negative random variable Y_i, nonnegative real numbers $\tilde{\mu}, \tilde{\nu}$ and an event $\mathcal{E}$.*

(*a*) *Suppose that either $\mathcal{E}$ does not occur or we have $\sum_{i=1}^n \mathbf{E}\big[Y_i \big| \mathcal{F}_{i-1}\big] \le \tilde{\mu}$, and $0 \le Y_i \le R$ for each $1 \le i \le n$. Then*

$$\mathbf{P}\left[\mathcal{E} \text{ and } \sum_{i=1}^n Y_i > 2\tilde{\mu}\right] \le \exp\Big(-\frac{\tilde{\mu}}{4R}\Big)\,.$$

(*b*) *Suppose that either $\mathcal{E}$ does not occur or we have $\sum_{i=1}^n \mathbf{E}\big[Y_i \big| \mathcal{F}_{i-1}\big] = \tilde{\mu} \pm \tilde{\nu}$, and $0 \le Y_i \le R$ for each $1 \le i \le n$. Then for each $\tilde{\varrho} > 0$ we have*

$$\mathbf{P}\left[\mathcal{E} \text{ and } \sum_{i=1}^n Y_i \neq \tilde{\mu} \pm (\tilde{\nu} + \tilde{\varrho})\right] \le 2\exp\Big(-\frac{\tilde{\varrho}^2}{2R(\tilde{\mu} + \tilde{\nu} + \tilde{\varrho})}\Big)\,.$$

In particular, if $\tilde{\nu} = \tilde{\varrho} = \tilde{\mu}\tilde{\eta} > 0$ and $\tilde{\eta} \le \frac{1}{2}$, then

$$\mathbf{P}\left[\mathcal{E} \text{ and } \sum_{i=1}^n Y_i \neq \tilde{\mu}(1 \pm 2\tilde{\eta})\right] \le 2\exp\Big(-\frac{\tilde{\mu}\tilde{\eta}^2}{4R}\Big)\,.$$

5.2. **Quasirandomness.** Here we collect the definitions of various auxiliary quasirandomness properties that we use in our proof. The last of these, index-quasirandomness, is the strongest. However, a large fraction of this paper will be taken up by the analysis of two randomised algorithms for which we need a less general notion of quasirandomness; in order to keep notation in these parts manageable, we also give weaker definitions at the level that these two pieces of analysis need.

We start with a generalisation of (γ, L)-quasirandomness, that was introduced already in [2]. We will need this notion to state various lemmas from [2] and to perform our own extra analysis of the randomised algorithm from [2].

Definition 26 (diet condition)**.** *Let H be a graph with n vertices and $p\binom{n}{2}$ edges, and let $X \subseteq V(H)$ be any vertex set. We say that the pair (H, X) satisfies the* (γ, L)-diet condition *if for every set $S \subseteq V(H)$ of at most L vertices we have*

$$|\mathsf{N}_H(S) \setminus X| = (1 \pm \gamma)p^{|S|}(n - |X|)\,.$$

Observe that if H is a (γ, L)-quasirandom graph, and X is a randomly chosen subset of vertices whose size is not too close to n, then it is very likely that (H, X) has the $((1 + o(1)\gamma, L)$-diet condition. The randomised algorithm of [2], which we briefly described at the end of Section 3, has the property that (H, X) is very likely to satisfy the diet condition, where X is the image of the currently embedded vertices of G_i at any given time in the embedding of G_i. This is our way of formalising the idea that the image of G_i looks like a random set of vertices.

An important tool in this paper is a randomised algorithm for packing path-forests whose leaves are all embedded, which we state and prove in Section 7. As discussed in the proof sketch of Section 3, when we need this algorithm we will be working with a graph whose vertices are split into two parts (the 'sides' mentioned in the proof sketch) and densities within and between the sides are not necessarily equal. The following definition is a partite version of quasirandomness and the diet condition.

Definition 27 (block-quasirandom, block-diet)**.** *Suppose that $L \in \mathbb{N}$ and $\gamma > 0$ are given. Let H be a graph, and $U \subseteq V(H)$ be a subset. We say that (H, U) is* (γ, L)-block-diet *on $V_\boxminus \dot\cup V_\boxplus = V(H)$ if for every pair of sets $S_\boxminus \subseteq V_\boxminus$, $S_\boxplus \subseteq V_\boxplus$ with $|S_\boxminus| + |S_\boxplus| \le L$ we have*

$$\Big|\big(\mathsf{N}_{H[V_\boxminus, V_\boxplus]}(S_\boxminus) \cap \mathsf{N}_{H[V_\boxplus]}(S_\boxplus)\big) \setminus U\Big| = (1 \pm \gamma)|V_\boxplus \setminus U| \cdot d(H[V_\boxplus])^{|S_\boxplus|} \cdot d(H[V_\boxminus, V_\boxplus])^{|S_\boxminus|}\,,$$
$$\Big|\big(\mathsf{N}_{H[V_\boxminus]}(S_\boxminus) \cap \mathsf{N}_{H[V_\boxplus, V_\boxminus]}(S_\boxplus)\big) \setminus U\Big| = (1 \pm \gamma)|V_\boxminus \setminus U| \cdot d(H[V_\boxminus])^{|S_\boxminus|} \cdot d(H[V_\boxminus, V_\boxplus])^{|S_\boxplus|}\,,$$

where $d(H[V_\boxplus])$ is the density of $H[V_\boxplus]$, $d(H[V_\boxminus])$ is the density of $H[V_\boxminus]$, and $d(H[V_\boxminus, V_\boxplus])$ is the bipartite density of $H[V_\boxminus, V_\boxplus]$. If $(H, \emptyset)$ is (γ, L)-block-diet, we say H is (γ, L)-block-quasirandom.

We remark that we shall only consider whether some (H, U) is block-diet when we already know that H is block-quasirandom.

Our final quasirandomness condition is a good deal more complicated. Recall from the proof sketch in Section 3 that eventually in Stage G we will draw an auxiliary *chest* and argue that a generalised design in this chest corresponds to completing our perfect packing. We need the chest to be regular and extendable (as defined in Section 4) and to obtain this we ask for a partite quasirandomness condition rather similar to that of block-quasirandomness. In particular, we should be able to control the size of common neighbourhoods (in specified colours) of several vertices of the chest. The following index-quasirandomness controls the sizes of these sets when they are within the part V whose vertices $i \in [\frac{n}{2}]$ correspond to terminal pairs $\{\boxminus_i, \boxplus_i\}$ of H.

Recall that a vertex $\ell \in U$ in the chest corresponds to a graph G_ℓ (and we still need to embed a path-forest in G_ℓ). There are edges of three different colours leaving ℓ in the chest, which tell us which vertices of $V_\boxminus$ and of $V_\boxplus$ have not been used to embed vertices of G_ℓ, and which terminal pairs need connecting by a path. We do not need to know the graph structure of G_ℓ in order to know where these edges go, we simply need to know the set U_ℓ of vertices of H used in the embedding, and the set A_ℓ which contains the indices $i \in [\frac{n}{2}]$ of terminal pairs $\{\boxminus_i, \boxplus_i\}$ to which paths of G_ℓ are anchored.

In order to stick to this definition throughout the stages of our packing, we need to enhance this a little. From Stage C onwards, we have a collection of path-forests in various different graphs, indexed by κ in the following definition, to pack. Each of these graphs has a used set U_ℓ. However only some of them, indexed by κ', will be the graphs whose packing we complete in Stage G and only these graphs have a set A_ℓ of terminal pairs.

In the following definition we have an error parameter γ and a size parameter L, which play much the same rôle as the similar parameters in the above quasirandomness conditions. In addition we have two density parameters d_1 and d_2. The density d_1 is (approximately) the density within each of $V_\boxminus$ and $V_\boxplus$; we will always have very nearly (but not necessarily exactly) the same number of edges within each of these sets. The density d_2 is the density between $V_\boxminus$ and $V_\boxplus$. We have sets S_1 and S_2 of vertices of H, whose common neighbourhoods in respectively $V_\boxminus$ and $V_\boxplus$ we want to consider. In addition, we have sets $T_1, T_2 \subseteq \kappa$ and $T_3 \subseteq \kappa'$. We want to know which vertices of $V_\boxminus$ are not used in the embedding of graphs G_ℓ with $\ell \in T_1$, and similarly vertices of $V_\boxplus$ for $\ell \in T_2$. In addition, we want to know which $i \in [\frac{n}{2}]$ are terminal pairs for each G_ℓ with $\ell \in T_3$.

A coloured common neighbourhood in the chest is thus the same thing as a set of vertices $i \in [\frac{n}{2}]$ which satisfy all of the following. $\boxminus_i$ is in the common neighbourhood of S_1, and not used by any graph G_ℓ with $\ell \in T_1$. Similarly $\boxplus_i$ in the common neighbourhood of S_2 and not used by any graph G_ℓ with $\ell \in T_2$. Furthermore $\{\boxminus_i, \boxplus_i\}$ is a terminal pair of G_ℓ for each $\ell \in T_3$. The following definition controls the sizes of all such sets.

Definition 28 (index-quasirandom)**.** *Let $\tilde{H}$ be a graph on n vertices for n even, and $V_\boxminus = \{\boxminus_i : i \in [\frac{n}{2}]\}$, $V_\boxplus = \{\boxplus_i : i \in [\frac{n}{2}]\}$ be disjoint vertex sets such that $V(\tilde{H}) = V_\boxminus \dot\cup V_\boxplus$. Further, let $(U_\ell)_{\ell\in\kappa}$ be a collection of vertex sets $U_\ell \subseteq V(\tilde{H})$, and $(A_\ell)_{\ell\in\kappa'}$ be a collection of index sets with $A_\ell \subseteq [\frac{n}{2}]$ and $\kappa' \subseteq \kappa$. For a vertex set $S \subseteq V(\tilde{H})$ and an index set $T \subseteq \kappa$ we define*

$$\mathbb{N}_{\tilde{H}}(S,T) := \mathsf{N}_{\tilde{H}}(S) \setminus \bigcup_{\ell\in T} U_\ell \,.$$

One should think of $\mathbb{N}_{\tilde{H}}(S,T)$ *as being the vertices of* $\tilde{H}$ *adjacent to all members of* S *and not used in the embedding of any* G_ℓ *with* $\ell \in T$.

We say that the triple $(\tilde{H}, V_\boxminus, V_\boxplus)$ *is* (L,γ,d_1,d_2)-index-quasirandom *with respect to* $(U_\ell)_{\ell\in\kappa}$ *and* $(A_\ell)_{\ell\in\kappa'}$*, if for all* $S_1,S_2 \subseteq V(\tilde{H})$*, for all* $T_3 \subseteq \kappa'$ *and all* $T_1,T_2 \subseteq \kappa$*,* T_1,T_2,T_3 *pairwise disjoint such that*

$$|S_1|,|S_2|,|T_1|,|T_2|,|T_3| \leq L$$

the set

$$\mathbb{U}_{\tilde{H}}(S_1,S_2,T_1,T_2,T_3) := \left\{ i \in [\tfrac{n}{2}] : \boxminus_i \in \mathbb{N}_{\tilde{H}}(S_1,T_1), \boxplus_i \in \mathbb{N}_{\tilde{H}}(S_2,T_2), i \in \bigcap_{\ell\in T_3} A_\ell \right\}$$

satisfies

$$|\mathbb{U}_{\tilde{H}}(S_1,S_2,T_1,T_2,T_3)| = (1\pm\gamma)d_1^{|S_1\cap V_\boxminus|+|S_2\cap V_\boxplus|}d_2^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}\cdot\frac{n}{2}\cdot\prod_{\ell\in T_1\cup T_2}\Big(1-\frac{|U_\ell|}{n}\Big)\cdot\prod_{\ell\in T_3}\frac{|A_\ell|}{n/2}\,.$$

As mentioned, through the Stages of our packing we will maintain index-quasirandomness. However to apply our path packing theorem, we need the weaker block-quasirandomness and block-diet. The following lemma checks that we indeed get it.

Lemma 29. *Suppose that the triple* $(H, V_\boxminus, V_\boxplus)$ *is* (L,γ,d_1,d_2)*-index-quasirandom with respect to sets* $(U_\ell)_{\ell\in\kappa}$ *and* $(A_\ell)_{\ell\in\kappa'}$*. Suppose that* γ *is sufficiently small compared to* L^{-1} *and that* $n = |V(H)|$ *is sufficiently large given* L,γ,d_1,d_2*. Then we have*

$$d(H[V_\boxminus]), d(H[V_\boxplus]) = (1\pm 2\gamma)d_1 \quad \textit{and} \quad d(H[V_\boxminus, V_\boxplus]) = (1\pm\gamma)d_2\,.$$

In addition, for each $\ell \in \kappa$*, we have*

$$|V_\boxminus \setminus U_\ell|, |V_\boxplus \setminus U_\ell| = (1\pm\gamma)\left(\tfrac{n}{2} - \tfrac{|U_\ell|}{2}\right),$$

and the pair (H, U_ℓ) *is* $\big((L+2)\gamma, L\big)$*-block diet.*

Proof. Observe that $\big|\mathsf{N}_H(v)\cap V_\boxminus\big| = \big|\mathbb{U}_H(\{v\},\emptyset,\emptyset,\emptyset,\emptyset)\big|$. So we have

$$d(H[V_\boxminus])\cdot\binom{n/2}{2} = \frac{1}{2}\sum_{v\in V_\boxminus}|\mathsf{N}_H(v)\cap V_\boxminus| = \frac{1}{2}\cdot\frac{n}{2}(1\pm\gamma)d_1\frac{n}{2} = (1\pm(\gamma+\tfrac{\gamma}{2L}))d_1\cdot\binom{n/2}{2}\,.$$

Analogously, we obtain $d(H[V_\boxplus]) = (1\pm(\gamma+\frac{\gamma}{2L}))d_1$, and $d(H[V_\boxminus,V_\boxplus]) = (1\pm\gamma)d_2$ (in the last case we have $|V_\boxminus||V_\boxplus| = \frac{1}{4}n^2$ as compared to the small order difference between $\binom{|V_\boxminus|}{2}$ and $\frac{1}{8}n^2$ which is responsible for the worse error parameter). For the second part, let $U = U_\ell$. We have

$$|V_\boxminus\setminus U| = |\mathbb{U}_H(\emptyset,\emptyset,\{\ell\},\emptyset,\emptyset)| = (1\pm\gamma)(1-\frac{|U|}{n})\cdot\frac{n}{2} = (1\pm\gamma)\left(\tfrac{n}{2}-\tfrac{|U|}{2}\right)\,.$$

Let $S_{\boxplus}$ and $S_{\boxminus}$ be as in the definition of block-diet. Setting $S_2 := S_{\boxplus} \cup S_{\boxminus}$ we have

$$\begin{aligned}|(\mathsf{N}_{H[V_{\boxminus},V_{\boxplus}]}(S_{\boxminus}) \cap \mathsf{N}_{H[V_{\boxplus}]}(S_{\boxplus}) \setminus U| &= |\mathbb{U}_H(\emptyset, S_2, \emptyset, \{\ell\}, \emptyset)| = (1 \pm \gamma) d_1^{|S_{\boxplus}|} d_2^{|S_{\boxminus}|} \cdot (1 - \frac{|U|}{n}) \frac{n}{2} \\ &= (1 \pm \gamma) \left(\frac{d(H[V_{\boxplus}])}{(1 \pm (\gamma + \frac{\gamma}{2L}))} \right)^{|S_{\boxplus}|} \left(\frac{d(H[V_{\boxminus}, V_{\boxplus}])}{(1 \pm \gamma)} \right)^{|S_{\boxminus}|} \cdot \left(\tfrac{n}{2} - \tfrac{|U|}{2} \right) \\ &= (1 \pm (L+2)\gamma) d(H[V_{\boxplus}])^{|S_{\boxplus}|} \cdot d(H[V_{\boxminus}, V_{\boxplus}])^{|S_{\boxminus}|} |V_{\boxplus} \setminus U| \,.\end{aligned}$$

The estimate for $|(\mathsf{N}_{H[V_{\boxminus}]}(S_{\boxminus}) \cap \mathsf{N}_{H[V_{\boxminus},V_{\boxplus}]}(S_{\boxplus}) \setminus U|$ is done analogously. □

6. The proof of the main theorem

In this section we give our proof of Theorem 10. We first set up constants and preprocess the graphs $(G_s)_{s \in \mathcal{G}}$. In the key Section 6.4 we introduce the seven packing stages in seven corresponding lemmas, which readily yield Theorem 10.

Before we start, we introduce some notation. Suppose that P is a path of length ℓ. For any such a path, we will tacitly (and arbitrarily) fix a left-right orientation. Now, given $h \in \{0, 1, \ldots, \ell\}$, let $\mathsf{leftpath}_h(P)$ be the leftmost subpath of P of length h (that is, with h edges and so $h+1$ vertices). If $h = 0$, then $\mathsf{leftpath}_h$ is the leftmost vertex of P (which we will typically view as a vertex rather than a 1-vertex graph). We define $\mathsf{rightpath}_h(P)$ analogously. If $h < \ell$ and h has the same parity as ℓ, we define $\mathsf{middlepath}_h(P)$ as $P - (\mathsf{leftpath}_{(\ell-h-2)/2}(P) \cup \mathsf{rightpath}_{(\ell-h-2)/2}(P))$. Hence the length of $\mathsf{middlepath}_h(P)$ is h.

6.1. Constants.

Recall that we are given D_0, d and δ. Throughout Sections 6–14, we shall use the following constants satisfying the following hierarchy, where the numbers D_0, $L_{\mathbf{F}}$, $L_{\mathbf{E}}$, $L_{\mathbf{D}}$, $L_{\mathbf{C}}$ and D are integers and the remaining constants are positive real numbers:

$$\begin{aligned}(1) \quad & \frac{1}{D_0}, d, \delta \gg \lambda \gg \frac{1}{L_{\mathbf{F}}} \gg \frac{1}{L_{\mathbf{E}}} \gg \frac{1}{L_{\mathbf{D}}} \\ & \gg \frac{1}{L_{\mathbf{C}}} \gg \frac{1}{D} \gg \sigma_0 \gg \sigma_1 \gg \gamma_{\mathbf{F}} \gg \gamma_{\mathbf{E}} \gg \gamma_{\mathbf{D}} \gg \gamma_{\mathbf{A}} \gg c, \xi \gg \frac{1}{n_0} \,.\end{aligned}$$

This hierarchy should be understood as follows. Given D_0^{-1}, d, δ, there is a function $f(D_0^{-1}), d, \delta)$ which is monotone decreasing as any of the parameters decreases. We may choose any $0 < \lambda < f(D_0^{-1}, d, \delta)$. Given this choice, we may then choose any $L_{\mathbf{F}}^{-1}$ sufficiently small (in the same sense), and so on. We will finally be supplied with a parameter $n \geq n_0$ by Theorem 10, and we will want to assume that λn, $\sigma_0 n$ and $\sigma_1 n$ are all integers. Observe that for any given $x > 0$ there is a real number y with $x - n_0^{-1} < y \leq x$ such that yn is an integer. So what we will in fact do is the following: we do not at this stage specify λ precisely, but rather an interval of the form $\left[\frac{1}{2}x, x\right]$ where x is sufficiently small given D_0^{-1}, d, δ. We then insist that $L_{\mathbf{F}}^{-1}$ is sufficiently small for $\lambda = \frac{1}{2}x$, and so on, again choosing intervals for σ_0 and σ_1. With this construction we in particular have fixed values for all the constants listed above except λ, σ_0 and σ_1, and a guarantee that whatever values we choose in the specified intervals for these three constants, our calculations will work. One requirement we make is that n_0^{-1} is smaller than the length of any of these three intervals.

The constants promised by Theorem 10 are n_0, $L := 2D + 3$, c, ξ. Suppose now that the remaining parameter $n \geq n_0$ of Theorem 10 is given. We now fix λ, σ_0 and σ_1 in their specified intervals such that λn, $\sigma_0 n$ and $\sigma_1 n$ are all integers. Similarly, we increase if necessary δ by less than $1/n$ such that δn is an integer. From this point and through the rest of the proof, we keep these constants fixed.

We now summarise the meanings of these various constants. We refer to sets $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ and use notation $\mathsf{SpecPaths}_s$ which are defined in Section 6.3 below in order to keep these descriptions in one place.

D_0, d: The graphs which we want to pack are all D_0-degenerate, and the graph H into which we pack has at least dn^2 edges.

δ: The graphs G_s with $s \in \mathcal{J} \cup \mathcal{K}$ have $(1-\delta)n$ vertices.

λ: The size of $\mathcal{J}$ (i.e. the number of graphs with many bare paths) is λn. For each $s \in \mathcal{J}_0 \cup \mathcal{J}_2$ we have $|\mathsf{SpecPaths}_s| = \lambda n$.

σ_0: The size of $\mathcal{J}_0$ is $\sigma_0 n$.

σ_1: The size of $\mathcal{J}_1$ is $\sigma_1 n$, and if $s \in \mathcal{J}_1$ then $|\mathsf{SpecPaths}_s| = \sigma_1 n$.

$D, L_{\mathbf{C}}, L_{\mathbf{D}}, L_{\mathbf{E}}, L_{\mathbf{F}}$: In quasirandomness conditions in our various stages, we bound above the number of vertices for which we control the size of a common neighbourhood and the number of guest graphs whose common image we look at. This bound is D after Stage A (for consistency with [2]), and then $L_{\mathbf{C}}$ after Stage C, and so on.

$\gamma_{\mathbf{A}}, \gamma_{\mathbf{D}}, \gamma_{\mathbf{E}}, \gamma_{\mathbf{F}}$: The accuracy of our quasirandomness conditions is measured by $\gamma_{\mathbf{A}}$ after Stage A (and by $2\gamma_{\mathbf{A}}$ after Stage B and $100\gamma_{\mathbf{A}}$ after Stage C), and is then $\gamma_{\mathbf{D}}$ after stage D, and so on.

c: We impose the bound $\Delta(G_s) \leq \frac{cn}{\log n}$ for all guest graphs G_s.

ξ: We require H to be $(\xi, 2D+3)$-quasirandom.

n_0: We require $n \geq n_0$.

6.2. **Correcting inequalities and sizes.** It is convenient to assume in our proof that various of the inequalities in Definition 9 hold with equality, and to reduce the size of some sets from what is guaranteed by Definition 9.

We want that $\sum_{s\in\mathcal{G}} e(G_s) = e(H)$, so that our aim will always be a perfect packing. By Definition 9(b), we have $\sum_{s\in\mathcal{G}} e(G_s) \leq e(H)$. So, we add $e(H) - \sum_{s\in\mathcal{G}} e(G_s)$ copies of the graph K_2 to our family and correspondingly increase the size of $\mathcal{G}$. Observe that a perfect packing of the new family contains a packing of the old family.

We want that

$$|\mathcal{J}| = \lambda n \,. \tag{2}$$

Since we are given to start $|\mathcal{J}| \geq \delta n > \lambda n$ by Definition 9, we remove arbitrarily indices from $\mathcal{J}$ until we obtain this.

We want that $(1+\delta)n \leq \left|\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s\right| \leq (1+2\delta)n$. The lower bound is given by Definition 9(*e*). If the upper bound is exceeded, we remove arbitrarily indices from $\mathcal{K}$ one at a time until it is no longer exceeded. Since removing one index decreases (by Definition 9(*e*)) the size of $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ by at most $\frac{cn}{\log n} < \delta n$, the result is the desired pair of inequalities.

We want that each G_s with $s \in \mathcal{J} \cup \mathcal{K}$ has exactly $(1-\delta)n$ vertices. So, we add $(1-\delta)n - v(G_s)$ isolated vertices to each G_s with $s \in \mathcal{J} \cup \mathcal{K}$. Observe that a packing of the new family of graphs immediately gives a packing of the old family by ignoring the added isolated vertices. Similarly, we obtain $v(G_s) = n$ for each $s \in \mathcal{G} \setminus (\mathcal{J} \cup \mathcal{K})$.

Abusing notation slightly, we will continue to use the same letters $G_s, \mathcal{J}$ and so on for the modified family of graphs.

6.3. **Subgraphs of $(G_s)_{s\in\mathcal{G}}$ used in the packing stages.** Our packing stages, detailed below, will refer to the following subgraphs of our guest graphs $(G_s)_{s\in\mathcal{G}}$. Fix two disjoint families $\mathcal{J}_0, \mathcal{J}_1 \subseteq \mathcal{J}$ with $|\mathcal{J}_0| = \sigma_0 n$ and $|\mathcal{J}_1| = \sigma_1 n$. Set $\mathcal{J}_2 := \mathcal{J} \setminus (\mathcal{J}_0 \cup \mathcal{J}_1)$, meaning that

$$|\mathcal{J}_2| = (\lambda - \sigma_0 - \sigma_1)n \; . \tag{3}$$

The graphs G_s indexed by these three families will play quite different roles in the packing. For $s \in \mathcal{J}_1$, let $\mathsf{BasicPaths}'_s \subseteq \mathsf{BasicPaths}_s$ be an arbitrary family of $\sigma_1 n$ paths. For $s \in \mathcal{J}$, we define

$$\mathsf{SpecPaths}_s = \begin{cases} \mathsf{BasicPaths}_s & \text{if } s \in \mathcal{J}_0\,, \\ \mathsf{BasicPaths}'_s & \text{if } s \in \mathcal{J}_1\,, \\ \left\{\mathsf{middlepath}_7(P) : P \in \mathsf{BasicPaths}_s\right\} & \text{if } s \in \mathcal{J}_2\,. \end{cases}$$

For $s \in \mathcal{K}$, let

$$G_s^{\spadesuit} := G_s - \mathsf{OddVert}_s \, .$$

Next, for each $s \in \mathcal{J}$, we are going to define a graph $G_s^{||} \subseteq G_s$ by trimming off parts of the bare paths, i.e., for $s \in \mathcal{J}$ let

$$G_s^{||} := G_s - \{\mathsf{middlepath}_{\ell-2}(P) : P \in \mathsf{SpecPaths}_s, P \text{ has length } \ell\} \, .$$

That is, we have

$$v(G_s) = n \quad \text{for each } s \in \mathcal{G} \setminus (\mathcal{J} \cup \mathcal{K}) \;, \tag{4}$$

$$v(G_s^{\spadesuit}) = (1-\delta)n - |\mathsf{OddVert}_s| \quad \text{for each } s \in \mathcal{K} \;, \tag{5}$$

$$v(G_s^{||}) = (1-\delta-10\sigma_1)n \quad \text{for each } s \in \mathcal{J}_1 \;, \tag{6}$$

$$v(G_s^{||}) = (1-\delta-10\lambda)n \quad \text{for each } s \in \mathcal{J}_0 \;, \tag{7}$$

$$v(G_s^{||}) = (1-\delta-6\lambda)n \quad \text{for each } s \in \mathcal{J}_2 \; . \tag{8}$$

6.4. **The packing stages.** Our packing of $(G_s)_{s\in\mathcal{G}}$ into H will be provided in seven stages, called Stages A–G. These stages are captured by Lemma 30 (Stage A), Lemma 31 (Stage B),

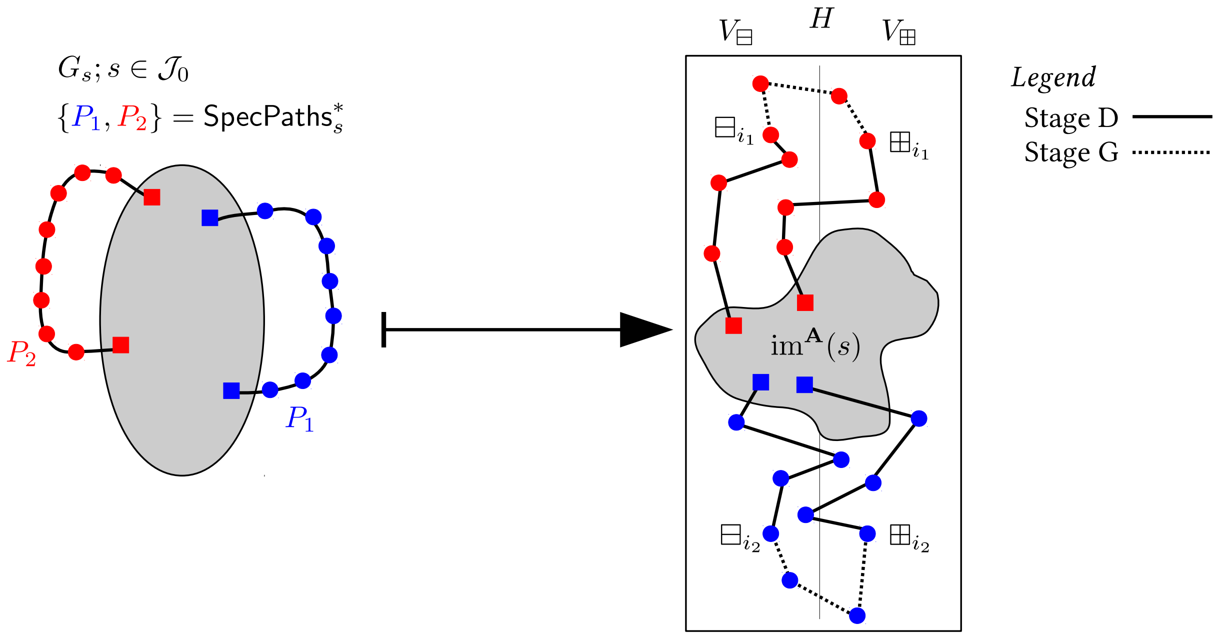

FIGURE 3. Embedding a graph G_s, $s \in \mathcal{J}_0$.

Lemma 33 (Stage C), Lemma 34 (Stage D), Lemma 36 (Stage E), Lemma 37 (Stage F) and Lemma 38 (Stage G). The proofs of these key lemmas are given in Sections 8–14.

Before turning to these lemmas, let us briefly summarise which parts of which graphs are embedded in which stage. The graphs $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}$ will be packed entirely in Stage A. In Stage A, we shall also pack $(G_s^{\spadesuit})_{s\in\mathcal{K}}$ and $(G_s^{\|})_{s\in\mathcal{J}}$. We shall add the odd-degree vertices $(\mathsf{OddVert}_s)_{s\in\mathcal{K}}$ to the packing in Stage B, hence finalising the packing of $(G_s)_{s\in\mathcal{K}}$. Stage C is devoted to splitting $V(H)$ into two vertex sets $V_{\boxminus}$ and $V_{\boxplus}$ of sizes $\lfloor \frac{n}{2} \rfloor$, and $\{\boxdot\}$ if n is odd. Further, for each $s \in \mathcal{J}$ we embed a subset of the paths $\mathsf{SpecPaths}_s$; the unembedded paths will be called $\mathsf{SpecPaths}^*_s$. In doing so, we use all the edges of the form $\boxminus_i\boxplus_i$. If n is odd, we further do away with having to deal with $\boxdot$ in the future by using all the edges incident with it. From Stage D to Stage G we pack the remaining paths from $(G_s)_{s\in\mathcal{J}}$. These packings differ for $\mathcal{J}_0$ (see Figure 3), $\mathcal{J}_1$ and $\mathcal{J}_2$. In each of these stages, we shall have some family, say $\mathcal{P}$, of paths we want to process during that stage. $\mathcal{P}$ are subgraphs of $\bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}^*_s$. We emphasise that being a subgraph can mean both that for paths of $\bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}^*_s$ we restrict to some shorter paths, or that we consider subfamilies of $\{\mathsf{SpecPaths}^*_s\}_{s\in\mathcal{J}}$ and take the corresponding paths of the full length. The paths of $\mathcal{P}$ will be *anchored*, by which we mean that prior to that stage, for each $P \in \mathcal{P}$ we have that the embedding of $\mathsf{leftpath}_0(P)$ and $\mathsf{rightpath}_0(P)$ is defined, while the rest of P is unembedded. When $\{\mathsf{leftpath}_0(P), \mathsf{rightpath}_0(P)\}$ is embedded on $\{u, v\} \subseteq V(H)$, we say that P *is anchored at* u *and* v. In that given stage we shall then embed $\mathcal{P}$. The progress of the embedding during the above stages is also outlined in Table 1.

Stage	$\mathcal{G} \setminus (\mathcal{K} \cup \mathcal{J})$	$\mathcal{K}$	$\mathcal{J}_0$	$\mathcal{J}_1$	$\mathcal{J}_2$
A	Packing G_s ✓ Packing complete	Packing $G_s^{\spadesuit}$ ✘ $\mathsf{OddVert}_s$ missing	Packing $G_s^{\parallel}$ ✘ 10-paths missing	Packing $G_s^{\parallel}$ ✘ 10-paths missing	Packing $G_s^{\parallel}$ ✘ 6-paths missing
B		Packing $\mathsf{OddVert}_s$ ✓ Packing complete			
C			Packing ≤ 1 path from $\mathsf{SpecPaths}_s$	Packing $\leq n^{0.6}$ paths from $\mathsf{SpecPaths}_s$	
D			Packing $\mathsf{leftpath}_4(P)$ and $\mathsf{rightpath}_4(P)$ for $P \in \mathsf{SpecPaths}_s^*$ ✘ 2-paths missing		
E					Packing $\mathsf{SpecPaths}_s^*$ ✓ Packing complete
F				$\mathsf{SpecPaths}_s^*$ ✓ Packing complete	
G			Packing $\mathsf{middlepath}_3(P)$ for $P \in \mathsf{SpecPaths}_s^*$ ✓ Packing complete		

TABLE 1. Packing of different parts of the graphs $\{G_s\}_{s\in\mathcal{G}}$ in Stage A-Stage G depending on the type s. The description is slightly imprecise. For the purpose of the table, a *missing ℓ-path* means that ℓ vertices on a bare path need to be embedded; since these are surrounded by 2 anchors, this means $\ell + 1$ edges to be packed.

We remark that each of the seven stages starts with a quasirandomness assumption, and indeed none of the seven packing steps would be possible without such an assumption. Hence, the outcome of each step needs to provide the quasirandomness conditions for the next step. These quasirandomness features are more complicated than the one formulated in Definition 4 which talks merely about the structure of the host graph. In particular, we need to guarantee quasirandomness properties for the images (as vertex sets) of the embedded graphs and for the anchors of the paths still to be embedded.

Let us now turn to the stages. Given embeddings $(\phi_s)_{s\in\mathcal{F}}$ of graphs $(F_s)_{s\in\mathcal{F}}$ into a graph H, the *leftover graph* is the graph on vertex set $V(H)$ containing precisely all edges of H not used by any ϕ_s. The *image in* H of the embedding ϕ_s is the set of vertices in $V(H)$ used by ϕ_s.

In Stage A we shall pack the family $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})} \cup (G_s^{\spadesuit})_{s\in\mathcal{K}} \cup (G_s^{\parallel})_{s\in\mathcal{J}}$ into the graph H (see $(A.i)$) with leftover graph $H_{\mathbf{A}}$. As mentioned above, in addition to providing the packing of the family $(G_s^{\spadesuit})_{s\in\mathcal{K}} \cup (G_s^{\parallel})_{s\in\mathcal{J}}$, we need to obtain certain quasirandomness conditions for $H_{\mathbf{A}}$ that will be used in later stages. Here, $(A.ii)$ and $(A.iii)$ guarantee that the individual images of the graphs $(G_s^{\spadesuit})_{s\in\mathcal{K}} \cup (G_s^{\parallel})_{s\in\mathcal{J}}$ are spread uniformly, $(A.iv)$ and $(A.v)$ guarantee that neighbours of the reserved odd degree vertices are embedded uniformly, while $(A.vi)$–$(A.ix)$ assert that anchors of paths that remain unembedded are spread uniformly.

Lemma 30 (Stage A, bulk embedding). *Given constants $D \in \mathbb{N}$, $\delta, \lambda, \sigma_1, \gamma_{\mathbf{A}} > 0$ there are constants $\xi, c > 0$ such that the following holds. Given graphs $(G_s)_{s\in\mathcal{G}}$ as described above, and a graph H which is $(\xi, 2D{+}3)$-quasirandom, there exist maps $(\phi_s^{\mathbf{A}})_{s\in\mathcal{G}}$ with the following properties.*

(A.i) *$(\phi^{\mathbf{A}}_s)_{s\in\mathcal{G}}$ packs $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}\cup(G^{\spadesuit}_s)_{s\in\mathcal{K}}\cup(G^{\|}_s)_{s\in\mathcal{J}}$ into H with leftover graph $H_{\mathbf{A}}$ of density $d_{\mathbf{A}}$ and with images $(\mathrm{im}^{\mathbf{A}}(s))_{s\in\mathcal{G}}$ in H.*

(A.ii) *For each $S\subseteq V(H)$, $|S|\le D$ and each $T\subseteq\mathcal{J}\cup\mathcal{K}$, $|T|\le D$ we have*

$$\Big|\mathsf{N}_{H_{\mathbf{A}}}(S)\setminus\bigcup_{s\in T}\mathrm{im}^{\mathbf{A}}(s)\Big|=(1\pm\gamma_{\mathbf{A}})d_{\mathbf{A}}^{|S|}n\prod_{s\in T}\Big(1-\frac{|\,\mathrm{im}^{\mathbf{A}}(s)|}{n}\Big)\,.$$

(A.iii) *For every pair of disjoint sets $S_1,S_2\subseteq V(H)$ with $|S_1|,|S_2|\le D$ we have with $\mathcal{X}:=\{s\in\mathcal{J}:\mathrm{im}^{\mathbf{A}}(s)\cap S_1=\emptyset,S_2\subseteq\mathrm{im}^{\mathbf{A}}(s)\}$ that*

$$\begin{aligned}|\mathcal{J}_0\cap\mathcal{X}|&=(1\pm\gamma_{\mathbf{A}})(\delta+10\lambda)^{|S_1|}(1-\delta-10\lambda)^{|S_2|}|\mathcal{J}_0|\,,\\|\mathcal{J}_1\cap\mathcal{X}|&=(1\pm\gamma_{\mathbf{A}})(\delta+10\sigma_1)^{|S_1|}(1-\delta-10\sigma_1)^{|S_2|}|\mathcal{J}_1|\,,\\|\mathcal{J}_2\cap\mathcal{X}|&=(1\pm\gamma_{\mathbf{A}})(\delta+6\lambda)^{|S_1|}(1-\delta-6\lambda)^{|S_2|}|\mathcal{J}_2|\,.\end{aligned}$$

(A.iv) *For every $v\in V(H)$ we have $\sum_{s\in\mathcal{K}}\deg_{G_s}\big((\phi^{\mathbf{A}}_s)^{-1}(v);\mathsf{OddVert}_s\big)\le\frac{20cn}{\log n}$, where if $v\notin\mathrm{im}^{\mathbf{A}}(s)$ we count these degrees as zero.*

(A.v) *For every $v\in V(H)$ we have*

$$\sum_{s\in\mathcal{K};v\notin\mathrm{im}^{\mathbf{A}}(s)}\;\sum_{x\in\mathsf{OddVert}_s}\;\prod_{y\in\mathsf{N}_{G_s}(x)}\mathbb{1}_{v\phi^{\mathbf{A}}_s(y)\in E(H_{\mathbf{A}})}\ge 4^{-50D^2d_{\mathbf{A}}^{-D}\delta^{-1}}n\,.$$

(A.vi) *For each $v\in V(H)$ and for every $s\in\mathcal{J}$ such that $v\notin\mathrm{im}^{\mathbf{A}}(s)$ we have that*

$$\Big|\Big\{x\in\bigcup_{P\in\mathsf{SpecPaths}_s}\{\mathsf{leftpath}_0(P),\mathsf{rightpath}_0(P)\}:v\in\mathsf{N}_{H_{\mathbf{A}}}\big(\phi^{\mathbf{A}}_s(x)\big)\Big\}\Big|$$
$$=(1\pm\gamma_{\mathbf{A}})d_{\mathbf{A}}\cdot 2|\mathsf{SpecPaths}_s|\,.$$

(A.vii) *For every $v\in V(H)$ the following hold. The number of $P\in\bigcup_{s\in\mathcal{J}_1}\mathsf{SpecPaths}_s$ for which $v\in\{\phi^{\mathbf{A}}_s(\mathsf{leftpath}_0(P)),\phi^{\mathbf{A}}_s(\mathsf{rightpath}_0(P))\}$ is equal to $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\sigma_1|\mathcal{J}_1|$. For every $\mathcal{J}^*\in\{\mathcal{J}_0,\mathcal{J}_2\}$ we have that the number of $P\in\bigcup_{s\in\mathcal{J}^*}\mathsf{SpecPaths}_s$ for which $v\in\{\phi^{\mathbf{A}}_s(\mathsf{leftpath}_0(P)),\phi^{\mathbf{A}}_s(\mathsf{rightpath}_0(P))\}$ is equal to $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\lambda|\mathcal{J}^*|$.*

(A.viii) *For every pair of distinct vertices $u,v\in V(H)$ and for every $\mathcal{J}^*\in\{\mathcal{J}_0,\mathcal{J}_1,\mathcal{J}_2\}$ the following hold. The number of $P\in\bigcup_{s\in\mathcal{J}^*}\mathsf{SpecPaths}_s$ for which $u\notin\mathrm{im}^{\mathbf{A}}(s)$ and $v\in\{\phi^{\mathbf{A}}_s(\mathsf{leftpath}_0(P)),\phi^{\mathbf{A}}_s(\mathsf{rightpath}_0(P))\}$ is equal to*

(*a*) $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\lambda(\delta+10\lambda)|\mathcal{J}_0|$, *if* $\mathcal{J}^*=\mathcal{J}_0$,

(*b*) $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\sigma_1(\delta+10\sigma_1)|\mathcal{J}_1|$, *if* $\mathcal{J}^*=\mathcal{J}_1$,

(*c*) $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\lambda(\delta+6\lambda)|\mathcal{J}_2|$, *if* $\mathcal{J}^*=\mathcal{J}_2$.

If u,u',v are distinct vertices of $V(H)$, the number of $P\in\bigcup_{s\in\mathcal{J}_0}\mathsf{SpecPaths}_s$ for which $u,u'\notin\mathrm{im}^{\mathbf{A}}(s)$ and $v\in\{\phi^{\mathbf{A}}_s(\mathsf{leftpath}_0(P)),\phi^{\mathbf{A}}_s(\mathsf{rightpath}_0(P))\}$ is equal to

(*d*) $2(1\pm\frac{\gamma_{\mathbf{A}}}{2})\lambda(\delta+10\lambda)^2|\mathcal{J}_0|$.

$(A.ix)$ *For every two distinct vertices* $v_1, v_2 \in V(H)$ *we have that the number of* $P \in \bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}_s$ *for which* $\{v_1, v_2\} = \{\phi_s^{\mathbf{A}}(\mathsf{leftpath}_0(P)), \phi_s^{\mathbf{A}}(\mathsf{rightpath}_0(P))\}$ *is at most* $n^{0.3}$.

The proof of this lemma, which we give in Section 8, relies on the analysis of the randomised algorithm *PackingProcess*, which was introduced in [2] and was also one of the key components in [1]. We use the main technical result of [2] as a black box to pack the graphs $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}$, after which the remaining edges still form a quasirandom graph. Unfortunately, we cannot use the results of [2] as a black box to pack the remaining graphs, all of which have at most $(1-\delta)n$ vertices. The reason is that, in addition to providing the packing of the family $(G_s^{\spadesuit})_{s\in\mathcal{K}} \cup (G_s^{\|})_{s\in\mathcal{J}}$ we also need to ensure various quasirandomness properties listed in $(A.ii)$–$(A.ix)$. Note that the density $d_{\mathbf{A}}$ of the leftover graph of this stage satisfies

$$d_{\mathbf{A}} = \binom{n}{2}^{-1} \cdot \Big(e(H) - \sum_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})} e(G_s) - \sum_{s\in\mathcal{K}} e(G_s^{\spadesuit}) - \sum_{s\in\mathcal{J}} e(G_s^{\|}) \Big) \tag{9}$$

$$\boxed{\text{Sec 6.2,6.3}} \quad = \binom{n}{2}^{-1} \cdot \Big(D_{\mathrm{odd}} \cdot |\cup_{s\in\mathcal{K}} \mathsf{OddVert}_s| + 11\lambda n|\mathcal{J}_0| + 10\sigma_1 n|\mathcal{J}_1| + 7\lambda n|\mathcal{J}_2| \Big)$$

$$\boxed{\text{Sec 6.3,(1)}} \quad = (14 \pm 0.1)\lambda^2. \tag{10}$$

We now turn to Stage B. By suitably embedding the reserved odd vertices $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ we shall ensure that the degrees of vertices of the leftover graph of this stage have parities suitable for the remaining stages of our packing process. Let us provide more details on this now. Suppose that $v \in V(H)$ is an arbitrary vertex. After Stage A, there are $\deg_{H_{\mathbf{A}}}(v)$ edges incident to v which are available for the remaining stages. We know that once the packing is completed, all these edges will be used (by our assumptions on equalities from Section 6.2). These host graph edges will be used to accommodate guest graph edges of the following types.

[et1] Edges of the type $xy \in E(G_s)$, where $y \in \mathsf{OddVert}_s$, $x \in \mathsf{N}_{G_s}(y)$, $\phi_s^{\mathbf{A}}(x) = v$ and $s \in \mathcal{K}$. The number of such edges is equal to

$$\mathrm{OddOut}(v) := \sum_{s\in\mathcal{K}} \deg_{G_s}\left((\phi_s^{\mathbf{A}})^{-1}(v), \mathsf{OddVert}_s\right).$$

[et2] Edges of the type $xy \in E(G_s)$, where $y \in \mathsf{OddVert}_s$, $x \in \mathsf{N}_{G_s}(y)$, $s \in \mathcal{K}$, and where y will be mapped on v.

[et3] Edges $\mathsf{leftpath}_1(P)$ and $\mathsf{rightpath}_1(P)$ of the paths $P \in \mathsf{SpecPaths}_s$ (for $s \in \mathcal{J}$), for those paths P for which $\phi_s^{\mathbf{A}}(\mathsf{leftpath}_0(P)) = v$ or $\phi_s^{\mathbf{A}}(\mathsf{rightpath}_0(P)) = v$. The number of such edges is equal to

$$\mathrm{PathTerm}(v) := \sum_{s\in\mathcal{J}} \left|\{v\} \cap \{\phi_s^{\mathbf{A}}(\mathsf{rightpath}_0(P)), \phi_s^{\mathbf{A}}(\mathsf{leftpath}_0(P)) : P \in \mathsf{SpecPaths}_s\}\right|. \tag{11}$$

[et4] Edges xy of the paths $P \in \mathsf{SpecPaths}_s$ (for $s \in \mathcal{J}$) with x or y that will be mapped to v, for which P is not anchored at v, i.e., $(\phi^{\mathbf{A}})^{-1}(v) \cap \{\mathsf{leftpath}_0(P), \mathsf{rightpath}_0(P)\} = \emptyset$.

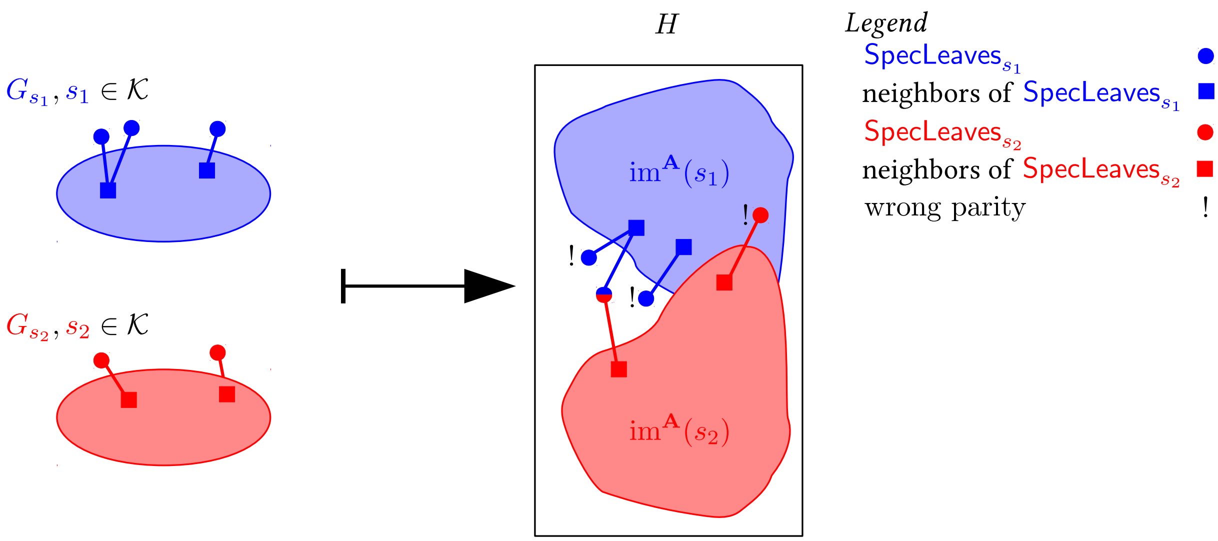

FIGURE 4. Embedding graphs G_{s_1} and G_{s_2}, $s_1, s_2 \in \mathcal{K}$. In this picture all vertices in $\mathsf{OddVert}_s$ are leaves.

Each path counted in [et4] will actually use 2 edges at v. In particular, the total number of edges counted in [et4] will be even. Since edges counted in [et1] and [et3] are already determined, we need to adjust the number of edges counted in [et2] so that it has the same parity as

$$\mathrm{Parity}(v) := \deg_{H_{\mathbf{A}}}(v) - \mathrm{OddOut}(v) - \mathrm{PathTerm}(v) \mod 2 .$$

In other words, we need to embed $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ in such a way that for each vertex v, the number of vertices from $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ embedded on v has the same parity at $\mathrm{Parity}(v)$ (see $(B.iii)$). An illustration is given in Figure 4.

The number of edges we are newly embedding in Stage B is at most $D_{\mathrm{odd}}\left|\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s\right|$ by our assumptions, which is linear in n. So the density $d_{\mathbf{B}}$ of the leftover graph $H_{\mathbf{B}}$ after the packing of $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})} \cup (G_s^{\|})_{s\in\mathcal{J}} \cup (G_s)_{s\in\mathcal{K}}$ is very close to $d_{\mathbf{A}}$. Because we are embedding so few edges in Stage B, we do not need to use a randomised algorithm for this stage but can rely on matching-type arguments. We shall show (see $(B.iv)$) that we can perform this packing stage so that each host graph vertex loses only a tiny fraction of incident edges. It follows that the leftover graph $H_{\mathbf{B}}$ automatically inherits the quasirandomness properties concerning the uniform spread of images and anchors from $H_{\mathbf{A}}$. Accordingly, the following lemma does not list most of these quasirandomness properties again. We prove this lemma in Section 9.

Lemma 31 (Stage B, parity correction)**.** *Given partial embeddings $(\phi_s^{\mathbf{A}})_{s\in\mathcal{G}}$ as described in Lemma 30, there exist maps $(\phi_s^{\mathbf{B}})_{s\in\mathcal{G}}$ with the following properties.*

$(B.i)$ *For each $s \in \mathcal{K}$, the map $\phi_s^{\mathbf{B}}$ is an embedding of the entire graph G_s into H which extends $\phi_s^{\mathbf{A}}$. For each $s \in \mathcal{G} \setminus \mathcal{K}$ we set $\phi_s^{\mathbf{B}} = \phi_s^{\mathbf{A}}$.*

(*B.ii*) $(\phi_s^{\mathbf{B}})_{s\in\mathcal{G}}$ *is a packing of* $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}\cup\left(G_s^{\|}\right)_{s\in\mathcal{J}}\cup(G_s)_{s\in\mathcal{K}}$ *into the graph* H *with leftover graph* $H_{\mathbf{B}}$ *of density* $d_{\mathbf{B}}$ *and images* $\left(\mathrm{im}^{\mathbf{B}}(s)\right)_{s\in\mathcal{G}}$.

(*B.iii*) *For each* $v\in V(H)$ *we have*

$$\deg_{H_{\mathbf{B}}}(v)\equiv \mathrm{PathTerm}(v) \mod 2\ .$$

(*B.iv*) *For each* $v\in V(H)$ *we have* $\deg_{H_{\mathbf{B}}}(v)\geq \deg_{H_{\mathbf{A}}}(v)-8Dcn$.

(*B.v*) *For each* $S\subseteq V(H)$, $|S|\leq L$ *and each* $T\subseteq\mathcal{J}$, $|T|\leq L$ *we have*

$$\left|\mathsf{N}_{H_{\mathbf{B}}}(S)\setminus\bigcup_{s\in T}\mathrm{im}^{\mathbf{B}}(s)\right|=(1\pm 2\gamma_{\mathbf{A}})d_{\mathbf{B}}^{|S|}n\prod_{s\in T}\left(1-\frac{|\mathrm{im}^{\mathbf{B}}(s)|}{n}\right)\ .$$

In Stage C, we randomly split $V(H)$ into two vertex sets $V_{\boxminus}$ and $V_{\boxplus}$ of size $\lfloor\frac{n}{2}\rfloor$, and a special vertex $\boxdot$ if n is odd. We label the vertices of $V_{\boxminus}$ as $\{\boxminus_i : i\in[\lfloor\frac{n}{2}\rfloor]\}$ and of $V_{\boxplus}$ as $\{\boxplus_i : i\in[\lfloor\frac{n}{2}\rfloor]\}$. We then embed a few of the paths in $\bigcup_{s\in\mathcal{J}}\mathsf{SpecPaths}_s$ so that the following three things happen, of which the last two apply only when n is odd. Firstly, all the edges of H which are of the form $\boxminus_i\boxplus_i$ are used. Secondly, all paths which are anchored at $\boxdot$ are embedded. Thirdly, all other edges leaving $\boxdot$ are used. After this, if n is odd then all edges incident to the vertex $\boxdot$ are used up. So, regardless of the parity of n in the remaining Stages D–G we effectively work with a host graph on an even number of vertices.

The key assumption of Theorem 10 was that of quasirandomness of the host graph, formalised in Definition 4. In $(A.ii)$–$(A.ix)$ we saw more complicated quasirandomness conditions which dealt for example with mutual positions of the images of different graphs G_s. In the following stages we will encapsulate our running quasirandomness properties under a common *quasirandom setup*, which we define now. That is, after each of the next three stages, we will argue that we have maintained the quasirandom setup (with quasirandomness parameter becoming a bit worse in each step) and gained certain additional good properties, which permit the final step to complete a perfect packing. In what follows, the reader should think of U_s as the vertices in $V(H)$ used in the current partial embedding of G_s; of F_s as the path-forest in G_s which remains to be embedded, with A_s the vertices to which the paths of F_s are anchored and ϕ_s the anchoring; and of I_s as the indices of terminal pairs to which we can connect (after Stage C) or have connected (after Stage D).

Definition 32 (Quasirandom setup)**.** *Let* H *be a graph on an even number of vertices partitioned into* $V_{\boxminus}=\{\boxminus_i : i\in[v(H)/2]\}$ *and* $V_{\boxplus}=\{\boxplus_i : i\in[v(H)/2]\}$. *Let* $\mathcal{J}=\mathcal{J}_0\dot\cup\mathcal{J}_1\dot\cup\mathcal{J}_2$ *be a set of indices. For each* i *and* $s\in\mathcal{J}_i$, *let* F_s *be a path-forest whose components all have at least 4 vertices. Let* U_s *be a subset of* $V(H)$, *and let* $A_s\subseteq U_s$. *Let* ϕ_s *be a bijection from the leaves of* F_s *to* A_s. *For each* $s\in\mathcal{J}$ *and* $a\in\{\boxplus,\boxminus\}$, *let* A_s^a *denote the set of* $x\in A_s$ *such that* $\phi_s(y)\in V_a$, *where* x *and* y *are the leaves of a path in* F_s. *Finally let for each* $s\in\mathcal{J}_0$ *the set* I_s *be a subset of* $[v(H)/2]$. *This is a* (γ,L,d_1,d_2)-quasirandom setup *if the following conditions are satisfied.*

(*Quasi* 1) *The triple* $(H, V_\boxminus, V_\boxplus)$ *is* (L,γ,d_1,d_2)*-index-quasirandom with respect to the sets* $(U_s)_{s\in\mathcal{J}}$ *and* $(I_s)_{s\in\mathcal{J}_0}$.

(*Quasi* 2) *For each* $a,b\in\{\boxminus,\boxplus\}$ *and each* $s\in\mathcal{J}_1\cup\mathcal{J}_2$ *we have* $|A_s^b\cap V_a|=(1\pm\gamma)\frac{1}{4}|A_s|$. *For each* $s\in\mathcal{J}_0$ *we have* $|A_s\cap V_a|=(1\pm\gamma)\frac{1}{2}|A_s|$.

(*Quasi* 3) *For every* $v\in V(H)$, $a\in\{\boxminus,\boxplus\}$ *and* $i\in\{1,2\}$ *we have that the number of* $s\in\mathcal{J}_i$ *such that* F_s *has a path with leaves* x *and* y *such that* $\phi_s(x)=v$ *and* $\phi_s(y)\in V_a$ *is equal to* $(1\pm\gamma)\frac{1}{2n}\sum_{s\in\mathcal{J}_i}|A_s|$.

(*Quasi* 4) *For each* $s\in\mathcal{J}_0$, *each* $u\in V_\boxminus\setminus U_s$ *and each* $v\in V_\boxplus\setminus U_s$ *we have*

$$\begin{aligned}&\big|\{\boxminus_i\in\mathsf{N}_H(u):i\in I_s\}\big|,\big|\{\boxplus_i\in\mathsf{N}_H(v):i\in I_s\}\big|=(1\pm\gamma)d_1|I_s|\\ \text{and}\quad&\big|\{\boxminus_i\in\mathsf{N}_H(v):i\in I_s\}\big|,\big|\{\boxplus_i\in\mathsf{N}_H(u):i\in I_s\}\big|=(1\pm\gamma)d_2|I_s|\,.\end{aligned}$$

(*Quasi* 5) *For each* $s\in\mathcal{J}$, *each* $a\in\{\boxplus,\boxminus\}$, *each* $u\in V_\boxminus\setminus U_s$ *and each* $v\in V_\boxplus\setminus U_s$ *we have*

$$\big|\{\boxminus_i\in\mathsf{N}_H(u)\cap A_s^a\}\big|=(1\pm\gamma)d_1|A_s^a\cap V_\boxminus|\,,\tag{12}$$

$$\big|\{\boxminus_i\in\mathsf{N}_H(v)\cap A_s^a\}\big|=(1\pm\gamma)d_2|A_s^a\cap V_\boxminus|\,,\tag{13}$$

$$\big|\{\boxplus_i\in\mathsf{N}_H(v)\cap A_s^a\}\big|=(1\pm\gamma)d_1|A_s^a\cap V_\boxplus|\,,\tag{14}$$

$$\text{and}\quad\big|\{\boxplus_i\in\mathsf{N}_H(u)\cap A_s^a\}\big|=(1\pm\gamma)d_2|A_s^a\cap V_\boxplus|\,.\tag{15}$$

(*Quasi* 6) *Given disjoint sets* $S_1,S_2\subseteq V(H)$ *with* $|S_1|,|S_2|\le L$, *for each* $i=1,2$ *we have*

$$\big|\{s\in\mathcal{J}_i\,:\,S_1\cap U_s=\emptyset,S_2\subseteq U_s\}\big|=(1\pm\gamma)\sum_{s\in\mathcal{J}_i}\frac{(n-|U_s|)^{|S_1|}|U_s|^{|S_2|}}{n^{|S_1|+|S_2|}}\,.$$

Given additionally $T\subseteq[v(H)/2]$ *with* $|T|\le L$ *such that* $\{\boxminus_i,\boxplus_i:i\in T\}$ *is disjoint from* $S_1\cup S_2$, *suppose that there is no* i *such that* $\{\boxminus_i,\boxplus_i\}\subseteq S_1\cup S_2$. *Then we have*

$$\big|\{s\in\mathcal{J}_0\,:\,S_1\cap U_s=\emptyset,S_2\subseteq U_s,T\subseteq I_s\}\big|=(1\pm\gamma)\sum_{s\in\mathcal{J}_0}\frac{(n-|U_s|)^{|S_1|}|U_s|^{|S_2|}(2|I_s|)^{|T|}}{n^{|S_1|+|S_2|+|T|}}\,.$$

(*Quasi* 7) *For each* $i=0,1,2$ *and each* $v\in V(H)$, *the number of* $s\in\mathcal{J}_i$ *such that* $v\in A_s$ *is* $(1\pm\gamma)\sum_{s\in\mathcal{J}_i}\frac{|A_s|}{n}$.

(*Quasi* 8) *For each* $i=1,2$ *and each* $u,v\in V(H)$ *with* $u\neq v$, *the number of* $s\in\mathcal{J}_i$ *such that* $u\in A_s$ *and* $v\notin U_s$ *is* $(1\pm\gamma)\sum_{s\in\mathcal{J}_i}\frac{|A_s|(n-|U_s|)}{n^2}$. *If in addition there is no* j *such that* $\{u,v\}=\{\boxminus_j,\boxplus_j\}$ *then the number of* $s\in\mathcal{J}_0$ *such that* $u\in A_s$ *and* $v\notin U_s$ *is* $(1\pm\gamma)\sum_{s\in\mathcal{J}_i}\frac{|A_s|(n-|U_s|)}{n^2}$.

Note that when we use this definition, in all the summations the summands will be constant or very nearly so, for example $|A_s|$ will vary by only a constant as s ranges over any one $\mathcal{J}_i$.

The following lemma now encapsulates what we do in Stage C. We prove it in Section 10.

Lemma 33 (Stage C, partite reduction)**.** *Given partial embeddings $(\phi_s^{\mathbf{B}})_{s\in\mathcal{G}}$ as described in Lemma 31 extending partial embeddings $(\phi_s^{\mathbf{A}})_{s\in\mathcal{G}}$ as described in Lemma 30, we can construct the following.*

If n is even we find a labelling $V(H)=\{\boxminus_i,\boxplus_i\}_{i\in[n/2]}$. If n is odd we find a suitable labelling $V(H)=\{\boxdot\}\cup\{\boxminus_i,\boxplus_i\}_{i\in[(n-1)/2]}$. Set $V_\boxminus:=\{\boxminus_i : i\in[\lfloor\frac{n}{2}\rfloor]\}$ and $V_\boxplus:=\{\boxplus_i : i\in[\lfloor\frac{n}{2}\rfloor]\}$.

We find partial embeddings $(\phi_s^{\mathbf{C}})_{s\in\mathcal{G}}$ into H extending the $(\phi_s^{\mathbf{B}})_{s\in\mathcal{G}}$, such that no edge of H is in the image of two different $\phi_s^{\mathbf{C}},\phi_{s'}^{\mathbf{C}}$ and such that the following holds. For each $s\in\mathcal{J}$, the map $\phi_s^{\mathbf{C}}$ embeds some entire paths from $\mathsf{SpecPaths}_s$, and no other vertices. For each $s\in\mathcal{J}_0$, the map $\phi_s^{\mathbf{C}}$ embeds exactly one path of $\mathsf{SpecPaths}_s$ if n is odd, and exactly zero paths if n is even. For each $s\in\mathcal{J}_1\cup\mathcal{J}_2$, $\phi_s^{\mathbf{C}}$ embeds at most $n^{0.6}$ paths. We denote the remaining unembedded paths in G_s by $\mathsf{SpecPaths}_s^*$ for each $s\in\mathcal{J}$. We write $\mathrm{im}^{\mathbf{C}}(s)=\mathrm{im}(\phi_s^{\mathbf{C}})$.

In addition, every edge of H of the form $\boxminus_i\boxplus_i$ for $i\in[\lfloor n/2\rfloor]$ is in the image of some $\phi_s^{\mathbf{C}}$. If n is odd, every edge of H incident with $\boxdot$ is in the image of some $\phi_s^{\mathbf{C}}$.

Let $H_{\mathbf{C}}$ be the graph on $V_\boxminus\dot\cup V_\boxplus$ of edges of H which are not in the image of any $\phi_s^{\mathbf{C}}$. Let $d_{\mathbf{C}}$ be the density of $H_{\mathbf{C}}$. With the given $\mathcal{J}_0,\mathcal{J}_1,\mathcal{J}_2$, let for each $i=0,1,2$ and $s\in\mathcal{J}_i$ the path-forest $F_s^{\mathbf{C}}$ be $\mathsf{SpecPaths}_s^*$, the set $U_s^{\mathbf{C}}=\mathrm{im}^{\mathbf{C}}(s)$, and the set $A_s^{\mathbf{C}}$ be the images of the leaves of $F_s^{\mathbf{C}}$ under $\phi_s^{\mathbf{C}}$, with ϕ_s the restriction of $\phi_s^{\mathbf{C}}$ to the leaves of $F_s^{\mathbf{C}}$. For each $s\in\mathcal{J}$ we have $A_s^{\mathbf{C}}\subseteq V(H_{\mathbf{C}})$, i.e. all unembedded paths have their anchors in $V_\boxminus\cup V_\boxplus$. Finally let for each $s\in\mathcal{J}_0$ the set $I_s^{\mathbf{C}}=\{i\in[\lfloor\frac{n}{2}\rfloor] : \boxminus_i,\boxplus_i\notin U_s^{\mathbf{C}}\}$. Then we have a $(100\gamma_{\mathbf{A}},L_{\mathbf{C}},d_{\mathbf{C}},d_{\mathbf{C}})$-quasirandom setup.

Furthermore, for each $s\in\mathcal{J}_0$ we have $|I_s|=(1\pm10\gamma_{\mathbf{A}})\frac{n}{2}\big(1-\frac{|U_s^{\mathbf{C}}|}{n}\big)^2$, and for each $u\in V(H_{\mathbf{C}})$ and index $j\in[\lfloor\frac{n}{2}\rfloor]$ such that $u\notin\{\boxminus_j,\boxplus_j)$, the number of $s\in\mathcal{J}_0$ such that $u\in A_s^{\mathbf{C}}$ and $j\in I_s^{\mathbf{C}}$ is $(1\pm100\gamma_{\mathbf{A}})\sum_{s\in\mathcal{J}_0}\frac{2|A_s^{\mathbf{C}}||I_s^{\mathbf{C}}|}{n^2}$.

In Stage D we shall extend our packing by outer parts of paths of $\bigcup_{s\in\mathcal{J}_0}\mathsf{SpecPaths}_s^*$; recall Figure 3. More precisely, for each $s\in\mathcal{J}_0$ we shall extend the packing to

$$G_s^{\boxminus\boxplus}:=G_s-\bigcup_{P\in\mathsf{SpecPaths}_s^*}\mathsf{middlepath}_1(P)\ .$$

Observe that $|\mathsf{SpecPaths}_s^*|$ has the same size for every $s\in\mathcal{J}_0$, which we denote by

$$|\mathsf{SpecPaths}_s^*|=\lambda^* n\ . \tag{16}$$

Recalling that $\mathsf{SpecPaths}_s=\mathsf{BasicPaths}_s$ (see Section 6.3) has size λn, we see that $\lambda^*=\lambda$, if n is even, and $\lambda^*=\lambda-1/n$, if n is odd. A crucial property guaranteed in Lemma 34 is that the unembedded edges $\mathsf{middlepath}_2(P)$ are always anchored on a *terminal pair*, that is, a pair of vertices of the form $\{\boxminus_i,\boxplus_i\}$. We shall call the graph consisting of edges that remain after this step as $H_{\mathbf{D}}$. Observe that the density of $H_{\mathbf{D}}$ will satisfy

$$d_{\mathbf{D}}:=\frac{e(H_{\mathbf{D}})}{\binom{|V_\boxminus\cup V_\boxplus|}{2}}=(1\pm\xi)\left(14(\lambda-\sigma_0-\sigma_1)\lambda+6\lambda\sigma_0+22\sigma_1^2\right)\ . \tag{17}$$

The following lemma is proved in Section 11.

Lemma 34 (Stage D, connecting to terminal pairs)**.** *Given partial embeddings $(\phi_s^{\mathbf{C}})_{s\in\mathcal{G}}$ as described in Lemma 33, we can construct the following.*

We find partial embeddings $(\phi_s^{\mathbf{D}})_{s\in\mathcal{G}}$ into H extending the $(\phi_s^{\mathbf{C}})_{s\in\mathcal{G}}$, such that no edge of H is in the image of two different $\phi_s^{\mathbf{D}}, \phi_{s'}^{\mathbf{D}}$ such that the following holds. For each $s \in \mathcal{J} \setminus \mathcal{J}_0$ we have $\phi_s^{\mathbf{D}} = \phi_s^{\mathbf{C}}$. For each $s \in \mathcal{J}_0$ and each $P \in \mathsf{SpecPaths}_s^*$, the map $\phi_s^{\mathbf{D}}$ embeds all vertices of P except for the two vertices $\mathsf{middlepath}_1(P)$, which are not embedded. Furthermore, the fifth and eighth vertices of P are embedded to $\{\boxminus_i, \boxplus_i\}$ for some $i \in [\lfloor \frac{n}{2} \rfloor]$ (not necessarily in this order). We write $\mathrm{im}^{\mathbf{D}}(s) = \mathrm{im}(\phi_s^{\mathbf{D}})$.

Let $H_{\mathbf{D}}$ be the graph on $V_\boxminus \cup V_\boxplus$ of edges of H which are not in the image of any $\phi_s^{\mathbf{D}}$. Let $d_{\mathbf{D}}$ be the density of $H_{\mathbf{D}}$. With the given $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$, let for each $i = 1, 2$ and $s \in \mathcal{J}_i$ the path-forest F_s be $\mathsf{SpecPaths}_s^*$, the set $U_s = \mathrm{im}^{\mathbf{D}}(s)$, and the set A_s be the images of the leaves of F_s under $\phi_s^{\mathbf{D}}$, with ϕ_s the restriction of $\phi_s^{\mathbf{D}}$ to the leaves of F_s. Let for each $s \in \mathcal{J}_0$ the path-forest F_s be $\mathsf{SpecShortPaths}_s := \{\mathsf{middlepath}_3(P) : P \in \mathsf{SpecPaths}_s^*\}$, the set $U_s = \mathrm{im}(\phi_s^{\mathbf{D}})$, and the set A_s be the images of the leaves of F_s under $\phi_s^{\mathbf{D}}$, with ϕ_s the restriction of $\phi_s^{\mathbf{D}}$ to the leaves of F_s. Let the set $I_s = \{i \in [\lfloor \frac{n}{2} \rfloor] : \boxminus_i, \boxplus_i \in A_s\}$. Then we have a $(\gamma_{\mathbf{D}}, L_{\mathbf{D}}, d_{\mathbf{D}}, d_{\mathbf{D}})$-quasirandom setup.

In Stage E we shall finish packing of $(G_s)_{s\in\mathcal{J}_2}$. To this end we need to pack $\mathsf{SpecPaths}_s^*$ for each $s \in \mathcal{J}_2$. The main feature is that we achieve a precise given count of edges of $H_{\mathbf{E}}$ within $V_\boxminus$, within $V_\boxplus$, and from $V_\boxminus$ to $V_\boxplus$. To motivate where these count requirements come from, we need to explain packings in Stage F and Stage G.

[fg1] In Stage F, we pack $\{\mathsf{SpecPaths}_s^*\}_{s\in\mathcal{J}_1}$. Recall that each path P from this set has 11 edges. If one endvertex of P was mapped (in Stage A) to $V_\boxminus$ and the other to $V_\boxplus$, then the packing in Stage F will be such that 3 edges of P will be mapped inside $V_\boxminus$, 3 edges will be mapped inside $V_\boxplus$ and 5 edges will be mapped into $H[V_\boxminus, V_\boxplus]$. If both endvertices of P were mapped (in Stage A) to $V_\boxminus$, then the packing in Stage F will be such that 2 edges of P will be mapped inside $V_\boxminus$, 3 edges will be mapped inside $V_\boxplus$ and 6 edges will be mapped into $H[V_\boxminus, V_\boxplus]$. The situation when both endvertices of P were mapped (in Stage A) to $V_\boxplus$ is symmetric.

[fg2] In Stage G, we pack $\{\mathsf{SpecShortPaths}_s\}_{s\in\mathcal{J}_0}$. Recall that each path P from this set has 3 edges. The packing will be such that 1 edge of P will be mapped inside $V_\boxminus$, 1 edge will be mapped inside $V_\boxplus$ and 1 edge will be mapped into $H[V_\boxminus, V_\boxplus]$. (Recall Figure 3.)

So, denote by $j_{\boxminus\boxplus}$, (resp. $j_{\boxminus\boxminus}$, and $j_{\boxplus\boxplus}$), the number of paths from $\{\mathsf{SpecPaths}_s^*, s \in \mathcal{J}_1\}$ with one anchor in $V_\boxminus$ (resp. in $V_\boxminus$, and in $V_\boxplus$) and the other one in $V_\boxplus$ (resp. in $V_\boxminus$, and in $V_\boxplus$). The above description explains why we require the counts (18).

Define $d_{\mathbf{E}}^* := 8(\lambda^*\sigma_0 + \frac{11}{4}\sigma_1^2)$ and $\bar{d_{\mathbf{E}}} := 4(\lambda^*\sigma_0 + \frac{11}{2}\sigma_1^2)$. The following fact is immediate.

Fact 35. *We have* $\min\{d_{\mathbf{E}}^*, \bar{d_{\mathbf{E}}}\} \geq 4\sigma_0\lambda^*$ *and* $d_{\mathbf{E}}^* - \bar{d_{\mathbf{E}}} = 4\lambda^*\sigma_0$.

The numbers $\bar{d_{\mathbf{E}}}$ and $d_{\mathbf{E}}^*$ will be close to the edge densities between the vertex sets $V_\boxminus$ and $V_\boxplus$, and within either of the sets $V_\boxminus$ and $V_\boxplus$, respectively, after Stage E finishes. This is expressed by the quasirandomness setup below. The proof of the following lemma can be found in Section 12.

Lemma 36 (Stage E, density correction)**.** *Given partial embeddings $(\phi_s^{\mathbf{D}})_{s\in\mathcal{G}}$ as described in Lemma 34, we can construct the following.*

We find partial embeddings $(\phi_s^{\mathbf{E}})_{s\in\mathcal{G}}$ into H extending the $(\phi_s^{\mathbf{D}})_{s\in\mathcal{G}}$, such that no edge of H is in the image of two different $\phi_s^{\mathbf{E}}, \phi_{s'}^{\mathbf{E}}$. Furthermore, the following holds. For each $s \in \mathcal{J} \setminus \mathcal{J}_2$ we have $\phi_s^{\mathbf{E}} = \phi_s^{\mathbf{D}}$. For each $s \in \mathcal{J}_2$, the map $\phi_s^{\mathbf{E}}$ embeds all vertices of G_s. We write $\mathrm{im}^{\mathbf{E}}(s) = \mathrm{im}(\phi_s^{\mathbf{E}})$. Let $H_{\mathbf{E}}$ be the graph on $V_{\boxminus} \cup V_{\boxplus}$ of edges of H which are not in the image of any $\phi_s^{\mathbf{E}}$. We have

$$\begin{aligned} e_{H_{\mathbf{E}}}(V_{\boxminus}) &= \sigma_0\lambda^* n^2 + 2j_{\boxminus\boxminus} + 3(j_{\boxminus\boxplus} + j_{\boxplus\boxplus})\ , \\ e_{H_{\mathbf{E}}}(V_{\boxplus}) &= \sigma_0\lambda^* n^2 + 2j_{\boxplus\boxplus} + 3(j_{\boxminus\boxplus} + j_{\boxminus\boxminus})\ , \\ e_{H_{\mathbf{E}}}(V_{\boxminus}, V_{\boxplus}) &= \sigma_0\lambda^* n^2 + 6(j_{\boxminus\boxminus} + j_{\boxplus\boxplus}) + 5j_{\boxminus\boxplus}\ . \end{aligned} \tag{18}$$

With the given $\mathcal{J}_0, \mathcal{J}_1$ and $\emptyset$ let for each $s \in \mathcal{J}_0$ the path-forest F_s be $\mathsf{SpecShortPaths}_s^*$, and for each $s \in \mathcal{J}_1$ the path-forest F_s be $\mathsf{SpecPaths}_s^*$. Let for each $i = 0, 1$ and $s \in \mathcal{J}_i$ the set $U_s = \mathrm{im}(\phi_s^{\mathbf{E}})$, and the set A_s be the images of the leaves of F_s under $\phi_s^{\mathbf{E}}$, with ϕ_s the restriction of $\phi_s^{\mathbf{E}}$ to the leaves of F_s. For each $s \in \mathcal{J}_0$ let the set $I_s = \{i \in [\lfloor \frac{n}{2} \rfloor] : \boxminus_i, \boxplus_i \in A_s\}$. Then we have a $(\gamma_{\mathbf{E}}, L_{\mathbf{E}}, d_{\mathbf{E}}^*, \bar{d_{\mathbf{E}}})$-quasirandom setup, for suitable numbers $d_{\mathbf{E}}^*, \bar{d_{\mathbf{F}}} > 0$.

At this moment, it only remains to embed $\bigcup_{s\in\mathcal{J}_1} \mathsf{SpecPaths}_s^*$ and $\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecShortPaths}_s$. In Stage F we embed the former family in such a way that it will allow embedding the latter family in Stage G. In order to see how the embedding should look in Stage F, we thus need to look at the situation we will be in before and during Stage G. For any $v \in V_{\boxminus} \cup V_{\boxplus}$, let us write $t(v)$ for the number of paths from $\{\mathsf{SpecShortPaths}_s, s \in \mathcal{J}_0\}$ anchored at v. Observe that since for each $s \in \mathcal{J}_0$ the set $\mathsf{SpecShortPaths}_s$ is a set of disjoint paths, we have $t(v)$ equal to the number of $s \in \mathcal{J}_0$ such that $v \in A_s$, which by $(\mathit{Quasi}\,7)$ is

$$t(v) = (1 \pm \gamma_{\mathbf{D}}) 2\lambda^* \sigma_0 n\ . \tag{19}$$

Since each path of $\mathsf{SpecShortPaths}_s$ is anchored to a terminal pair by Lemma 34 for each $i \in [\lfloor \frac{n}{2} \rfloor]$ we have $t(\boxminus_i) = t(\boxplus_i)$.

In Stage G, we will embed all paths as follows: given a path $abcd \in \mathsf{SpecShortPaths}_s$, with $\phi_s^{\mathbf{F}}(a) = \boxminus_i$ and $\phi_s^{\mathbf{F}}(d) = \boxplus_i$, we will embed b into $V_{\boxminus}$ and c into $V_{\boxplus}$. Consider some $\boxminus_i$. In Stage G, we will use $t(\boxminus_i)$ edges from $\boxminus_i$ into $V_{\boxminus}$ to deal with the paths anchored at $\boxminus_i$, and (since we complete a perfect packing in Stage G) all the remaining edges from $\boxminus_i$ to $V_{\boxminus}$ are used to accommodate inner vertices of paths in $\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecShortPaths}_s$, and these paths also use all the edges from $\boxminus_i$ to $V_{\boxplus}$. A similar statement holds for $\boxplus_i$. It follows that in $H_{\mathbf{F}}$ we need, for each $i \in [\lfloor \frac{n}{2} \rfloor]$,

$$\deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxplus}) = \deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxminus}) + t(\boxminus_i) \ \text{ and } \ \deg_{H_{\mathbf{F}}}(\boxplus_i, V_{\boxminus}) = \deg_{H_{\mathbf{F}}}(\boxplus_i, V_{\boxplus}) + t(\boxplus_i)\,. \tag{20}$$

We will see that this is essentially the only requirement for Stage F. That is, the fact that we have after Stage E a quasirandom setup, together with the fact that we will use only very few edges at any given vertex in Stage F, ensures that we still have a quasirandom setup after

Stage F, which is one of the key properties needed for the existence of designs in Stage G. The following lemma is proved in Section 13.

Lemma 37 (Stage F, degree correction)**.** *Given partial embeddings $(\phi_s^{\mathbf{E}})_{s\in\mathcal{G}}$ as described in Lemma 36, we can construct the following.*

We find partial embeddings $(\phi_s^{\mathbf{F}})_{s\in\mathcal{G}}$ into H extending the $(\phi_s^{\mathbf{E}})_{s\in\mathcal{G}}$, such that no edge of H is in the image of two different $\phi_s^{\mathbf{F}}, \phi_{s'}^{\mathbf{F}}$ such that the following holds. For each $s \in \mathcal{J}_0$ we have $\phi_s^{\mathbf{F}} = \phi_s^{\mathbf{E}}$. For each $s \notin \mathcal{J}_0$, the map $\phi_s^{\mathbf{F}}$ embeds all vertices of G_s. Let $H_{\mathbf{F}}$ be the graph on $V_\boxminus \cup V_\boxplus$ of edges of H which are not in the image of any $\phi_s^{\mathbf{F}}$. The for each $i \in \lfloor n/2 \rfloor$ the equation (20) holds. We write $\mathrm{im}^{\mathbf{F}}(s) = \mathrm{im}(\phi_s^{\mathbf{F}})$.

With the given $\mathcal{J}_0$, $\emptyset$ and $\emptyset$ let for each $s \in \mathcal{J}_0$ the path-forest F_s be $\mathsf{SpecShortPaths}_s^*$. Let for each $s \in \mathcal{J}_0$ the set $U_s = \mathrm{im}^{\mathbf{F}}(s)$, and the set A_s be the images of the leaves of F_s under $\phi_s^{\mathbf{F}}$, with ϕ_s the restriction of $\phi_s^{\mathbf{F}}$ to the leaves of F_s. For each $s \in \mathcal{J}_0$ let the set $I_s = \{i \in [\lfloor n/2 \rfloor] : \boxminus_i, \boxplus_i \in A_s\}$. Then we have a $(\gamma_{\mathbf{F}}, L_{\mathbf{E}}, d_{\mathbf{F}}^*, \bar{d_{\mathbf{F}}})$-quasirandom setup, for suitable numbers $d_{\mathbf{F}}^*, \bar{d_{\mathbf{F}}} > 0$.

It remains to pack $\{\mathsf{SpecShortPaths}_s\}_{s\in\mathcal{J}_0}$. This is done in our last stage, encapsulated by the following lemma, which we prove in Section 14.

Lemma 38 (Stage G, designs completion)**.** *Given partial embeddings $(\phi_s^{\mathbf{F}})_{s\in\mathcal{G}}$ as described in Lemma 37, there exist maps $(\phi_s^{\mathbf{G}})_{s\in\mathcal{G}}$ with the following properties.*

- (*G.i*) *For each $s \in \mathcal{J}_0$, the map $\phi_s^{\mathbf{G}}$ is an embedding of G_s into H which extends $\phi_s^{\mathbf{F}}$. For each $s \in \mathcal{G} \setminus \mathcal{J}_0$, set $\phi_s^{\mathbf{G}} = \phi_s^{\mathbf{F}}$.*
- (*G.ii*) *The collection $(\phi_s^{\mathbf{G}})_{s\in\mathcal{G}}$ is a packing of $(G_s)_{s\in\mathcal{G}}$ into the graph H.*

The key tool of the proof of Lemma 38 is Keevash's machinery of generalised designs as we introduced in a tailored setting in Proposition 14. Again, recall that the key features for this machinery to work are the $(\gamma_{\mathbf{F}}, L_{\mathbf{E}}, d_{\mathbf{F}}^*, \bar{d_{\mathbf{F}}})$-quasirandom setup, (20) and that no terminal pair forms an edge (which we obtained in Stage C).

Once all these stages are complete, we obtained the desired packing of $(G_s)_{s\in\mathcal{G}}$ into H. This finishes the proof of Theorem 10.

7. Packing paths

A *path-forest* is a nonempty collection of components, each of which is a path of length at least 2. In this section we show how to pack a collection of path-forests into a quasirandom graph H, with several constraints. First, the quasirandom graph will be partitioned into two *sides* $V_\boxminus$ and $V_\boxplus$ of equal size, and the vertices of each path in each path forest will be pre-assigned to one side or the other. Furthermore, the path-forests are 'anchored', i.e. each leaf is pre-embedded to a specific vertex of H, and each path-forest is required to avoid a set of 'used' vertices given for that path-forest. What is helpful, the path-forests we pack will always contain only paths of bounded length, they will never be spanning (in fact they will always miss a positive proportion of the unused vertices in each part of H), and the assignment of sides is such that we finish the packing with a positive density of edges in each side and between the two sides of H unused.

Such a packing is not generally possible; however we will assume that the anchors and used vertices are quasirandomly distributed, and under this additional condition we show the following simple random algorithm works. We go through the path-forests in turn and in each path-forest pack one (randomly chosen) path after another, in each case choosing one of the possible paths consistent with the assigned sides uniformly at random. Our aim is to show that the resulting packing 'looks random' in the sense that in each path-forest the vertices used, and in total the edges in and between sides used, look like uniform random sets of the appropriate sizes. The main result of this section is Lemma 42.

7.1. Embedding one path-forest.

To begin with, we formally state the algorithm we use to embed one path-forest, and give the analysis of this algorithm.

Algorithm 1: *RandomPathEmbedding*

Input:
- a path-forest F with anchors A,
- a graph H on $V_{\boxminus}\dot\cup V_{\boxplus}$,
- a set $U_0 \subseteq V(H)$ of used vertices,
- an assignment $\xi : V(F) \setminus A \to \{V_{\boxminus}, V_{\boxplus}\}$,
- an embedding $\phi_0 : A \to V(H)$ such that $\operatorname{im} \phi_0 \subseteq U_0$

choose a uniform random order $F_1, \ldots, F_{t^*}$ of the components of F;
$\psi_0 := \phi_0$;
for $t = 1$ **to** t^* **do**
  let $x_1, \ldots, x_k$ be the path F_t;
  let $v_1 = \phi_0(x_1)$, and $v_k = \phi_0(x_k)$;
  let Paths_t be the set of v_1-v_k-paths $v_1, v_2, \ldots, v_k$ of length $k-1$ in $H - U_{t-1}$ such that $v_i \in \xi(x_i)$ for each $1 < i < k$;
  if Paths_t *is empty* **then** halt with failure;
  choose a path P uniformly at random in Paths_t;
  $\psi_t := \psi_{t-1} \cup \{F_t \hookrightarrow P\}$;
  $U_t := U_0 \cup \operatorname{im} \psi_t$;
end
return $\phi = \psi_{t^*}$;

In order to analyse this algorithm, we only need to specify what precisely we mean by a quasirandom distribution of the anchors.

Definition 39 (anchor distribution property)**.** *Suppose that we are given an n-vertex graph H with $V(H) = V_{\boxminus}\dot\cup V_{\boxplus}$, a subset U of $V(H)$, a path-forest F whose degree-1 vertices are the set A. For $\gamma \geq 0$, we say an injective map $\phi_0 : A \to U$ and assignment $\xi : V(F) \setminus A \to \{V_{\boxminus}, V_{\boxplus}\}$ has the* γ-anchor distribution property *if the following holds.*

Given $a, b, c \in \{\boxminus, \boxplus\}$ let $A_{a,b,c}$ denote the collection of $x \in A$ such that $\phi_0(x) \in V_a$, and such that the neighbour y of x satisfies $\xi(y) = V_b$, and the next vertex z (adjacent to y) satisfies

$\xi(z) = V_c$. *Define* $d_{\boxminus,\boxplus} = d_{\boxplus,\boxminus}$ *to be the bipartite density of* $H[V_\boxplus, V_\boxminus]$, $d_{\boxplus,\boxplus}$ *to be the density of* $H[V_\boxplus]$, *and* $d_{\boxminus,\boxminus}$ *to be the density of* $H[V_\boxminus]$. *For each* $v \in V_b \setminus U$ *we have*

$$\big|\big\{x \in A_{a,b,c} : \phi_0(x) \in \mathsf{N}_H(v)\big\}\big| = (1 \pm \gamma)d_{a,b}|A_{a,b,c}| \pm \tfrac{1}{2}\gamma n^{0.99} .$$

We write $A_{a,b} = A_{a,b,\boxminus} \cup A_{a,b,\boxplus}$, *and normally we will only need to refer to* $A_{a,b}$; *the* γ*-anchor distribution property implies*

$$\big|\big\{x \in A_{a,b} : \phi_0(x) \in \mathsf{N}_H(v)\big\}\big| = (1 \pm \gamma)d_{a,b}|A_{a,b}| \pm \gamma n^{0.99} .$$

The following lemma contains all the facts we need about the running of Algorithm 1.

Lemma 40 (Embedding a path forest)**.** *Given an integer* $L \geq 5$ *and real* $\nu > 0$, *there is a constant* $C > 0$ *such that the following holds for all* $0 < \gamma < C^{-1}$ *and all sufficiently large* n *(depending on* γ*).*

Let F *be a path-forest with leaves* A *(also called anchors), in which each path has at least* 5 *and at most* L *vertices. Suppose* $\nu n \leq |V_\boxplus|, |V_\boxminus| \leq n$, *and let* H *be* (γ, L)*-block-quasirandom on* $V_\boxminus \dot\cup V_\boxplus$ *with densities* $d(H[V_\boxminus]) = d_{\boxminus\boxminus}$, $d(H[V_\boxplus]) = d_{\boxplus\boxplus}$, $d(H[V_\boxminus, V_\boxplus]) = d_{\boxminus\boxplus}$, *each at least* ν. *Let* $\phi_0 : A \to V(H)$ *be an embedding of the anchors and* $\xi : V(F) \to \{V_\boxminus, V_\boxplus\}$ *be an assignment of the vertices of the paths to the parts of* H *such that* $\phi_0(x) \in \xi(x)$ *for each* $x \in A$. *For each* $a \in \{\boxminus, \boxplus\}$ *let* $X_a := \{x \in V(F) \,:\, \xi(x) = V_a\}$, *and suppose that* $|X_a \setminus A| \geq \nu n$. *Let* $U \subseteq V(H)$ *be a set of "used" vertices such that* $\operatorname{im}\phi_0 \subseteq U$, *and suppose that for each* $a \in \{\boxminus, \boxplus\}$ *we have*

$$\big|V_a \setminus U\big| - \big|\big\{x \in V(F) \setminus A : \xi(x) = V_a\big\}\big| \geq \nu n .$$

Suppose that we have the γ*-anchor distribution property, and* (H, U) *is* (γ, L)*-block-diet. Suppose that* $\big|\big\{(x, y) \,:\, xy \in E(F), x \in X_a \setminus A, y \in X_b \setminus A\big\}\big| \geq \nu n$ *for each* $a, b \in \{\boxminus, \boxplus\}$. *Let* V' *be a set of specified vertices with* $|V'| \geq \nu n$ *and* $V' \subseteq V_\boxminus \setminus U$ *or* $V' \subseteq V_\boxplus \setminus U$.

When we execute RandomPathEmbedding *(Algorithm 1), then*

- $(PE1)$ *with probability at least* $1 - \exp\big(-n^{-0.3}\big)$, *we obtain an embedding* $\phi : V(F) \to \operatorname{im}\phi_0 \cup \big(V(H) \setminus U\big)$ *of* F *in* H *extending* ϕ_0 *and such that* $\phi(x) \in \xi(x)$ *for each* $x \in V(F) \setminus A$.

Moreover, if we get such an embedding ϕ, *the following hold.*

- $(PE2)$ *If* $a \in \{\boxminus, \boxplus\}$ *is such that* $V' \subseteq V_a \setminus U$, *then*

$$\mathbf{P}\Big[|\operatorname{im}\phi \cap V'| \neq (1 \pm C\gamma)|V'|\tfrac{|X_a|}{|V_a \setminus U|}\Big] \leq \exp\big(-n^{0.3}\big) .$$

- $(PE3)$ *For each* $S_\boxminus \subseteq V_\boxminus \setminus U, S_\boxplus \subseteq V_\boxplus \setminus U$ *with* $|S_a| \leq L$ *for each* a, *we have*

$$\mathbf{P}\big[(S_\boxminus \cup S_\boxplus) \cap \operatorname{im}\phi = \emptyset\big] = (1 \pm C\gamma) \prod_{a \in \{\boxminus,\boxplus\}} \Big(\tfrac{|V_a \setminus U| - |\{x \in V(F) \setminus A:\ \xi(x) = V_a\}|}{|V_a \setminus U|}\Big)^{|S_a|} .$$

- $(PE4)$ *For each edge* $e \in E(H - U)$, *setting* $a, b \in \{\boxminus, \boxplus\}$ *such that* $e \in E(H[V_a, V_b])$, *we have*

$$\mathbf{P}\big[\phi \text{ uses } e\big] = (1 \pm C\gamma)\frac{\big|\big\{(x, y) \,:\, xy \in E(F), x \in X_a \setminus A, y \in X_b \setminus A\big\}\big|}{d_{ab}|V_a \setminus U||V_b \setminus U|} .$$

$(PE5)$ *For each $a,b\in\{\boxminus,\boxplus\}$, each $x\in A$ such that $x\in X_a$ and the neighbour of x in F is in X_b, and each $v\in V_b\setminus U$ with $\phi_0(x)v\in E(H)$ we have*

$$\mathbf{P}\big[\phi \text{ uses } \phi_0(x)v\big]=(1\pm C\gamma)\frac{1}{d_{ab}|V_b\setminus U|}\,.$$

$(PE6)$ *For each pair of edges $uv,u'v'\in E(H)$, we have*

$$\mathbf{P}[\phi \text{ uses } uv \text{ and } u'v']\le n^{-3/2}\,.$$

Proof. Given an integer $L\ge 2$ and $\nu>0$, we set

$$C:=\exp\left(10^4L^4\nu^{-6}\right),$$

and given $0<\gamma<1/C$, for each $x\in\mathbb{R}$ we let t^* be the number of components of F, and let

$$\beta_x:=\gamma\exp\left(\tfrac{1000L^2x}{\nu^3t^*}\right).$$

We set $\varepsilon=\gamma^2C^{-1}$, and note that $\beta_{t^*}\le\sqrt{C}\gamma$.

First, we show that after randomly ordering the paths of F, each interval of εn consecutive paths from this order has the anchor-distribution property. This avoids the possibility that neighbours of anchors are distributed in a non-uniform way for most of the process: this imbalance would eventually be corrected, but it would complicate the analysis.

Claim 40.1. *In the uniform random order $F_1,\dots,F_{t^*}$ of the components of F, with probability at least $1-\exp\left(-n^{0.4}\right)$ for each $1\le i\le t^*+1-\varepsilon n$ the forest F_i^* (with the same U, and ξ and ϕ_0 restricted to F_i^*) whose components are $F_i,F_{i+1},\dots,F_{i+\varepsilon n-1}$ has the γ-anchor distribution property. Furthermore, for each $a\in\{\boxminus,\boxplus\}$, we have $\frac{|X_a\cap V(F_i^*)\setminus A|}{|V(F_i^*)\setminus A|}=(1\pm\varepsilon)\frac{|X_a\setminus A|}{|V(F)\setminus A|}$.*

Proof. Since for each $1\le i\le t^*+1-\varepsilon n$, the distribution of paths in F selected to F_i^* is a uniform random set of εn paths, by the union bound it suffices to show the above properties of F_i^* hold with probability $1-\exp\left(-n^{0.5}\right)$ for a uniform random set F^* of εn paths of F. Note that $|X_a|\ge\nu n$ for each a, and so in particular $t^*\ge\nu n/L>2\varepsilon n$.

To begin with, we show that the vertices in $X_a\setminus A$ are well distributed. Given $a\in\{\boxminus,\boxplus\}$, we partition the path components of F according to their number of vertices in $X_a\setminus A$. Suppose there are u_j paths with j vertices in $X_a\setminus A$ for each $0\le j\le L$ (and note no path has more than L vertices). Then in expectation we choose $\frac{u_j\varepsilon n}{t^*}$ paths with j vertices in $X_a\setminus A$ to F^*, and so by Fact 23 the probability we do not pick $\frac{u_j\varepsilon n}{t^*}\pm n^{0.98}$ such paths is at most $\exp\left(-n^{0.9}\right)$. Assuming none of these bad events occur, we conclude

$$|X_a\cap V(F^*)\setminus A|=\sum_{j=0}^{L}j\cdot\left(\tfrac{u_j\varepsilon n}{t^*}\pm n^{0.98}\right)=\tfrac{|X_a\cap V(F)\setminus A|\varepsilon n}{t^*}\pm L^2n^{0.98}\,.$$

So with probability at least $1-\exp\left(-n^{0.8}\right)$ the above estimate holds for each $a\in\{\boxminus,\boxplus\}$, which implies $\frac{|X_a\cap V(F^*)\setminus A|}{|V(F^*)\setminus A|}=(1\pm\varepsilon)\frac{|X_a\setminus A|}{|V(F)\setminus A|}$ for each $a\in\{\boxminus,\boxplus\}$.

We now show F^* is likely to have the γ-anchor distribution property. To that end, fix a vertex $v\in V(H)\setminus U$. For any $a,b,c\in\{\boxminus,\boxplus\}$, consider the set $A_{a,b,c}$. Each path of F may have zero, one, or two endpoints $x \in A_{a,b,c}$ such that $\phi_0(x) \in \mathsf{N}_H(v)$, and we partition the paths of F accordingly into sets of sizes u_0, u_1, u_2 respectively. By the given γ-anchor distribution property we have $\sum_{j=0}^{2} ju_j = (1 \pm \gamma)d_{ab}|A_{a,b,c}| \pm \frac{1}{2}\gamma n^{0.99}$. Moreover, as before, the expected number of paths in the jth set we select to F^* is $\frac{u_j \varepsilon n}{t^*}$, and by Fact 23 the probability of failing to select $\frac{u_j \varepsilon n}{t^*} \pm n^{0.98}$ of these paths is at most $\exp\big(-n^{0.8}\big)$. Assuming none of these bad events occur, we see that the total number of endpoints $x \in A_{a,b,c}$ of paths of F^* such that $\phi_0(x) \in \mathsf{N}_H(v)$ is

$$\sum_{j=0}^{2} j \cdot \Big(\tfrac{u_j \varepsilon n}{t^*} \pm n^{0.98}\Big) = \tfrac{(1\pm\gamma)d_{ab}|A_{a,b,c}|\varepsilon n \pm \frac{1}{2}\gamma\varepsilon n \cdot n^{0.99}}{t^*} \pm 3n^{0.98}.$$

By a similar argument, we have

$$\big|A_{a,b,c} \cap V(F^*)\big| = \tfrac{|A_{a,b,c}|\varepsilon n}{t^*} \pm n^{0.98}$$

with probability at least $1 - \exp\big(-n^{0.8}\big)$. It follows that the number of endpoints $x \in A_{a,b,c}$ of paths of F^* such that $\phi_0(x) \in \mathsf{N}_H(v)$ is

$$\big(1 \pm \gamma\big)d_{a,b}\big|A_{a,b,c} \cap V(F^*)\big| \pm \tfrac{1}{4}\gamma n^{0.99} \pm 5n^{0.98}$$

with probability at least $1 - \exp\big(-n^{0.7}\big)$, where we use that $t^* > 2\varepsilon n$. Taking the union bound over choices of a, b, c and v we see the probability that this part of the γ-anchor distribution property holds for F^* is at least $1 - \exp\big(-n^{0.6}\big)$. □

From this point on we will assume that the outcome of the uniform random choice of $P_1, \dots, P_{t^*}$ is fixed and the likely statements of the above claim hold. In particular, this means that (unless Algorithm 1 fails before time t) the number $|V_a \setminus U_{t-1}|$ is fixed for each $a \in \{\boxminus, \boxplus\}$ (although the sets themselves are not). We next argue that vertices are used with a predictable probability in each path.

Claim 40.2. *Given any $t \in [t^*]$ and $0 < \varrho < 2^{-4L}$, let $\mathscr{H}_{t-1}$ denote the event that the embedding of the first $t-1$ paths succeeds and furthermore (H, U_{t-1}) is $(\varrho, 2)$-block-diet. Given any $y \in V(P_t) \setminus A$, let x and z be the neighbours of y on P_t; if exactly one of these is in A suppose $x \in A$. Let $a, b, c \in \{\boxminus, \boxplus\}$ be such that $\xi(x) = V_a$, $\xi(y) = V_b$ and $\xi(z) = V_c$. Given any $v \in V_b \setminus U_{t-1}$, we have*

$$\mathbf{P}\big[\psi_t(y) = v \big| \mathscr{H}_{t-1}\big] = \begin{cases} (1 \pm 5L\varrho)\frac{1}{|V_b \setminus U_{t-1}|} & \text{if } x, z \notin A \\ (1 \pm 5L\varrho)\frac{1}{d_{ab}|V_b \setminus U_{t-1}|} & \text{if } x \in A,\ z \notin A,\ v \in \mathsf{N}_H\big(\phi_0(x)\big) \end{cases}.$$

In all other cases, the probability of embedding y to v is zero.

Proof. Observe that since $0 < \varrho \leq 2^{-4L}$ we have $(1 \pm 2\varrho)^L (1 \pm 2\varrho)^{-L} = 1 \pm 5L\varrho$.

The probability zero statement of the claim is trivial, since y must be embedded to $V_b \setminus U_{t-1}$ and since it must be embedded to a neighbour of the images of x and z.

Given any three vertices p, q, r of $V(H)$ and any $3 \leq k \leq \ell - 2$, we can count the number of ℓ-vertex paths from p to r in $H - U_{t-1}$ whose kth vertex is q and whose ith vertex (for each

$1 \le i \le \ell$) is in $V_i \in \{V_{\boxminus}, V_{\boxplus}\}$, for any given choice of the V_i. We do this as follows. Mark p, q, r used. We count the number of vertices of $V_2 \setminus U_{t-1}$ adjacent to p and not so far used, for each of these the number of vertices of $V_3 \setminus U_{t-1}$ adjacent to the first and not so far used, and so on up to the $(k-1)$st vertex which we require to be adjacent to both the $(k-2)$nd and to q; we repeat this (avoiding previously used vertices) for the path from q to r. Denote with d_{ij} the bipartite density $d[V_i, V_j]$ if $V_i \neq V_j$, or the density $d[V_i]$ if $V_i = V_j$. Then, by the block-diet condition we have $(1 \pm \varrho)d_{12}|V_2 \setminus U_{t-1}| \pm \ell$ choices for the second vertex, for each of these $(1 \pm \varrho)d_{23}|V_3 \setminus U_{t-1}| \pm \ell$ choices for the third vertex, and so on until we get to $\big(1 \pm \varrho\big)d_{(k-2)(k-1)}d_{(k-1)k}|V_{k-1} \setminus U_{t-1}| \pm \ell$ choices for the $(k-1)$st vertex, and similarly for the path from q to r. Here the $\pm\ell$ comes from the vertices previously used on the path we construct. Since n is sufficiently large, and since $V_a \setminus U_t$ is linear in n for each $a \in \{\boxminus, \boxplus\}$ while $\ell \le L$, we can absorb the $\pm\ell$ into the multiplicative error to obtain $(1 \pm 2\varrho)d_{12}|V_2 \setminus U_{t-1}|$ choices for the second vertex, and so on. Putting this together, the total number of such paths is

$$(1 \pm 2\varrho)^{\ell-2} \prod_{i=1}^{\ell} d_{i(i+1)} \prod_{i=2}^{k-1} |V_i \setminus U_{t-1}| \prod_{i=k+1}^{\ell-1} |V_i \setminus U_{t-1}| \,.$$

If $x, z \notin A$, then using the above with p and r being the embeddings of the ends of P_t, letting q range over all vertices of $\xi(y) \setminus U_{t-1}$, and letting $\ell = v(P_t)$ with y being the kth vertex of P_t and assignments as specified in the lemma statement, we obtain the total number of paths from p to r from which Algorithm 1 chooses uniformly at random. We can thus compute the probability of $\phi_t(y) = v$. Most of the terms cancel, leaving

$$\mathbf{P}\big[\psi_t(y) = v \big| \mathscr{H}_{t-1}\big] = (1 \pm 2\varrho)^{\ell-2}(1 \pm 2\varrho)^{2-\ell}\tfrac{1}{|V_b \setminus U_{t-1}|}\,,$$

which by choice of ϱ is as desired.

Suppose $x \in A$ and $y \notin A$, and that $v \in \mathsf{N}_H\big(\phi_0(x)\big)$. By a similar argument, for each vertex $u \in \mathsf{N}_H\big(\phi_0(x)\big) \cap V_b$ we can estimate the number of correctly assigned paths in $H - U_{t-1}$ between the embedded anchors of P_t. Summing over the $(1 \pm \varrho)d_{ab}|V_b \setminus U_{t-1}| \pm 3$ choices of u given by the block-diet condition, we find the total number of such paths and hence

$$\mathbf{P}\big[\psi_t(y) = v \big| \mathscr{H}_{t-1}\big] = (1 \pm 2\varrho)^{\ell-1}(1 \pm 2\varrho)^{1-\ell}\tfrac{1}{d_{ab}|V_b \setminus U_{t-1}|}\,,$$

which again is as desired. □

In light of this claim, given $v \in V(H) \setminus U_0$ and $y \in V(P_t) \setminus A$, it is natural to define a quantity $p_{v,y}$ which approximates the probability that y is embedded to v, conditioned on the first $t-1$ paths being embedded in such a way that v is not used and the block-diet condition is maintained. To that end, suppose $\xi(y) = V_b$ and the neighbours of y on P_t are x and z; if exactly one of these is in A, suppose it is x. Suppose $\xi(x) = V_a$ and $\xi(z) = V_c$. We define

$$p_{v,y} := \begin{cases} \frac{1}{|V_b \setminus U_{t-1}|} & \text{if } x, z \notin A \\ \frac{1}{d_{ab}|V_b \setminus U_{t-1}|} & \text{if } x \in A,\, z \notin A,\, v \in \mathsf{N}_H\big(\phi_0(x)\big) \end{cases} . \tag{21}$$

We also define

$$p^*_{v,t} := \sum_{t'=t}^{t+\varepsilon n-1} \sum_{\substack{y\in V(P_{t'})\setminus A \\ \xi(y)=V_b}} p_{v,y}\,,$$

which (as we will see) is roughly the probability that v is used in embedding one of the first εn paths from P_t onwards, given it was not used earlier. For now, we will use the anchor distribution property established in Claim 40.1 to evaluate $p^*_{v,t}$.

Claim 40.3. *Given $1 \le t \le t^* - \varepsilon n + 1$, $b \in \{\boxminus, \boxplus\}$ and $v \in V_b \setminus U_{t-1}$, let $X^{\mathrm{int}}_{b,t}$ denote the set of vertices y which are internal to $P_{t'}$ for some $t \le t' < t + \varepsilon n$ and furthermore satisfy $\xi(y) = V_b$. Then we have*

$$p^*_{v,t} = (1 \pm 2\gamma)\tfrac{|X^{int}_{b,t}|}{|V_b\setminus U_{t-1}|}\,.$$

Proof. We split $X^{\mathrm{int}}_{b,t}$ into parts corresponding to the cases of the definition of $p_{v,y}$, including the choice of a in the line where this appears. Let X^{zero} be the subset of $y \in X^{\mathrm{int}}_{b,t}$ whose neighbours are not in A. For each $a \in \{\boxminus, \boxplus\}$ let $A_{a,b}$ be the set of $y \in X^{\mathrm{int}}_{b,t}$ whose neighbours are x, z with $x \in A$ and $z \notin A$ and where $x \in V_a$. Note that these sets partition $X^{\mathrm{int}}_{b,t}$.

The contribution to $p^*_{v,t}$ from vertices of X^{zero} is $\frac{|X^{\mathrm{zero}}|}{|V_b\setminus U_{t-1}|}$. Now fix $a \in \{\boxminus, \boxplus\}$. The contribution to $p^*_{v,t}$ from vertices of $A_{a,b}$ is, by the γ-anchor distribution property from Claim 40.1,

$$\frac{(1\pm\gamma)d_{ab}|A_{a,b}| \pm \gamma n^{0.99}}{d_{ab}|V_b \setminus U_{t-1}|} = (1\pm\gamma)\tfrac{|A_{a,b}|}{|V_b\setminus U_{t-1}|} \pm n^{-0.001}\,,$$

where the equality uses $d_{ab} \ge \nu$ and $|V_b \setminus U_{t-1}| \ge |V_b \setminus U| - |\{x \in V(F) \setminus A : \xi(x) = V_b\}| \ge \nu n$, the last inequality being an assumption of the lemma. Summing up we obtain

$$p^*_{v,t} = \tfrac{|X^{\mathrm{zero}}|}{|V_b\setminus U_{t-1}|} + (1\pm\gamma)\tfrac{|A_{\boxminus,b}|}{|V_b\setminus U_{t-1}|} + (1\pm\gamma)\tfrac{|A_{\boxplus,b}|}{|V_b\setminus U_{t-1}|} \pm 2n^{0.001} = (1\pm\gamma)\tfrac{|X^{\mathrm{int}}_{b,t}|}{|V_b\setminus U_{t-1}|} \pm 2n^{-0.001}\,.$$

By the 'furthermore' part of Claim 40.1, and since each path of F has at least 5 vertices, we have

$$|X^{\mathrm{int}}_{b,t}| \ge (1-\varepsilon)\tfrac{|X_a\setminus A|}{|V(F)\setminus A|}\cdot 3\varepsilon n \ge \tfrac{\nu n}{2n}\cdot 3\varepsilon n \ge \varepsilon\nu n\,,$$

and this justifies absorbing the $2n^{-0.001}$ additive error to the relative error as claimed. □

We are now in a position to estimate the number of vertices embedded to a given subset X of $V_a \setminus U_{t-1}$ when $P_t, \dots, P_{t+\varepsilon n-1}$ are embedded by Algorithm 1, where $a \in \{\boxminus, \boxplus\}$.

Claim 40.4. *Given any $1 \le t \le t^* - \varepsilon n + 1$, let $\mathscr{H}_{t-1}$ denote the event that embedding of the first $t-1$ paths succeeds and furthermore (H, U_{t-1}) is $(\beta_{t-1}, 2)$-block-diet. Given any $b \in \{\boxminus, \boxplus\}$ and $X \subseteq V_b \setminus U_{t-1}$ such that $|X| \ge \nu^4 n$, when Algorithm 1 embeds paths $P_t, \dots, P_{t+\varepsilon n-1}$, we have*

$$\mathbf{P}\left[\big|X \cap U_{t+\varepsilon n-1}\big| = (1 \pm 50L\beta_{t-1})|X|\tfrac{|V_b\cap(U_{t+\varepsilon n-1}\setminus U_{t-1})|}{|V_b\setminus U_{t-1}|}\,\middle|\,\mathscr{H}_{t-1}\right] \ge 1 - \exp\big(-n^{0.5}\big)\,.$$

Proof. Suppose that $\mathscr{H}_{t-1}$ occurs, and consider the running of Algorithm 1 to embed P_t onwards. Fix $b \in \{\boxminus, \boxplus\}$ and $X \subseteq V_b \setminus U_{t-1}$.

Observe that in embedding $P_t,\dots,P_{t+\varepsilon n-1}$ we embed in total at most $L\varepsilon n$ vertices. Thus the estimates provided by the $(\beta_{t-1},2)$-block-diet condition for (H,U_{t-1}) are changed by at most $L\varepsilon n$ for any $t-1\le t'<t+\varepsilon n$. Since $d_{pq}\ge\nu$ and $|V_p\setminus U_{t-1}|\ge\nu n$ for each $p,q\in\{\boxminus,\boxplus\}$, and by choice of ε, we see that for any $t-1\le t'<t+\varepsilon n$, the pair $(H,U_{t'-1})$ satisfies the $(2\beta_{t-1},2)$-block-diet condition. This in particular implies that Algorithm 1 cannot fail at any time in this interval, and that the conclusions of Claim 40.2 are valid with $\varrho=2\beta_{t-1}$ for each of these paths.

Given $v\in X$ and $t\le t'\le t+\varepsilon n-1$, the probability that a vertex of $P_{t'}$ is embedded to v, conditioned on the embedding of $P_t,\dots,P_{t'-1}$, is either zero (if a vertex of $P_t,\dots,P_{t'-1}$ has been embedded to v) or given by summing, over all internal vertices y of $P_{t'}$ such that $\xi(y)=V_b$, the probability $\big(1\pm 10L\beta_{t-1}\big)p_{v,y}$ given by Claim 40.2. The latter statement holds because the events $\psi_{t'}(y)=v$ for different y in $P_{t'}$ are disjoint. We wish to estimate

$$Z=\sum_{t'=t}^{t+\varepsilon n-1}\mathbf{E}\Big[\big|\psi_{t'}\big(V(P_{t'})\big)\cap X\big|\,\Big|\,\text{embeddings of } P_t,\dots,P_{t'-1}\Big]\,,$$

and it follows that an upper bound for this sum is $(1\pm 10L\beta_{t-1})\sum_{v\in X}p^*_{v,t}$. Furthermore, the difference between this upper bound and the correct value of the (random variable) Z is that vertices v to which we embed at some time cease to contribute to the sum; there are at most $L\varepsilon n$ such vertices, and so we obtain

$$Z=(1\pm 10L\beta_{t-1})\sum_{v\in X}p^*_{v,t}\;\pm\;L\varepsilon n\cdot 2\tfrac{|X^{\mathrm{int}}_{b,t}|}{|V_b\setminus U_{t-1}|}=(1\pm 20L\beta_{t-1})\tfrac{|X|\cdot|X^{\mathrm{int}}_{b,t}|}{|V_b\setminus U_{t-1}|}\,,$$

where we use Claim 40.3 to estimate $p^*_{v,t}$.

Now consider the process of embedding $P_t,\dots,P_{t+\varepsilon n-1}$ following Algorithm 1. By Corollary 25(*b*), since each path has at most L vertices, we obtain

$$\mathbf{P}\Big[\big|X\cap U_{t+\varepsilon n-1}\big|\neq(1\pm 40L\beta_{t-1})\tfrac{|X|\cdot|X^{\mathrm{int}}_{b,t}|}{|V_b\setminus U_{t-1}|}\Big|\mathscr{H}_{t-1}\Big]\le\exp(-n^{0.5})\,.$$

Since $\big|X^{\mathrm{int}}_{b,t}\big|=\big|V_b\cap(U_{t+\varepsilon n-1}\setminus U_{t-1})\big|$, this completes the proof. □

Using the above claim, we can show that, given any $1<t'\le t^*$, any $b\in\{\boxminus,\boxplus\}$ and any large enough $X\subseteq V_b\setminus U_0$, with very high probability the following holds when Algorithm 1 is run. Either the $(\beta_{t-1},2)$-block-diet condition fails for some (H,U_{t-1}) with $1\le t<t'\le t^*$, or $|X\cap U_{t'}|$ is roughly $|X|\frac{|V_b\cap(U_{t'}\setminus U_0)|}{|V_b\setminus U_0|}$.

Claim 40.5. *Given any $t\ge 1$, any $b\in\{\boxminus,\boxplus\}$ and any $X\subseteq V_b\setminus U_0$ such that $|X|\ge\nu^2 n$, when Algorithm 1 embeds paths $P_1,\dots,P_t$, we have*

$$\mathbf{P}\Big[\big|X\setminus U_t\big|=\big(1\pm\tfrac34\beta_t\big)|X|\tfrac{|V_b\setminus U_t|}{|V_b\setminus U_0|}\ \text{or}\ \exists t'\in[t-1]\colon (H,U_{t'})\ \text{is not}\ (\beta_{t'},2)\text{-block-diet}\Big]\ge 1-\exp\big(-n^{0.4}\big)\,.$$

Proof. The proof of this claim is simply an iteration of Claim 40.4, similar to the proof of [2, Lemma 26]. For this, observe that $\big|X \setminus U_{t'}\big| = (1 \pm \beta_{t'})|X|\frac{|V_b \setminus U_{t'}|}{|V_b \setminus U_0|}$ implies $|X \setminus U_{t'}| \geq \nu^4 n$, since $|X| \geq \nu^2 n$ and $|V_b \setminus U_{t'}| \geq \nu n$, and hence we are allowed to apply Claim 40.4 iteratively.

Now, given $t \geq 1$, $b \in \{\boxminus, \boxplus\}$ and $X \subseteq V_b \setminus U_0$, we suppose that the likely event of Claim 40.4 holds for each time t' with $1 \leq t' \leq t - \varepsilon n + 1$, with the input set $X \setminus U_{t'-1}$ (provided this set has size at least $\nu^4 n$). The probability that this likely event does not occur, but $(H, U_{t'})$ is $(\beta_{t'}, 2)$-block-diet for each such t', is by the union bound at most $n \exp\big(-n^{0.5}\big) < \exp\big(-n^{0.4}\big)$. We claim that outside of the unlikely event the claim statement holds.

Let $s = \lfloor \frac{t}{\varepsilon n} \rfloor$. We require specifically the above likely event for each $t' = 1 + k\varepsilon n$, where k is an integer between 0 and $s-1$ inclusive; this yields

$$
\begin{aligned}
\big|X \setminus U_{s\varepsilon n}\big| &= \big|X\big| \cdot \prod_{k=0}^{s-1} \Big(1 - (1 \pm 50L\beta_{k\varepsilon n})\tfrac{|V_b \cap (U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n})|}{|V_b \setminus U_{k\varepsilon n}|}\Big) \\
&= \big|X\big| \cdot \prod_{k=0}^{s-1} \Big(\tfrac{|V_b \setminus U_{(k+1)\varepsilon n}|}{|V_b \setminus U_{k\varepsilon n}|} \pm 50L\beta_{k\varepsilon n}\tfrac{|V_b \cap (U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n})|}{|V_b \setminus U_{k\varepsilon n}|}\Big) \\
&= \big|X\big| \cdot \prod_{k=0}^{s-1} \tfrac{|V_b \setminus U_{(k+1)\varepsilon n}|}{|V_b \setminus U_{k\varepsilon n}|} \Big(1 \pm 50L\beta_{k\varepsilon n}\tfrac{|V_b \cap (U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n})|}{|V_b \setminus U_{(k+1)\varepsilon n}|}\Big) \\
&= \big|X\big| \cdot \tfrac{|V_b \setminus U_{s\varepsilon n}|}{|V_b \setminus U_0|} \cdot \prod_{k=0}^{s-1} \Big(1 \pm 50L\beta_{k\varepsilon n}\tfrac{|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}|}{|V_b \setminus U_{(k+1)\varepsilon n}|}\Big) .
\end{aligned}
$$

Taking logs, we obtain

$$
\begin{aligned}
\log\big|X \setminus U_{s\varepsilon n}\big| &= \log\big|X\big| + \log\tfrac{|V_b \setminus U_{s\varepsilon n}|}{|V_b \setminus U_0|} + \sum_{k=0}^{s-1} \log\big(1 \pm \tfrac{50L\beta_{k\varepsilon n}|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}|}{|V_b \setminus U_{(k+1)\varepsilon n}|}\big) \\
&= \log\big|X\big| + \log\tfrac{|V_b \setminus U_{s\varepsilon n}|}{|V_b \setminus U_0|} \pm \sum_{k=0}^{s-1} \tfrac{100L\beta_{k\varepsilon n}|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}|}{|V_b \setminus U_{(k+1)\varepsilon n}|}
\end{aligned}
$$

where the last line uses the approximation to the logarithm $\log(1 \pm x) = \pm 2x$, valid for all sufficiently small x. In this case the term is sufficiently small because $\big|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}\big| \leq L\varepsilon n$, because ε is sufficiently small, and because $|V_b \setminus U_t| \geq \nu n$. Now we have

$$
\begin{aligned}
\sum_{k=0}^{s-1} \tfrac{100L\beta_{k\varepsilon n}|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}|}{|V_b \setminus U_{(k+1)\varepsilon n}|} &\leq \frac{100L}{\nu n} \sum_{k=0}^{s-1} \beta_{k\varepsilon n}|U_{(k+1)\varepsilon n} \setminus U_{k\varepsilon n}| \leq \frac{100L}{\nu n} \sum_{k=0}^{s-1} \beta_{k\varepsilon n} L\varepsilon n \\
&\leq \frac{100L^2}{\nu n} \int_{-\infty}^{s\varepsilon n} \beta_x \mathrm{d}x \leq \tfrac{\nu^2 t^*}{10n}\beta_{s\varepsilon n} \leq 0.1\nu^2\beta_{s\varepsilon n} ,
\end{aligned}
$$

where the third inequality uses the fact that β_x is increasing in x and nonnegative. We obtain

$$
\log\big|X \setminus U_{s\varepsilon n}\big| = \log\big|X\big| + \log\tfrac{|V_b \setminus U_{s\varepsilon n}|}{|V_b \setminus U_0|} \pm \tfrac{1}{4}\beta_{s\varepsilon n} = \log\big|X\big| + \log\tfrac{|V_b \setminus U_{s\varepsilon n}|}{|V_b \setminus U_0|} + \log\big(1 \pm \tfrac{1}{2}\beta_{s\varepsilon n}\big) ,
$$

hence $\big|X \setminus U_{s\varepsilon n}\big| = (1 \pm \frac{1}{2}\beta_{s\varepsilon n})|X|\frac{|V_b\setminus U_{s\varepsilon n}|}{|V_b\setminus U_0|}$. Since $s\varepsilon n \le t \le (s+1)\varepsilon n$ and since β_x is increasing in x, we have

$$\big|X \setminus U_t\big| = (1 \pm \tfrac{1}{2}\beta_t)|X|\tfrac{|V_b\setminus U_t|\pm L\varepsilon n}{|V_b\setminus U_0|} \pm L\varepsilon n\,,$$

which by choice of ε gives the desired conclusion. □

If at some time t Algorithm 1 gives us a U_t such that the $(\beta_t, 2)$-block-diet condition fails for (H, U_t), then there is a first such time t. Suppose that $b \in \{\boxminus, \boxplus\}$ and S (which is a set of either one or two vertices) is a witness of the failure. Because (H, U_0) is $(\gamma, 2)$-block-diet, we have

$$\big|\mathsf{N}_H(S) \setminus U_0\big| = (1 \pm \gamma)d^{i_1}_{\boxminus b}d^{i_2}_{\boxplus b}|V_b \setminus U_0|$$

where i_1 and i_2 are the number of vertices of S in $\boxminus$ and $\boxplus$ respectively. Now by Claim 40.5, with probability at least $1 - \exp(n^{-0.4})$ we have

$$\big|\mathsf{N}_H(S) \setminus U_t\big| = \big(1 \pm \tfrac{3}{4}\beta_t\big)(1 \pm \gamma)d^{i_1}_{\boxminus b}d^{i_2}_{\boxplus b}|V_b \setminus U_t|$$

which by choice of γ is a contradiction to b and S witnessing failure of $(\beta_t, 2)$-block-diet.

Taking the union bound over $1 \le t \le t^*$ and witnesses to failure of block-diet (i.e. singletons or pairs of vertices and $b \in \{\boxplus, \boxminus\}$), the probability that such a first time t exists is by Claim 40.5 at most $n^4 \exp\big(-n^{0.4}\big)$. In other words, with probability at least $1-\exp\big(-n^{0.3}\big)$, for each $1 \le t \le t^*$ the pair (H, U_t) is $(\beta_t, 2)$-block-diet (and so Algorithm 1 does not fail). This establishes $(PE1)$ and by Claim 40.5 also $(PE2)$.

Next, we find the probability that any one of a given small collection of vertices is embedded to in some interval of time by Algorithm 1. To begin with, we deal with intervals of length exactly εn.

Claim 40.6. *Given any $1 \le t \le t^* - \varepsilon n$, let $\mathscr{H}_{t-1}$ be a history of Algorithm 1 up to and including the embedding of P_{t-1}. Suppose $\mathscr{H}_{t-1}$ is such that the pair (H, U_{t-1}) is $(\beta_{t-1}, 2)$-block-diet. Let for each $b \in \{\boxminus, \boxplus\}$ the set $S_b \subseteq V_b \setminus U_{t-1}$ contain at most L elements. Then we have*

$$\mathbf{P}\big[(S_\boxminus \cup S_\boxplus) \cap U_{t+\varepsilon n-1} \neq \emptyset \big| \mathscr{H}_{t-1}\big] = \big(1 \pm 30L\beta_{t-1}\big) \sum_{b\in\{\boxminus,\boxplus\}} |S_b|\tfrac{|V_b\cap(U_{t+\varepsilon n-1}\setminus U_{t-1})|}{|V_b\setminus U_{t-1}|}\,.$$

Proof. Fix a history $\mathscr{H}_{t-1}$ as in the claim statement; throughout we condition on $\mathscr{H}_{t-1}$. Let $S = S_\boxminus \cup S_\boxplus$. We write

$$\begin{aligned}\mathbf{P}\big[|S \cap U_{t+\varepsilon n-1}| \ge 1\big] &= \sum_{t'=t}^{t+\varepsilon n-1} \mathbf{P}\big[|S \cap \phi\big(V(P_{t'})\big)| \ge 1 \text{ and } S \cap \operatorname{im}\psi_{t'-1} = \emptyset\big]\\ &= \sum_{t'=t}^{t+\varepsilon n-1} \sum_{\psi_{t'-1}} \mathbf{P}\big[|S \cap \phi\big(V(P_{t'})\big)| \ge 1 \big| \psi_{t'-1}\big] \cdot \mathbf{P}[\psi_{t'-1}]\,,\end{aligned}$$

where the sum over $\psi_{t'-1}$ runs only over embeddings where $S \cap \operatorname{im}\psi_{t'-1} = \emptyset$. Note that since by choice of $\mathscr{H}_{t-1}$ the pair (H, U_{t-1}) is $(\beta_{t-1}, 2)$-block-diet, it follows as before that $(H, U_{t'-1})$ is $(2\beta_{t-1}, 2)$-block-diet for each t' we consider, and in particular the embeddings we assume exist indeed do. Observe that the expected number of vertices of P' embedded to S is

$$\sum_{j=1}^{2L}\mathbf{P}\big[|S\cap\phi\big(V(P_{t'})\big)|\geq j\,\big|\psi_{t'-1}\big]\,,$$

and rearranging this we get

$$\mathbf{P}\big[|S\cap\phi\big(V(P_{t'})\big)|\geq 1\big|\psi_{t'-1}\big]=\mathbf{E}\big[\big|S\cap\phi\big(V(P_{t'})\big)\big|\,\big|\psi_{t'-1}\big]\pm|S|\mathbf{P}\big[|S\cap\phi\big(V(P_{t'})\big)|\geq 2\big|\psi_{t'-1}\big]\,.$$

Now by Claim 40.2, and since $|S|\leq 2L$, we have

$$\begin{aligned}&\mathbf{P}\big[|S\cap\phi\big(V(P_{t'})\big)|\geq 1\big|\psi_{t'-1}\big]\\&\qquad=(1\pm 10L\beta_{t-1})\Bigg(\sum_{b\in\{\boxminus,\boxplus\}}\ \sum_{u\in S_b}\ \sum_{\substack{y\in V(P_{t'})\setminus A\\ \xi(y)=V_b}}p_{u,y}\Bigg)\pm 2L\mathbf{P}\Big[\big|S\cap\phi\big(V(P_{t'})\big)\big|\geq 2\Big|\psi_{t'-1}\Big]\,.\end{aligned}$$

We can bound above $2L\mathbf{P}\big[\big|S\cap\phi\big(V(P_{t'})\big)\big|\geq 2\big|\psi_{t'-1}\big]$ by $20L^5\nu^{-L}n^{-2}<n^{-1.5}$, by similar logic as in Claim 40.2. Since the $p_{u,y}$ do not depend on $\psi_{t'-1}$, we get

$$\mathbf{P}\big[|S\cap U_{t+\varepsilon n-1}|\geq 1\big]=(1\pm 10L\beta_{t-1})\Big(\sum_{u\in S}p^*_{u,t}\pm 2n^{-0.5}\Big)\mathbf{P}[S\cap\operatorname{im}\psi_{t'-1}=\emptyset]\,.$$

This expression is trivially bounded above by $(1+10L\beta_{t-1})\big(\sum_{u\in S}p^*_{u,t}+2n^{-0.5}\big)<8L^2\varepsilon\nu^{-1}$, so $\mathbf{P}[S\cap\operatorname{im}\psi_{t'-1}=\emptyset]=1\pm 8L^2\varepsilon\nu^{-1}$ and we get

$$\begin{aligned}\mathbf{P}\big[|S\cap U_{t+\varepsilon n-1}|\geq 1\big]&=(1\pm 10L\beta_{t-1})\Big(\sum_{u\in S}p^*_{u,t}\pm 2n^{-0.5}\Big)\cdot\big(1\pm 8L^2\varepsilon\nu^{-1}\big)\\&=(1\pm 20L\beta_{t-1})\sum_{u\in S}p^*_{u,t}=(1\pm 30L\beta_{t-1})\sum_{b\in\{\boxminus,\boxplus\}}|S_b|\tfrac{|X^{\rm int}_{b,t}|}{|V_b\setminus U_{t-1}|}\,,\end{aligned}$$

where the last equality uses Claim 40.3. Since $|X^{\rm int}_{b,t}|=\big|V_b\cap(U_{t+\varepsilon n-1}\setminus U_{t-1})\big|$, we get the claimed probability. □

Using this repeatedly we can deduce similar estimates for general intervals.

Claim 40.7. *Given any $1\leq t<t'\leq t^*$, let $\mathscr{H}_{t-1}$ be a history of Algorithm 1 up to and including embedding P_{t-1}. Suppose $\mathscr{H}_{t-1}$ is such that the pair (H,U_{t-1}) is $(\beta_{t-1},2)$-block-diet. Let for each $b\in\{\boxminus,\boxplus\}$ the set $S_b\subseteq V_b\setminus U_{t-1}$ contain at most L elements. Then we have*

$$\begin{aligned}&\mathbf{P}\big[(S_\boxminus\cup S_\boxplus)\cap U_{t'-1}=\emptyset\big|\mathscr{H}_{t-1}\big]\\&\qquad=\big(1\pm 2\beta_{t'-1}\big)\Bigg(\prod_{a\in\{\boxminus,\boxplus\}}\Big(\tfrac{|V_a\setminus U_{t'-1}|}{|V_a\setminus U_{t-1}|}\Big)^{|S_a|}\Bigg)\pm\varepsilon^{-1}\mathbf{P}[\textit{block-diet fails}|\mathscr{H}_{t-1}]\,,\end{aligned}$$

where the last probability refers to the event that, at some time t'' between t and t^*, the pair $(H,U_{t''-1})$ is not $(\beta_{t''-1},2)$-block-diet.

Proof. We condition throughout on $\mathscr{H}_{t-1}$ as given in the claim statement. We allow an error term of size $\varepsilon^{-1}\mathbf{P}[\text{block-diet fails}|\mathscr{H}_{t-1}]$, so we can throughout assume that block-diet does not fail (for the purposes of applying Claim 40.6 at most ε^{-1} times) since the errors caused by its failure are subsumed in this error term. Because ε is small, it is enough to prove the desired statements with relative error $1 \pm 99L\beta_{t'-1}$ under the additional assumption that $t'-t$ is an integer multiple of εn (the change in $\beta_{t'-1}$ and the change in the main term caused by increasing t' by a further εn is covered by the increase in the relative error). So we may assume $t' = t + k\varepsilon n$ for some integer $k \geq 1$. We write $S = S_{\boxminus} \cup S_{\boxplus}$. By Claim 40.6, we have

$$\begin{aligned}
\mathbf{P}\big[S \cap U_{t'-1} = \emptyset\big] &= \prod_{\ell=0}^{k-1} \left(1 - \big(1 \pm 30L\beta_{t+\ell\varepsilon n-1}\big) \sum_{a\in\{\boxminus,\boxplus\}} |S_a| \tfrac{|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}|}{|V_a \setminus U_{t+\ell\varepsilon n-1}|}\right) \\
&= \prod_{\ell=0}^{k-1} (1 \pm 65L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}) \left(1 - \sum_{a\in\{\boxminus,\boxplus\}} |S_a| \tfrac{|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}|}{|V_a \setminus U_{t+\ell\varepsilon n-1}|}\right) \\
&= \prod_{\ell=0}^{k-1} (1 \pm 70L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}) \prod_{a\in\{\boxminus,\boxplus\}} \left(1 - \tfrac{|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}|}{|V_a \setminus U_{t+\ell\varepsilon n-1}|}\right)^{|S_a|} \\
&= \prod_{\ell=0}^{k-1} \big(1 \pm 70L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}\big) \prod_{a\in\{\boxminus,\boxplus\}} \left(\tfrac{|V_a \setminus U_{t+(\ell+1)\varepsilon n-1}|}{|V_a \setminus U_{t+\ell\varepsilon n-1}|}\right)^{|S_a|} \\
&= \left(\prod_{\ell=0}^{k-1} \big(1 \pm 70L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}\big)\right) \prod_{a\in\{\boxminus,\boxplus\}} \left(\tfrac{|V_a \setminus U_{t+k\varepsilon n-1}|}{|V_a \setminus U_{t-1}|}\right)^{|S_a|}.
\end{aligned}$$

To see that the second line is valid, we use $|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}| \leq \varepsilon n$ and $|V_a \setminus U_{t+\ell\varepsilon n-1}| \geq \nu n$. For the third line, observe that expanding out all the brackets in the product on the third line, we get the brackets of the second line plus a collection of at most 2^{2L} products in which $\frac{|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}|}{|V_a \setminus U_{t+\ell\varepsilon n-1}|}$ terms appear at least twice. Since $|X^{\mathrm{int}}_{a,t+\ell\varepsilon n}| \leq \varepsilon n$, by choice of ε all of these terms are tiny compared to β_0 and hence are accounted for by the increase in the error term. The fourth and fifth lines are straightforward equalities.

It remains to show that the product of error terms is sufficiently close to 1. Taking logs and using the Taylor series approximation (valid since ε is sufficiently small), we have

$$\log \prod_{\ell=0}^{k-1} \big(1 \pm 70L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}\big) = \pm \sum_{\ell=0}^{k-1} 80L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1} = \pm \int_{x=-\infty}^{t'} \tfrac{80L^2\nu^{-1}\beta_x}{n} \mathrm{d}x < \beta_{t'-1}$$

where the final equality uses the definition of β_x and the fact $t^* \leq n$. We obtain

$$\prod_{\ell=0}^{k-1} \big(1 \pm 70L^2\varepsilon\nu^{-1}\beta_{t+\ell\varepsilon n-1}\big) = \exp\big(\pm \beta_{t'-1}\big) = 1 \pm 2\beta_{t'-1}\,,$$

as desired. □

From this point we will no longer require the detailed definition of β_{t-1}; it will suffice that $\beta_{t-1} \leq \beta_{t^*} \leq \sqrt{C}\gamma$, and we will simply replace error bounds from the earlier claims with the latter. The remaining parts of Lemma 40 are relatively straightforward at this point.

As noted earlier, the probability of block-diet failing is at most $\exp\big(-n^{0.3}\big)$. Now, taking in particular the case $t=1$ and $t'=t^*$ of Claim 40.7, and observing that $\big|\big\{x \in V(F)\setminus A : \xi(x) = V_a\big\}\big| = |V_a \cap (U_{t^*-1}\setminus U_0)| \pm L$, we obtain $(PE3)$.

For $(PE4)$, fix $a,b \in \{\boxminus,\boxplus\}$ and let $e=uv$ with $u\in V_a$ and $v \in V_b$. It is convenient to give to each directed edge (y,z) of F a weight, which will relate to the probability that y is embedded to u and z to v. We do this as follows. If one of y and z is in A, we set $q_{y,z}=0$. Otherwise, suppose x,y,z,w are adjacent in F in that order, and suppose $\xi(x)=V_c$, $\xi(y)=V_a$, $\xi(z)=V_b$ and $\xi(w)=V_d$, where $c,d\in\{\boxminus,\boxplus\}$ and a,b are as above. If x is in A, and $\phi_0(x)u$ is an edge of H, we set $q_{y,z}=\frac{1}{d_{ac}}$. If w is in A, and $\phi_0(w)v$ is an edge of H, we set $q_{y,z}=\frac{1}{d_{bd}}$. Note that we cannot have both x and w in A, since all paths have at least four edges. If neither x nor w is in A, set $q_{y,z}=1$, and in all other cases set $q_{y,z}=0$.

Fix $c\in\{\boxminus,\boxplus\}$ and consider the sum $\sum_{y,z} q_{y,z}$ where the sum runs over pairs (y,z) such that $x\in A$ and $\xi(x)=V_c$. By the γ-anchor distribution property, there are

$$(1\pm\gamma)d_{ca}|A_{c,a,b}| \pm \tfrac{1}{2}\gamma n^{0.99}$$

summands equal to $\frac{1}{d_{ac}}$, and so this sum evaluates to

$$(1\pm\gamma)|A_{c,a,b}| \pm n^{0.999}\,.$$

Similarly, given $d\in\{\boxminus,\boxplus\}$, considering the sum over pairs (y,z) such that $w\in A$ has $\xi(w)=V_d$, we have

$$\sum_{y,z} q_{y,z} = (1\pm\gamma)|A_{d,b,a}| \pm n^{0.999}\,.$$

Finally, the set $\big\{(y,z)\,:\, yz\in E(F), y\in X_a\setminus A, z\in X_b\setminus A\big\}$ splits up into $|A_{c,a,b}|$ pairs (y,z) such that $x\in A$ and $\xi(x)=V_c$, $|A_{d,b,a}|$ pairs (y,z) such that $w\in A$ and $\xi(w)=V_d$, for each $c,d\in\{\boxminus,\boxplus\}$, and the remaining pairs (y,z) each of which has $q_{y,z}=1$. We conclude

$$\begin{aligned}\sum_{(y,z):yz\in E(F)} q_{y,z} &= (1\pm\gamma)\Big|\big\{(y,z)\,:\, yz\in E(F), y\in X_a\setminus A, z\in X_b\setminus A\big\}\Big| \pm 4n^{0.999} \\ &= (1\pm2\gamma)\Big|\big\{(y,z)\,:\, yz\in E(F), y\in X_a\setminus A, z\in X_b\setminus A\big\}\Big|\,,\end{aligned} \tag{22}$$

where the second equality uses the lemma assumption that there are at least νn pairs in the set on the right hand side of (22).

We observe that

$$\mathbf{P}[\phi \text{ uses } e] = \sum_{(y,z):yz\in E(F)} \mathbf{P}[\phi(y)=u,\phi(z)=v]$$

and letting t be such that $y,z\in V(P_t)$ we can write

$$\mathbf{P}[\phi(y)=u,\phi(z)=v] = \sum_{\psi_{t-1}} \mathbf{P}\big[\phi(y)=u,\phi(z)=v\big|\psi_{t-1}\big]\mathbf{P}[\psi_{t-1}]\,.$$

Observe that if u or v is in $\operatorname{im}\psi_{t-1}$, then $\mathbf{P}\big[\phi(y)=u,\phi(z)=v\big|\psi_{t-1}\big]=0$. Furthermore, the total probability of ψ_{t-1} such that (H,U_{t-1}) is not $(\beta_{t-1},2)$-block-diet is at most $\exp\big(-n^{0.3}\big)$; so we are mainly interested in estimating $\mathbf{P}\big[\phi(y)=u,\phi(z)=v\big|\psi_{t-1}\big]$ when ψ_{t-1} is $(\beta_{t-1},2)$-block-diet and $u,v\notin\operatorname{im}\psi_{t-1}$. We can do this by much the same approach as in Claim 40.2; we obtain

$$\mathbf{P}\big[\phi(y)=u,\phi(z)=v\big|\psi_{t-1}\big]=\tfrac{(1\pm 5L\beta_{t-1})q_{y,z}}{d_{ab}|V_a\setminus U_{t-1}||V_b\setminus U_{t-1}|}\,.$$

Summing over all ψ_{t-1}, we obtain

$$\begin{aligned}\mathbf{P}[\phi(y)=u,\phi(z)=v]&\\ &=\tfrac{(1\pm 5L\beta_{t-1})q_{y,z}}{d_{ab}|V_a\setminus U_{t-1}||V_b\setminus U_{t-1}|}\cdot\big(1\pm 100L\beta_{t-1}\big)\tfrac{|V_a\setminus U_{t-1}||V_b\setminus U_{t-1}|}{|V_a\setminus U_0||V_b\setminus U_0|}\pm 2\varepsilon^{-1}\exp\big(-n^{0.3}\big)\\ &=\big(1\pm 200L\beta_{t-1}\big)\tfrac{q_{y,z}}{d_{ab}|V_a\setminus U_0||V_b\setminus U_0|}\end{aligned}$$

where the probability $\mathbf{P}[u,v\notin\operatorname{im}\psi_{t-1}]$ is estimated using Claim 40.7. Summing over all (y,z) such that $yz\in E(F)$, and using (22), we obtain $(PE4)$. Note that we do not need to be careful estimating a sum of β_t here; the error bounds are larger than any of the β_t (and this will be the case for the next few properties, too).

For $(PE5)$, observe that we use $\phi_0(x)v$, where $x\in V(P_t)$ is in X_a, if and only if $v\notin\operatorname{im}\psi_{t-1}$ and then v is selected as the vertex to which we embed the neighbour y of x, where $y\in X_b$. The probability $\mathbf{P}[v\notin\operatorname{im}\psi_{t-1}]$ is $(1\pm 100L\beta_{t-1})\frac{|V_b\setminus U_{t-1}|}{|V_b\setminus U_0|}\pm\varepsilon^{-1}\exp\big(-n^{0.3}\big)$ by Claim 40.7. Among ψ_{t-1} in this event, the probability that (H,U_{t-1}) is not $(\beta_{t-1},2)$-block-diet is at most $\exp\big(-n^{0.3}\big)$, and for the remaining ψ_{t-1}, the probability of $\psi(y)=v$, conditioned on ψ_{t-1}, is $(1\pm 5L\beta_{t-1})\frac{1}{d_{ab}|V_b\setminus U_{t-1}|}$ according to Claim 40.2. Putting these facts together we get $(PE5)$.

Finally, for $(PE6)$, we separate two cases. If uv and $u'v'$ share a vertex, suppose $v=v'$. Observe that we use both uv and $u'v$ if and only if there is some path P_t which uses both edges. There are two subcases to consider. First, one or both of u and u' is in $\phi_0(A)$, in which case there is at most one possible choice of t to consider, and second, neither is in $\phi_0(A)$, in which case we need to consider all paths P_t.

We deal with the first subcase first; assume without loss of generality $u\in\phi_0(A)$. If ψ_{t-1} is such that (H,U_{t-1}) is $(\beta_{t-1},2)$-block-diet, then the probability of using both uv and $u'v$ is easily estimated to be $O(n^{-2})$ (it is $\Theta(n^{-2})$ if neither v nor u' is in $\operatorname{im}\psi_{t-1}$, but this event need not occur). Moreover, the probability that ψ_{t-1} fails to give block-diet is at most $\exp\big(-n^{0.3}\big)$. Hence we conclude $(PE6)$.

For the second subcase, there are at most n paths P_t to consider and for each one, using both uv and $u'v$ fixes three vertices in $\phi\big(V(P_t)\big)$. When ψ_{t-1} is such that (H,U_{t-1}) is $(\beta_{t-1},2)$-block-diet, the probability of this is $O(n^{-3})$, so the probability a given P_t uses uv and $u'v$ is at most $n^{-2.5}$; taking the union bound over at most n choices of t we again get $(PE6)$.

If now uv and $u'v'$ are disjoint edges, we have a few more subcases to consider. If both uv and $u'v'$ intersect $\phi_0(A)$, then this fixes the path or paths that can use uv and $u'v'$. If there is only one path (i.e. uv and $u'v'$ between them contain both anchors of that path) then using uv and $u'v'$ fixes two additional vertices on that path, and hence the probability of picking it is $O(n^{-2})$.

If there are two such paths (i.e. uv and $u'v'$ both contain an anchor but of different paths) then each path has one additional vertex on it fixed and again the probability of choosing such a pair of paths is $O(n^{-2})$. If uv intersects $\phi_0(A)$, but $u'v'$ does not, then there is only one path which can contain uv and doing so fixes one additional vertex on this path. While there are up to n paths which might contain $u'v'$, for any given one of them to do so fixes two additional vertices on that path. By the union bound the probability of choosing any one such path is $O(n^{-1})$, and again the probability of choosing both uv and $u'v'$ is $O(n^{-2})$. The probability of choosing uv and $u'v'$ on the same path is $O(n^{-3})$ since doing so fixes three vertices on that unique path as uv and $u'v'$ are disjoint. Finally, if uv and $u'v'$ are both disjoint from $\phi_0(A)$, then either they can both be chosen on one path, or on two different paths. In the first case, there are n possible paths and for any one to contain both uv and $u'v'$ fixes four vertices, for a probability $O(n^{-3})$ by the union bound. In the second case, there are n^2 possible pairs of paths, on each of which two vertices are fixed for a probability $O(n^{-2})$ by the union bound. In all these cases we duly conclude $(PE6)$. □

7.2. **Packing path-forests.** Having analysed Algorithm 1, we now state the complete path-forest packing algorithm, which simply runs Algorithm 1 repeatedly.

Algorithm 2: *PathPacking*

Input:
- path-forests $F_1, \dots, F_{s^*}$, such that F_s has anchors A_s,
- a graph H_0 on $V_\boxminus \dot\cup V_\boxplus$,
- sets $U_s \subseteq V(H_0)$ of used vertices for $s \in [s^*]$,
- assignments $\xi_s : V(F_s) \to \{V_\boxminus, V_\boxplus\}$,
- embeddings $\phi_s : A_s \to V(H_0)$ such that $\operatorname{im} \phi_s \subseteq U_s$

Randomly permute the F_s ;
for $s = 1$ **to** s^* **do**
  run *RandomPathEmbedding*$(F_s,H_{s-1},U_s,\xi_s,\phi_s)$ to get an embedding ϕ'_s of F_s into H_{s-1};
  let H_s be the graph obtained from H_{s-1} by removing the edges of $\phi'_s(F_s)$;
end

We (abusing notation) still refer to F_s after the uniform random permutation of the path-forests; what we really mean is the sth forest in the random permutation, and we will continue this abuse of notation in our analysis. However the statements we make about the algorithm in Lemma 42 below are invariant under permutation.

We are now in a position to analyse Algorithm 2, and give the important result of this section. Briefly, if we have a collection of anchored path-forests satisfying the conditions of Lemma 40, and in addition we have the following *pair distribution property*, we can complete the analysis. We should stress that this property depends only weakly on the graph structure of H: in particular, if edges are removed from H it cannot cause the pair distribution property to fail.

Definition 41 (Pair distribution property)**.** *Given a graph H with $V(H) = V_{\boxminus} \dot\cup V_{\boxplus}$ and a collection $F_1, \dots, F_{s^*}$ of path-forests all of whose paths have at least four edges, where for each $s \in [s^*]$ the endvertices of F_s is the set A_s, and we have a used set $U_s \subseteq V(H)$ and an injective map $\phi_s : A_s \to U_s$. We say that a collection of maps $(\xi_s : V(F_s) \to \{V_{\boxminus}, V_{\boxplus}\})_{s\in\mathcal{J}_2}$ has the* γ-pair distribution property *if for each $a, b \in \{\boxminus, \boxplus\}$ and pair $u \in V_a$, $v \in V_b$ of distinct vertices of H the following holds. Define $w_{uv;s}$ by*[1]

$$w_{uv;s} = \begin{cases} \frac{|\{(x,y):xy\in E(F_s),\xi_s(x)=V_a,\xi_s(y)=V_b \text{ and } x,y\notin A_s\}|}{|V_a\setminus U_s||V_b\setminus U_s|} & \text{if } u,v \notin U_s & \text{(23a)} \\ \frac{1}{|V_b\setminus U_s|} \text{ if } \phi_s^{-1}(u) = x \in A_s,\ v \notin U_s,\ xy \in E(F_s) \text{ with } \xi_s(y) = V_b & & \text{(23b)} \\ \frac{1}{|V_a\setminus U_s|} \text{ if } \phi_s^{-1}(v) = x \in A_s,\ u \notin U_s,\ xy \in E(F_s) \text{ with } \xi_s(y) = V_a & & \text{(23c)} \end{cases}$$

and otherwise $w_{uv;s} = 0$. Then if $uv \in E(H)$ we have

$$\sum_{s=1}^{s^*} w_{uv;s} = (1\pm\gamma)\frac{\sum_{s=1}^{s^*}|\{(x,y) : xy \in E(F_s), \xi_s(x) = V_a, \xi_s(y) = V_b\}|}{|V_a||V_b|} \pm \gamma n^{-0.01}\,. \tag{24}$$

Before stating the lemma, we need a slightly simplified version of the set defined by index-quasirandomness. Given any sets $S_1, S_2 \subseteq V(H)$, and any disjoint $T_1, T_2 \subseteq [s^*]$, and any pairing $V_{\boxminus} = \{\boxminus_i\}_{i\in[n/2]}$ and $V_{\boxplus} = \{\boxplus_i\}_{i\in[n/2]}$, let $\mathbb{U}_H(S_1, S_2, T_1, T_2)$ denote the set of $i \in [n/2]$ such that $\boxminus_i s \in E(H)$ for each $s \in S_1$, and $\boxplus_i s \in E(H)$ for each $s \in S_2$, and $\boxminus_i \notin U_t$ for each $t \in T_1$, and $\boxplus_i \notin U_t$ for each $t \in T_2$. The simplification as compared to index-quasirandomness is that we have no set T_3. We similarly define $\mathbb{U}'_{H'}(S_1, S_2, T_1, T_2)$ by replacing each U_t with $U_t \cup \operatorname{im} \phi'_t$ (where ϕ'_t is the embedding of F_t by Algorithm 2) and H with $H' := H_{s^*}$ the final graph left by Algorithm 2.

Lemma 42 (Path packing lemma)**.** *Given $\nu > 0$ and $L \geq 2$, there exists a constant C' such that for all $0 < \gamma < 1/C'$ the following holds.*

Let n be even and sufficiently large, and H be (γ, L)-block-quasirandom on $V_{\boxminus} \dot\cup V_{\boxplus}$, each of which has $n/2$ vertices, with densities $d(H[V_{\boxminus}]) = d_{\boxminus\boxminus}$, $d(H[V_{\boxplus}]) = d_{\boxplus\boxplus}$, $d(H[V_{\boxminus}, V_{\boxplus}]) = d_{\boxminus\boxplus}$. Suppose $s^* \geq \nu n$. For each $1 \leq s \leq s^*$, suppose F_s is a path-forest with leaves (anchors) A_s in which each path has between 5 and L vertices inclusive, and suppose $\phi_s : A_s \to V(H)$ is an embedding of the anchors. Let $\xi_s : V(F_s) \to \{V_{\boxminus}, V_{\boxplus}\}$ be an assignment of the vertices of F_s to sides of H such that $\phi_s(x) \in \xi_s(x)$ for each $x \in A_s$. Finally let U_s be a set of vertices containing $\phi_s(A_s)$. Suppose that for each s the pair (H, U_s) is (γ, L)-block-diet, and we have the γ-anchor distribution property for each F_s with respect to H. Suppose that for each $a \in \{\boxminus, \boxplus\}$ and each $1 \leq s \leq s^*$ we have $|V_a \setminus U_s| - \big|\{x \in V(F_s) \setminus A_s : \xi_s(x) \in V_a\}\big| \geq \nu n$ and $\big|\xi_s^{-1}(\{V_a\}) \setminus A_s\big| \geq \nu n$. Suppose that for each $1 \leq s \leq s^*$ we have $\big|\{(x, y) : xy \in E(F_s), x \in \xi_s^{-1}(\{V_a\}) \setminus A_s, y \in \xi_s^{-1}(\{V_b\}) \setminus A_s\}\big| \geq \nu n$ for each $a, b \in \{\boxminus, \boxplus\}$. Suppose in addition that we have the γ-pair

[1]We emphasise that in the formulae below, a single edge xy can be counted twice, as (x, y) and (y, x).

distribution property. For each $a \in \{\boxminus, \boxplus\}$ *define*

$$d'_{aa} := \tfrac{1}{\binom{n/2}{2}}\Big(e_H(V_a) - \sum_{s=1}^{s^*} e_{F_s}\big(\xi_s^{-1}(\{V_a\})\big)\Big) \quad \text{and}$$

$$d'_{\boxminus\boxplus} := \tfrac{4}{n^2}\Big(e_H(V_\boxminus, V_\boxplus) - \sum_{s=1}^{s^*} e_{F_s}\big(\xi_s^{-1}(\{V_\boxminus\}), \xi_s^{-1}(\{V_\boxplus\})\big)\Big)\,.$$

Suppose $d'_{\boxminus\boxminus}, d'_{\boxminus\boxplus}, d'_{\boxplus\boxplus} \geq \nu$.

Let $w : V(H) \to [0, \nu^{-1}]$ *be any weight function such that for each* $a \in \{\boxminus, \boxplus\}$ *we have* $\sum_{u \in V_a} w(u)$ *either equal to zero or at least* νn. *Given any* $S_1, S_2 \subseteq V(H)$ *and any disjoint* $T_1, T_2 \subseteq [s^*]$ *such that* $|S_1|, |S_2|, |T_1|, |T_2| \leq L$, *and any pairing* $V_\boxminus = \{\boxminus_i\}_{i \in [n/2]}$ *and* $V_\boxplus = \{\boxplus_i\}_{i \in [n/2]}$, *let* $X \subseteq \mathbb{U}_H(S_1, S_2, T_1, T_2)$ *be any set of size at least* νn. *Finally, given sets* $S_\boxminus \subseteq V_\boxminus$ *and* $S_\boxplus \subseteq V_\boxplus$ *with* $|S_\boxminus|, |S_\boxplus| \leq L$, *let* Y *be any subset of* $\{s \in [s^*] : (S_\boxminus \cup S_\boxplus) \cap \operatorname{im} \phi_s = \emptyset\}$ *such that* $|Y| \geq \nu n$.

When we execute PathPacking *(Algorithm 2), then*

- $(PP1)$ *with probability at least* $1 - \exp\big(-n^{0.2}\big)$, *for each* $1 \leq s \leq s^*$ *we obtain an embedding* ϕ'_s *of* F_s *in* $H\big[\big(V(H) \setminus U_s\big) \cup \phi_s(A_s)\big]$ *extending* ϕ_s *and such that* $\phi'_s(x) \in \xi_s(x)$ *for each* $x \in V(F_s)$. *Furthermore, if we obtain such embeddings, the following hold for* $H' := H_{s^*}$.
- $(PP2)$ *For each* $uv \in E(H)$ *there is at most one* s *such that* ϕ'_s *uses the edge* uv.
- $(PP3)$ *The graph* H' *is* $(C'\gamma, L)$*-block-quasirandom on* $V_\boxminus \dot\cup V_\boxplus$ *with densities* $d'_{\boxminus\boxminus}, d'_{\boxminus\boxplus}, d'_{\boxplus\boxplus}$.
- $(PP4)$ *With probability at least* $1 - \exp\big(-n^{0.2}\big)$, *for each* $a, b \in \{\boxminus, \boxplus\}$ *and* $u \in V_a$ *we have*

$$\sum_{v \in \mathsf{N}_{H'}(u; V_b)} \omega(v) = \big(1 \pm C'\gamma\big)\tfrac{d'_{ab}}{d_{ab}} \sum_{v \in \mathsf{N}_H(u; V_b)} \omega(v)\,.$$

- $(PP5)$ *With probability at least* $1 - \exp\big(-n^{0.2}\big)$ *we have*

$$\big|X \cap \mathbb{U}'_{H'}(S_1, S_2, T_1, T_2)\big| = (1 \pm C'\gamma)\Big(\frac{d'_{\boxminus\boxminus}}{d_{\boxminus\boxminus}}\Big)^{|S_1 \cap V_\boxminus|}\Big(\frac{d'_{\boxplus\boxplus}}{d_{\boxplus\boxplus}}\Big)^{|S_1 \cap V_\boxplus| + |S_2 \cap V_\boxminus|}\Big(\frac{d'_{\boxplus\boxplus}}{d_{\boxplus\boxplus}}\Big)^{|S_2 \cap V_\boxplus|}$$
$$\cdot \Big(\prod_{t \in T_1} \frac{\big|V_\boxminus \setminus (U_t \cup \operatorname{im} \phi'_t)\big|}{\big|V_\boxminus \setminus U_t\big|}\Big)\Big(\prod_{t \in T_2} \frac{\big|V_\boxplus \setminus (U_t \cup \operatorname{im} \phi'_t)\big|}{\big|V_\boxplus \setminus U_t\big|}\Big)|X|\,.$$

- $(PP6)$ *With probability at least* $1 - \exp\big(-n^{0.2}\big)$ *we have*

$$\big|\{s \in Y : (S_\boxminus \cup S_\boxplus) \cap \operatorname{im} \phi'_s = \emptyset\}\big| = (1 \pm C'\gamma) \sum_{s \in Y} \prod_{a \in \{\boxminus, \boxplus\}} \Big(\tfrac{|V_a \setminus (U_s \cup \operatorname{im} \phi'_s)|}{|V_a \setminus U_s|}\Big)^{|S_a|}\,.$$

The proof of this lemma mainly amounts to collecting the probabilistic estimates of Lemma 40 and our assumptions on the quasirandom distribution of the anchor sets and used sets of the various path-forests to obtain expected values, then applying Corollary 25 to show these expected values are likely all roughly correct. To begin with, we need to show that H_{s-1} is likely to satisfy the required block-diet condition for Lemma 40 with the used set of F_s at each stage s and the required anchor distribution property; for this purpose we will define an exponentially (in s) increasing error parameter α_s, much as in [2]. We should note that $(PP5)$ holds for any pairing, and in particular it can be the case for some choices of S_i, T_i that it holds vacuously (i.e. $\mathbb{U}_H(S_1, S_2, T_1, T_2) = \emptyset$). In our applications, we will fix a single pairing and it will be such that there are no vacuous cases.

Proof. Given ν and L, let $\nu_{\mathrm{L40}} = \nu^{10L}$ and let C be returned by Lemma 40 for input ν_{L40} and L. We set

$$C' = \exp\left(2000CL\nu^{-20L}\right).$$

Furthermore, for any $0 < \gamma < 1/C'$, we define

$$\alpha_x := \gamma \exp\left(\tfrac{1000CL\nu^{-20L}x}{s^*}\right)$$

for $x \in \mathbb{R}$, and we set $\varepsilon = \gamma^2 (C')^{-1}$.

We first argue that the random permutation of the F_s ensures that the pair distribution property holds not just globally but also on short intervals.

Claim 42.1. *With probability at least $1 - \exp\big(-n^{0.5}\big)$, for each $1 \le s \le s^* + 1 - \varepsilon n$, after the uniform permutation, the path-forests $F_s, F_{s+1}, \dots, F_{s+\varepsilon n-1}$ have the 2γ-pair distribution property.*

Proof. By the union bound, it is enough to show that with probability at least $1 - \exp\big(-n^{0.55}\big)$ a uniform random choice of εn path-forests has the 2γ-pair distribution property, since each interval of εn paths in the uniform random permutation has this distribution. For this, it is enough to show that a given pair of vertices $u \in V_a, v \in V_b$ with $uv \in E(H)$ is, with probability at least $1 - \exp\big(-n^{0.6}\big)$, not a witness of failure of the 2γ-pair distribution property.

We partition the set of path-forests according to the weight they assign to uv (as in Definition 41), rounded down to the nearest multiple of $n^{-1.02}$. Since a path-forest has at most n edges, and since $|V_{\boxminus} \setminus U|, |V_{\boxplus} \setminus U| \ge \nu n$ for U the used set of any path-forest, the maximum weight assigned to uv by any path-forest is $2\nu^{-2}n^{-1}$ and hence there are at most $3\nu^{-2}n^{0.02}$ parts in our partition. By Fact 23, with probability at least $1 - \exp\big(-n^{0.8}\big)$, in any given part P with p path-forests, the uniform random choice of εn path-forests contains $\frac{\varepsilon n}{s^*}p \pm n^{0.95}$ of them. Let Y_P denote the random variable which is the sum of weights assigned to uv by paths of P in the uniform random εn path-forests; then assuming the above likely event occurs, we have

$$\big|Y_P - \mathbf{E}[Y_p]\big| \le n^{-1.02}p + 2\nu^{-2}n^{-0.05}.$$

With probability at least $\exp\big(-n^{0.7}\big)$, by the union bound, the likely event occurs for each P, and in this case we therefore have

$$\Big|\sum_P (Y_P - \mathbf{E}[Y_P])\Big| \le n^{-1.02}s^* + 2\nu^{-2}n^{-0.05} \cdot 3\nu^{-2}n^{0.02} \le 2n^{-1.02}s^*,$$

where the final inequality is since $s^* \ge \nu n$.

Note that $\sum_P \mathbf{E}[Y_P]$ is simply $\frac{\varepsilon n}{s^*}$ times the total weight assigned to uv by all path-forests, which is as stated in Definition 41. By a similar argument (partitioning path-forests according to the number of edges between V_a and V_b, i.e. edges xy such that $\xi_s(x) = V_a$ and $\xi_s(y) = V_b$, rounded down to the nearest multiple of $n^{0.98}$) we conclude that if there are in total e edges between V_a and V_b in all path-forests, then with probability at least $1 - \exp\big(-n^{0.7}\big)$, there are $\frac{\varepsilon n}{s^*}e \pm 2n^{0.98}$ edges between V_a and V_b in the uniform random εn path-forests. If both this and the previous likely event occur, then we conclude uv does not witness failure of the 2γ-path distribution property for the uniform random εn path forests, as desired. □

As remarked above, from this point when we write F_s we mean the sth path-forest in the chosen permutation (and so on). We assume from this point that the likely event of Claim 42.1 occurs.

For each $a \in \{\boxminus, \boxplus\}$ we define

$$d_{aa;s} := d_{aa} - \binom{|V_a|}{2}^{-1} \sum_{i=1}^{s} e\big(F_i\big[\xi_i^{-1}(\{V_a\})\big]\big) \quad \text{and}$$

$$d_{\boxminus\boxplus;s} := d_{\boxminus\boxplus} - |V_\boxminus|^{-1}|V_\boxplus|^{-1} \sum_{i=1}^{s} e\big(F_i\big[\xi_i^{-1}(\{V_\boxminus\}), \xi_i^{-1}(\{V_\boxplus\})\big]\big)\,.$$

Note that by definition the $d_{ab;s}$ give the densities within and between the sides of H_s, assuming this graph is constructed by Algorithm 2. Moreover, $d'_{ab} = d_{ab;s^*}$ for each $a, b \in \{\boxminus, \boxplus\}$.

To begin with, much as in the proof of Lemma 40, we define a quantity $p_{uv;s}$ which approximates the probability that uv is used in the embedding of F_s, on the assumption that $uv \in E(H_{s-1})$ and that the conditions of Lemma 40 are met when embedding F_s. Suppose $u \in V_a$ and $v \in V_b$, where $a, b \in \{\boxminus, \boxplus\}$.

If neither u nor v is in U_s, we set

$$p_{uv;s} := \frac{|\{(x,y) : xy \in E(F_s), \xi_s(x) = V_a, \xi_s(y) = V_b \text{ and } x, y \notin A_s\}|}{d_{ab;s-1}|V_a \setminus U_s||V_b \setminus U_s|}\,.$$

If $u = \phi_s(x)$ for some $x \in A_s$, and $v \notin U_s$, and the neighbour y of x has $\xi(y) = V_b$, we set

$$p_{uv;s} := \frac{1}{d_{ab;s-1}|V_b \setminus U_s|}\,,$$

and similarly if $v = \phi_s(y)$ for some $y \in A_s$, and $u \notin U_s$, and the neighbour y of x has $\xi(y) = V_a$, we set

$$p_{uv;s} := \frac{1}{d_{ab;s-1}|V_a \setminus U_s|}\,.$$

In all other cases we set $p_{uv;s} := 0$. Note that Lemma 40 parts $(PE4)$ and $(PE5)$ state that, conditioning on running of Algorithm 2 up to creating H_{s-1} and on the assumption that the conditions of Lemma 40 are satisfied at this point, if uv is any edge of H_{s-1} then the probability that the embedding of F_s uses uv is $\big(1 \pm C\alpha_{s-1}\big)p_{uv;s}$. We also note that, since $d_{ab;s-1} \geq \nu$ and $|V_a \setminus U_s|, |V_b \setminus U_s| \geq \nu n$, and since $E(F_s) \leq n$, we always have $p_{uv;s} \leq \nu^{-3}n^{-1}$.

Using the pair distribution property, we can estimate the sum $\sum_{i=s+1}^{s+\varepsilon n} p_{uv;i}$ for any $0 \le s \le s^* - \varepsilon n$ and any pair $uv \in E(H)$; one should think of this as being the probability that uv is used in the embedding of some graph in this interval, assuming it is in H_s and assuming the process is well-behaved.

Claim 42.2. *For each $0 \le s \le s^* - \varepsilon n$, each $a, b \in \{\boxminus, \boxplus\}$ and each pair of distinct vertices $u \in V_a$ and $v \in V_b$ with $uv \in E(H)$, we have*

$$\sum_{s'=s+1}^{s+\varepsilon n} p_{uv;s'} = (1 \pm 3\gamma)\frac{d_{ab;s} - d_{ab;s+\varepsilon n}}{d_{ab;s}} \pm 3\gamma n^{-0.01}\,.$$

Proof. Because $d_{ab;s^*} \ge \nu$, and because at most εn^2 edges are in the path-forests $F_{s+1}, \dots, F_{s+\varepsilon n}$, we have $d_{ab;s'} = (1 \pm 10\varepsilon\nu^{-1})d_{ab;s}$ for each $s+1 \le s' \le s+\varepsilon n$. Because this collection of path-forests has the 2γ-pair distribution property, we have

$$\sum_{s'=s+1}^{s+\varepsilon n} p_{uv;s'} = \tfrac{(1\pm 2\gamma)}{(1\pm 10\varepsilon\nu^{-1})d_{ab;s}} \frac{\sum_{s=1}^{s^*} \big|\{(x,y) : xy \in E(F_s), \xi_s(x) = V_a, \xi_s(y) = V_b\}\big|}{|V_a||V_b|} \pm 2\gamma n^{-0.01}\,,$$

and by choice of ε and definition of $d_{ab;s}$ we obtain the claim statement. Note that the big fraction in the equation above is exactly $\frac{d_{ab;s} - d_{ab;s+\varepsilon n}}{d_{ab;s}}$ if $a \ne b$; if $a = b$ then it differs in that $|V_a|^2$ is not $2\binom{|V_a|}{2}$, but the ratio between these quantities is $1 + O(1/n)$ as n goes to infinity and this is absorbed to the error term. □

We now argue that neighbourhoods of sets of not too many vertices in H_s into reasonably large sets decrease roughly as one would expect if edges were removed uniformly at random (rather than as path-forests). For purposes of establishing $(PP5)$ we need to consider a fixed pairing $V_\boxminus = \{\boxminus_i : i \in [n/2]\}$ and $V_\boxplus = \{\boxplus_i : i \in [n/2]\}$. We will not need to assume any special properties of this pairing.

Claim 42.3. *Fix each $1 \le s < s'' \le s^*$, and run Algorithm 2 up to the point where H_s is defined (i.e. F_s has been embedded); condition on this H_s. Now choose sets $W_1, W_2 \subseteq V(H)$ with $|W_1|, |W_2| \le L$ and a set $U \subseteq [n/2]$ with $|U| \ge \nu^{5L} n$ such that for each $i \in U$, $w \in W_1$ and $w' \in W_2$ we have $w\boxminus_i, w'\boxplus_i \in E(H_s)$. We now run Algorithm 2 further until $H_{s''}$ is defined. With probability at least $1 - \exp\big(-n^{0.4}\big)$ one of the following occurs. First, we have*

$$\begin{aligned}&\Big|\big\{i \in U : \forall w \in W_1, w' \in W_2 \text{ we have } w\boxminus_i, w'\boxplus_i \in E(H_{s''})\big\}\Big| \\ &\quad = \big(1 \pm \tfrac{1}{2}\alpha_{s''}\big)|U|\Big(\tfrac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|}\Big(\tfrac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\Big(\tfrac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|}\,,\end{aligned}\tag{25}$$

Second, there is some stage s' between s and s'' inclusive such that either the conditions of Lemma 40 are not met, or its low-probability event occurs.

Proof. Let $U_s = U$, and for each $s' > s$ let $U_{s'}$ be the set of $i \in U_s$ such that $w\boxminus_i, w'\boxplus_i \in E(H_{s'})$ for all $w \in W_1$ and $w' \in W_2$. Observe that it is enough to prove that a given s'' is unlikely to be the first s'' such that (25) fails. So let $\mathcal{E}$ be the event that the conditions of Lemma 40 are met, and the low-probability event of that lemma does not occur, and (25) does not fail at any stage before s''. We aim to show that the probability that $\mathcal{E}$ occurs and (25) fails at stage s'' is at most $\exp\big(-n^{-0.5}\big)$, since then taking the union bound over choices of s'' establishes the claim statement.

Let $s < s' \leq s''$, and condition for a moment on the history of Algorithm 2 up to and including the embedding of $F_{s'-1}$. Given $i \in U$ such that for each $w \in W_1$ and $w' \in W_2$ we have $w\boxminus_i, w'\boxplus_i \in E(H_{s'-1})$, let E_i denote the edges $w\boxminus_i$ for $w \in W_1$ and $w'\boxplus_i$ for $w' \in W_2$. If $|E_i| = |W_1| + |W_2|$ we say i is *normal*, otherwise it is *special*. Note that if i is special, this means that $\boxminus_i \in W_2$ and $\boxplus_i \in W_1$; there are therefore at most L special i.

By Lemma 40, property $(PE4)$, the conditional probability that the embedding of $F_{s'}$ uses any given $uv \in E_i$ is $(1 \pm C\alpha_{s'-1})p_{uv;s'}$. Furthermore the conditional probability that any given pair of edges in E_i are used is, by $(PE6)$, at most $n^{-3/2}$. It follows that, provided at least one $p_{uv;s'}$ is non-zero with $uv \in E_i$, the conditional probability that at least one edge of E_i is used by the embedding of $F_{s'}$ is

$$(1 \pm C\alpha_{s'-1}) \sum_{uv \in E_i} p_{uv;s'} \pm \binom{2L}{2} n^{-3/2} = (1 \pm 2C\alpha_{s'-1}) \sum_{uv \in E_i} p_{uv;s'} \,,$$

where we justify the equality as follows. If some $p_{uv;s'}$ is non-zero, then by definition of $p_{uv;s'}$ it is at least νn^{-1} (here we use the assumption of the lemma that at least νn edges of $F_{s'}$ are assigned within each of $V_\boxminus$, $V_\boxplus$ and between $V_\boxminus$ and $V_\boxplus$) and hence the error term, which is $O(n^{-3/2})$, can be absorbed in the relative error. If all $p_{uv;s'}$ with $uv \in E_i$ are zero, then the $\binom{2L}{2}n^{-3/2}$ error term can be removed from the left side of the above: the probability that an edge of E_i is used is zero. Thus also in this case the right side is a valid estimate.

By linearity of expectation, we see that the conditional expected number of $i \in U_{s'-1}$ such that at least one edge of E_i is used in embedding $F_{s'}$, is

$$\big(1 \pm 2C\alpha_{s'-1}\big) \sum_{i \in U_{s'-1}} \sum_{uv \in E_i} p_{uv;s'} \,.$$

Now since embedding any one graph $F_{s'}$ uses at most two edges at any given vertex (in particular those in $W_1 \cup W_2$), there are at most $4L$ values of i such that an edge of E_i is used in embedding $F_{s'}$. So Corollary 25 tells us that with probability at most $\exp\big(-n^{0.5}\big)$ the event $\mathcal{E}$ occurs and nevertheless we have

$$\big|U_s \setminus U_{s''}\big| \neq \sum_{s'=s+1}^{s''} \big(1 \pm 3C\alpha_{s'-1}\big) \sum_{i \in U_{s'-1}} \sum_{uv \in E_i} p_{uv;s'} \pm \varepsilon n \,.$$

Observe that it is enough to show that within the event $\mathcal{E}$ we have

$$\begin{aligned}\sum_{s'=s+1}^{s''}\big(1\pm 3C\alpha_{s'-1}\big)\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'} = \\ \Bigg(1-\big(1\pm\tfrac14\alpha_{s''}\big)\Big(\tfrac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|}\Big(\tfrac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\Big(\tfrac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|}\Bigg)|U|\,,\end{aligned}\tag{26}$$

since by choice of ε and ν the error term εn is absorbed into the larger error in (25).

Note that the left side of (26) is a random variable; we will now argue that within $\mathcal{E}$ it is surely in the claimed interval. To that end, we split the sum over s' into disjoint intervals of length εn, together with a final interval of length less than εn. We deal first with the final interval: in this interval, we use at most $2\varepsilon n$ edges at any given vertex of $W_1\cup W_2$, hence the sum over this interval is at most $(|W_1|+|W_2|)\cdot 2\varepsilon n\le 4L\varepsilon n$, which we can absorb in the error term. We can thus assume $s''-s$ is a multiple of εn, and we only need to show

$$\begin{aligned}&\sum_{s'=s+1}^{s''}\big(1\pm 3C\alpha_{s'-1}\big)\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}\\ &\qquad=|U|\Bigg(1-\Big(\tfrac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|}\Big(\tfrac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\Big(\tfrac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|}\Bigg)\pm\tfrac18\alpha_{s''}|U|\nu^{2L}\,.\end{aligned}$$

Consider the sum

$$\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n}\big(1\pm 3C\alpha_{s'-1}\big)\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}$$

where $0\le\ell\le\frac{s''-s}{\varepsilon n}-1$. To estimate this sum, fix an ℓ. We separate the main term

$$\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n}\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}$$

and the error term

$$\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n}3C\alpha_{s'-1}\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}\le\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n}6C\alpha_{s'-1}|U|L\nu^{-3}n^{-1}\,,\tag{27}$$

where the inequality uses that each $p_{uv;s'}$ is bounded above by $\nu^{-3}n^{-1}$ and $|E_i|\le 2L$.

To estimate the main term, observe that the set $U_{s'-1}$ decreases, as s' ranges over the interval from $s+1+\ell\varepsilon n$ to $s+(\ell+1)\varepsilon n$, by at most $4L\varepsilon n$ indices. Each normal i which is in $U_{s+(\ell+1)\varepsilon n}$ contributes, by Claim 42.2,

$$\begin{aligned}(1\pm3\gamma)\Big(|W_1\cap V_\boxminus|\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}-d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s+\ell\varepsilon n}}+\big(|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|\big)\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}-d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s+\ell\varepsilon n}}\\ +|W_2\cap V_\boxplus|\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}-d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s+\ell\varepsilon n}}\Big)\pm 6L\gamma n^{-0.01}\end{aligned}$$

to the sum. Each special i (of which there are at most L) and each normal i which is not in $U_{s+(\ell+1)\varepsilon n}$ contributes at most this amount. Thus our main term is $|U_{s+\ell\varepsilon n}| \pm (4L\varepsilon n + L)$ times the above estimate. Since we are in $\mathcal{E}$, we know the size of $U_{s+\ell\varepsilon n}$; in particular it is of size at least $\frac{1}{2}\nu^{3L}n$, and so the error $4L\varepsilon n + L$ is tiny compared to α_0 by choice of ε. Putting this together, we have

$$\begin{aligned}(28)\quad &\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n} \sum_{i\in U_{s'-1}} \sum_{uv\in E_i} p_{uv;s'} \\ &= \big(1\pm\alpha_{s+\ell\varepsilon n}\big)|U| \Big(\tfrac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|} \Big(\tfrac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|} \Big(\tfrac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|} \\ &\quad\cdot \Bigg(|W_1\cap V_\boxminus| \tfrac{d_{\boxminus\boxminus;s+\ell\varepsilon n}-d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s+\ell\varepsilon n}} + \big(|W_1\cap V_\boxplus| + |W_2\cap V_\boxminus|\big)\tfrac{d_{\boxminus\boxplus;s+\ell\varepsilon n}-d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s+\ell\varepsilon n}} \\ &\quad + |W_2\cap V_\boxplus| \tfrac{d_{\boxplus\boxplus;s+\ell\varepsilon n}-d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s+\ell\varepsilon n}}\Bigg) \pm 6L\gamma n^{0.99}\end{aligned}$$

To understand the right-hand side of (28), consider expanding out

$$\begin{aligned}&\Big(\tfrac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|} \Big(\tfrac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|} \Big(\tfrac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|} \\ &= \Big(\tfrac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}} + \tfrac{d_{\boxminus\boxminus;s+\ell\varepsilon n}-d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|} \\ &\quad\cdot \Big(\tfrac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}} + \tfrac{d_{\boxminus\boxplus;s+\ell\varepsilon n}-d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|} \\ &\quad\cdot \Big(\tfrac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}} + \tfrac{d_{\boxplus\boxplus;s+\ell\varepsilon n}-d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|}.\end{aligned}$$

We will split the above expression into three terms: the first term will be obtained by multiplying all the left sides of the brackets; the second term will be given by the sum of the products which are obtained by multiplying one term from the right sides of the brackets and the remainder from the left; and the third term will consist of the left-over.

This way, the first term equals

$$\Big(\tfrac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\Big)^{|W_1\cap V_\boxminus|} \cdot \Big(\tfrac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\Big)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|} \cdot \Big(\tfrac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\Big)^{|W_2\cap V_\boxplus|}.$$

Because $d_{ab;s+(\ell+1)\varepsilon n} \geq \nu$, and because the path-forests $F_{s+1+\ell\varepsilon n}, \dots, F_{s+(\ell+1)\varepsilon n}$ have at most εn^2 edges in total, we have that $d_{ab;s+(\ell+1)\varepsilon n}$ and $d_{ab;s+\ell\varepsilon n}$ differ at most by a factor $(1\pm 10\varepsilon\nu^{-1})$.

Therefore, the second term equals

$$\left(\frac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\left(\frac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\left(\frac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$\cdot\Bigg(|W_1\cap V_\boxminus|\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}-d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}+\big(|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|\big)\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}-d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}$$
$$+|W_2\cap V_\boxplus|\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}-d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}\Bigg),$$

which equals

$$(1\pm 30L\varepsilon\nu^{-1})\left(\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\left(\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\left(\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$\cdot\Bigg(|W_1\cap V_\boxminus|\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}-d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s+\ell\varepsilon n}}+\big(|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|\big)\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}-d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s+\ell\varepsilon n}}$$
$$+|W_2\cap V_\boxplus|\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}-d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s+\ell\varepsilon n}}\Bigg),\tag{29}$$

which is, except from the factors $(1\pm 30L\varepsilon\nu^{-1})$ and $(1\pm a_{s+\ell\varepsilon n})|U|$, the same as the formula on the right-hand side of (28).

For the third term observe that there are less than 2^{2L} summands each of which is a product of at least two terms from the right sides of the brackets. Using again that $d_{ab;s+(\ell+1)\varepsilon n}$ and $d_{ab;s+\ell\varepsilon n}$ differ at most by a factor $(1\pm 10\varepsilon\nu^{-1})$, we see that this third term amounts to at most $2^{2L}(10\varepsilon\nu^{-1})^2$.

Putting this together, we see that the term in (29), equals

$$\left(\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\left(\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\left(\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$-\left(\frac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$\pm 2^{2L}(10\varepsilon\nu^{-1})^2\,.$$

Plugging this into (28), we finally obtain that $\sum_{s'=s+1+\ell\varepsilon n}^{s+(\ell+1)\varepsilon n}\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}$ equals

$$(1\pm 2\alpha_{s+\ell\varepsilon n})|U|\Bigg(\left(\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$-\left(\frac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}$$
$$\pm 2^{2L}(10\varepsilon\nu^{-1})^2\Bigg),$$

which in turn equals

$$|U|\left(\left(\frac{d_{\boxminus\boxminus;s+\ell\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s+\ell\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s+\ell\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}\right.$$
$$\left.-\left(\frac{d_{\boxminus\boxminus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s+(\ell+1)\varepsilon n}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}\right)$$
$$\pm 200\nu^{-4L}\varepsilon\alpha_{s+\ell\varepsilon n}|U|\,.$$

At last, we can sum this over ℓ, together with the error term bounded in (27), to obtain

$$\sum_{s'=s+1}^{s''}\big(1\pm 3C\alpha_{s'-1}\big)\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}$$
$$=|U|\left(\left(\frac{d_{\boxminus\boxminus;s}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}\right.$$
$$\left.-\left(\frac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}\right)$$
$$\pm\sum_{s'=s}^{s''}(200\nu^{-4L}n^{-1}+6CL\nu^{-3}n^{-1})|U|\alpha_{s'-1}$$

and hence

$$\sum_{s'=s+1}^{s''}\big(1\pm 3C\alpha_{s'-1}\big)\sum_{i\in U_{s'-1}}\sum_{uv\in E_i}p_{uv;s'}$$
$$=|U|\left(1-\left(\frac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\right)^{|W_1\cap V_\boxminus|}\cdot\left(\frac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\right)^{|W_1\cap V_\boxplus|+|W_2\cap V_\boxminus|}\cdot\left(\frac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\right)^{|W_2\cap V_\boxplus|}\right)$$
$$\pm\sum_{s'=s}^{s''}200CL\nu^{-4L}n^{-1}\alpha_{s'-1}|U|\,.$$

For the last error term, we observe that it can be bounded by

$$\int_{x=-\infty}^{s''}200CL\nu^{-4L}n^{-1}\alpha_x|U|\mathrm{d}x=0.2\nu^{16L}s^*n^{-1}\alpha_{s''}|U|<\tfrac{1}{8}\alpha_{s''}|U|\nu^{2L}\,,$$

which completes the proof of the claim. □

This claim implies (we will see a proof later) that the block-quasirandomness and block-diet requirements of Lemma 40 are satisfied at each stage. What we in addition need is the anchor distribution property, which we will show follows from the next claim.

Claim 42.4. *For each $1\le s\le s^*$, any $a,b\in\{\boxminus,\boxplus\}$ and $v\in V_a$, and any weight function $\omega:V(H_0)\to[0,\nu^{-1}]$ such that $\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\ge\nu n$, the following experiment has probability* *at least* $1-\exp\big(-n^{0.4}\big)$ *of success. We run Algorithm 2 up to the point at which* ϕ_s *and* H_s *are defined. We declare the experiment a success if either*

$$\sum_{u\in \mathsf{N}_{H_s}(v;V_b)}\omega(u)=(1\pm\tfrac12\alpha_s)\tfrac{d_{ab;s}}{d_{ab}}\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\,, \tag{30}$$

or at some stage $1\le s'<s$ *the conditions of Lemma 40 fail or the low-probability event of that lemma occurs.*

Proof. Fix a weight function ω. As with the previous claim, we show that any given s has probability at most $\exp\big(-n^{0.5}\big)$ of being the first s for which the claim statement fails; the claim then follows by the union bound over s. So fix s, and let $\mathcal{E}$ be the event that at each stage from 1 to s inclusive the conditions of Lemma 40 are met and its low-probability event does not occur, and in addition at each stage $1\le s'\le s-1$ we have

$$\sum_{u\in\mathsf{N}_{H_{s'}}(v;V_b)}\omega(u)=(1\pm\tfrac12\alpha_{s'})\tfrac{d_{ab;s'}}{d_{ab}}\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\,.$$

Note that Lemma 40 states that within $\mathcal{E}$ the probability that the embedding of $F_{s'}$ uses uv is $\big(1\pm C\alpha_{s'-1}\big)p_{uv;s'}$. Let $\mathscr{H}_{s'-1}$ denote the history of Algorithm 2 up to and including the embedding of $F_{s'-1}$. Thus by linearity of expectation, within $\mathcal{E}$ we have

$$\mathbf{E}\Big[\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)\setminus\mathsf{N}_{H_{s'}}(v;V_b)}\omega(u)\Big|\mathscr{H}_{s'-1}\Big]=\big(1\pm C\alpha_{s'-1}\big)\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}\,.$$

Observe that when we embed any given $F_{s'}$, we use at most two edges leaving u, so the maximum change to the weight of the neighbourhood of u is at most $2\nu^{-1}$. Thus by Corollary 25, the probability that $\mathcal{E}$ occurs and

$$\sum_{u\in\mathsf{N}_{H_0}(v;V_b)\setminus\mathsf{N}_{H_s}(v;V_b)}\omega(u)\neq\sum_{s'=1}^{s}\big(1\pm C\alpha_{s'-1}\big)\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}\pm n^{0.99}$$

is at most $\exp\big(-n^{0.5}\big)$. As with the previous claim, it thus suffices to show that within $\mathcal{E}$ we have

$$\sum_{s'=1}^{s}\big(1\pm C\alpha_{s'-1}\big)\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}=\big(1-(1\pm\tfrac14\alpha_s)\tfrac{d_{ab;s}}{d_{ab}}\big)\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\,.$$

As with the previous claim, note that the left side of this is a random variable which in $\mathcal{E}$ we will see surely lies in the claimed interval; as there, we show this by splitting the sum over s' into intervals of length εn together with a final interval of length less than εn. In the final interval at most $2\varepsilon n$ edges are removed from v, and the weight of vertices at these edges is thus at most $2\varepsilon n\nu^{-1}$; this upper bounds the sum of expectations over this interval. Thus it is enough to assume s is a multiple of εn and show

$$\sum_{s'=1}^{s}\big(1\pm C\alpha_{s'-1}\big)\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}=\big(1-(1\pm\tfrac{1}{8}\alpha_s)\tfrac{d_{ab;s}}{d_{ab}}\big)\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\,. \tag{31}$$

As before, we split this sum into a main term

$$\sum_{s'=1}^{s}\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}$$

and an error term

$$\begin{aligned}\sum_{s'=1}^{s}C\alpha_{s'-1}\sum_{u\in\mathsf{N}_{H_{s'-1}}(v;V_b)}\omega(u)p_{uv;s'}&\le\sum_{s'=1}^{s}C\alpha_{s'-1}\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\nu^{-3}n^{-1}\\&\le\sum_{s'=1}^{s}C\alpha_{s'-1}\nu^{-4}\le\nu^4\alpha_s n\,.\end{aligned}$$

In the main term, consider, for some integer $\ell\ge 0$, the interval $\ell\varepsilon n<s'\le(\ell+1)\varepsilon n$. Each vertex u which is in $\mathsf{N}_{H_{s'}}(v;V_b)$ for the entire interval contributes by Claim 42.2

$$\Big((1\pm 3\gamma)\tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab;\ell\varepsilon n}}\pm 3\gamma n^{-0.01}\Big)\omega(u)$$

to the sum. Thus an upper bound for this interval of the main term is

$$\Big((1\pm 3\gamma)\tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab;\ell\varepsilon n}}\pm 3\gamma n^{-0.01}\Big)\sum_{u\in\mathsf{N}_{H_{\ell\varepsilon n}}(v;V_b)}\omega(u)\,.$$

Since at most $2\varepsilon n$ vertices are removed from the neighbourhood during this interval, each of which removes at most its weight times $\nu^{-3}\varepsilon$ (by the general upper bound on $p_{uv;s'}$) from the sum, this upper bound is off from the correct answer by at most $2\nu^{-4}\varepsilon^2 n$. Furthermore, since we are in $\mathcal{E}$ we know

$$\sum_{u\in\mathsf{N}_{H_{\ell\varepsilon n}}(v;V_b)}\omega(u)=\big(1\pm\tfrac{1}{2}\alpha_{\ell\varepsilon n}\big)\tfrac{d_{ab;\ell\varepsilon n}}{d_{ab}}\sum_{u\in\mathsf{N}_{H_0}(v;V_b)}\omega(u)\,,$$

and we conclude

$$\sum_{s'=\ell\varepsilon n+1}^{(\ell+1)\varepsilon n} \sum_{u\in \mathsf{N}_{H_{s'-1}}(v;V_b)} \omega(u)p_{uv;s'}$$
$$= \Big((1\pm 3\gamma)\tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab;\ell\varepsilon n}} \pm 3\gamma n^{-0.01}\Big)\big(1\pm \tfrac{1}{2}\alpha_{\ell\varepsilon n}\big)\tfrac{d_{ab;\ell\varepsilon n}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm 2\nu^{-4}\varepsilon^2 n$$
$$= (1\pm\alpha_{\ell\varepsilon n})\tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm 3\nu^{-4}\varepsilon^2 n$$
$$= \tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm \alpha_{\ell\varepsilon n}\nu^{-2}2\varepsilon n \pm 3\nu^{-4}\varepsilon^2 n\,.$$

Summing this over the at most ε^{-1} choices of ℓ, we have

$$\sum_{s'=1}^{s} \sum_{u\in \mathsf{N}_{H_{s'-1}}(v;V_b)} \omega(u)p_{uv;s'}$$
$$= \sum_{\ell=0}^{\frac{s}{\varepsilon n}-1} \tfrac{d_{ab;\ell\varepsilon n}-d_{ab;(\ell+1)\varepsilon n}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm \sum_{\ell=0}^{\frac{s}{\varepsilon n}-1} \alpha_{\ell\varepsilon n}\nu^{-2}2\varepsilon n \pm 3\nu^{-4}\varepsilon n$$
$$= \tfrac{d_{ab}-d_{ab;s}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm \nu^{4}\alpha_s n\,,$$

where the final line uses our definition of α_x and summing the error terms as in previous claims.

Putting back the error term, we find that the left side of (31) is in the interval

$$\tfrac{d_{ab}-d_{ab;s}}{d_{ab}} \sum_{u\in \mathsf{N}_{H_0}(v;V_b)} \omega(u) \pm 2\nu^{4}\alpha_s n$$

which in particular shows (31) holds, proving the claim. □

We are now in a position to prove that it is likely that Algorithm 2 completes, from which $(PP1)$ and $(PP2)$ follow. What we need to do is show that it is likely that at each stage s the conditions of Lemma 40 are met, since then it is unlikely (by Lemma 40 and the union bound) that at any stage Lemma 40 fails to produce an embedding. As before, it is enough to show that, for any given $1\le s'\le s^*$, with probability at most $\exp\big(-n^{0.3}\big)$ stage s' is the first stage at which the conditions of Lemma 40 are not met or its low-probability event occurs.

Fix vertices $u_1,\dots,u_p\in V(H_0)$ for some $p\le L$ and $a\in\{\boxminus,\boxplus\}$. The probability that these vertices and side of $H_{s'-1}$ witness a failure of either $(\alpha_{s'-1},L)$-block quasirandomness of $H_{s'-1}$, or of the $(\alpha_{s'-1},L)$-block diet condition for $(H_{s'-1},U_{s'})$, is by Claim 42.3 at most $2\exp\big(-n^{0.4}\big)$; in both cases we apply the claim with $s=1$ and s', in the first place with $U=\{i\in[n/2]:\ a_i\in \mathsf{N}_{H_0}(u_1,\dots,u_p)\cap V_a\}$, and in the second place with $U=\{i\in[n/2]:\ a_i\in \mathsf{N}_{H_0}(u_1,\dots,u_p)\cap V_a\setminus U_{s'}\}$; in either case, if $a=\boxminus$ we use $W_1=\{u_1,\dots,u_p\}$ and $W_2=\emptyset$, while if $a = \boxplus$ we use $W_2 = \{u_1, \dots, u_p\}$ and $W_1 = \emptyset$. Taking the union bound over choices of p, a and $u_1, \dots, u_p$ we see that the probability such a witness exists is at most $\exp\big(-n^{0.3}\big)$.

Now fix $a, b, c \in \{\boxminus, \boxplus\}$, a vertex $v \in V_b$ and let ω be the weight function on $V(H_0)$ where for $u \in V_a$ we set $\omega(u) = 1$ if there is $x \in A_{s'}$ such that $\phi_{s'}(x) = u$ and the neighbour y of x in $F_{s'}$ has $\xi(y) = V_b$ and the next neighbour z of y in $F_{s'}$ has $\xi(z) = V_c$; and otherwise $\omega(u) = 0$. The γ-anchor distribution property with respect to H_0 (which is an assumption of Lemma 42) states that

$$\sum_{u \in \mathsf{N}_{H_0}(v;V_a)} \omega(u) = (1 \pm \gamma) d_{ab} \sum_{u \in V_a} \omega(u) \pm \gamma n^{0.99} .$$

Now Claim 42.4 tells us that with probability at least $1 - \exp\big(-n^{0.4}\big)$ we have

$$\sum_{u \in \mathsf{N}_{H_{s'-1}}(v;V_a)} \omega(u) = (1 \pm \gamma)(1 \pm \tfrac{1}{2}\alpha_{s'-1}) d_{ab;s'-1} \sum_{u \in V_a} \omega(u) \pm \gamma n^{0.99}$$

and hence in particular this choice of a, b, c, v does not witness a failure of the $\alpha_{s'-1}$-anchor distribution property for $F_{s'}$ in $H_{s'-1}$. Taking a union bound over the choices of a, b, c, v we see that with probability at most $\exp\big(-n^{0.3}\big)$ the $\alpha_{s'-1}$-anchor distribution property fails for $H_{s'-1}$ and $F_{s'}$.

Now fix $a, b \in \{\boxminus, \boxplus\}$, a vertex $v \in V_b$ and let ω be the weight function on $V(H_0)$ where $\omega(u) = 1$ if there is $x \in A_{s'}$ such that $\phi_{s'}(x) = u$ and the neighbour y of x in $F_{s'}$ has $\xi(y) = V_b$, and otherwise $\omega(u) = 0$. The γ-anchor distribution property with respect to H_0 (which is an assumption of Lemma 42) states that

$$\sum_{u \in \mathsf{N}_{H_0}(v;V_b)} \omega(u) = (1 \pm \gamma) d_{ab} \sum_{u \in V_b} \omega(u) \pm \gamma n^{0.99} .$$

Now Claim 42.4 tells us that with probability at least $1 - \exp\big(-n^{0.4}\big)$ we have

$$\sum_{u \in \mathsf{N}_{H_{s'-1}}(v;V_b)} \omega(u) = (1 \pm \gamma)(1 \pm \tfrac{1}{2}\alpha_{s'-1}) d_{ab;s'-1} \sum_{u \in V_b} \omega(u) \pm \gamma n^{0.99}$$

and hence in particular this choice of a, b, v does not witness a failure of the $\alpha_{s'-1}$-anchor distribution property for $F_{s'}$ in $H_{s'-1}$. Taking a union bound over the choices of a, b, v we see that with probability at most $\exp\big(-n^{0.3}\big)$ the $\alpha_{s'-1}$-anchor distribution property fails for $H_{s'-1}$ and $F_{s'}$.

Putting these together, the probability that the conditions of Lemma 40 are first not met at stage s' is at most $2\exp\big(n^{0.3}\big)$, and the probability that the conditions of Lemma 40 are met but Algorithm 2 fails at stage s' is at most $\exp\big(-n^{0.3}\big)$. Taking the union bound over the choices of s', we conclude that as desired the probability of any of these events occurring is at most $\exp\big(-n^{0.25}\big)$.

By choice of C', we immediately see that if at each stage s', including s^*, we have the above claimed quasirandomness then $(PP3)$ holds. Furthermore Claim 42.4 states that $(PP4)$ holds.

Next we prove $(PP5)$. To that end, fix S_1, S_2 sets of at most L vertices, and disjoint $T_1, T_2 \subseteq [s^*]$ of size at most L, and a pairing $V_\boxminus = \{\boxminus_i : i \in [n/2]\}$ and $V_\boxplus = \{\boxplus_i : i \in [n/2]\}$. Let X be any subset of $\mathbb{U}_H(S_1, S_2, T_1, T_2)$ of size at least νn. Observe that the elements of $T_1 \cup T_2$ are natural numbers and hence ordered, and the corresponding path-forests are embedded in that order by Algorithm 2. We let $X_0 := X$. We now define some subsets X'_j and X_j of X recursively. For $1 \le j \le |T_1| + |T_2|$, let X'_j denote the set of indices $i \in X_{j-1}$ such that $w\boxminus_i, w'\boxplus_i \in E(H_s)$ for each $w \in S_1$ and $w' \in S_2$, where s is the stage immediately before embedding the jth element of $T_1 \cup T_2$; let $X'_{|T_1|+|T_2|+1}$ be similarly defined, with $s = s^*$. For $1 \le j \le |T_1| + |T_2|$, we define $X_j \subseteq X'_j$ as follows. Let the jth element of $T_1 \cup T_2$ be s. If s is in T_1, then X_j is the set of $i \in X'_j$ such that $\boxminus_i \notin \operatorname{im}\phi_s$. If s is in T_2, then X_j is the set of $i \in X'_j$ such that $\boxplus_i \notin \operatorname{im}\phi_s$.

We show $(PP5)$ by finding ratios between sizes of each X_{j-1} and X'_j, and each X'_j and X_j. Note that $X'_{|T_1|+|T_2|+1} = \mathbb{U}'_{H_{s^*}}(S_1, S_2, T_1, T_2)$ is the set whose size we want to find, by definition. For the former, we use Claim 42.3 to estimate the effect of the sequence of embeddings of path-forests strictly between the $(j-1)$st and jth elements of $T_1 \cup T_2$. We also need to estimate the effect of removing edges when the $(j-1)$st path-forest of $T_1 \cup T_2$ is embedded (edges may be removed going to a $\boxplus_i$ while the $(j-1)$st path-forest is in T_1, for example, which is not accounted for in X_{j-1}) but these can be responsible for removing at most $2L$ vertices from X_{j-1}. For the latter, we use $(PE2)$. In each case, it will be enough to use α_{s^*} to bound error terms, which we will do for simplicity.

We claim inductively that the sets X_j, X'_j are of size at least $\nu^{5L} n$; this justifies the following applications of Claim 42.3 and $(PE2)$. First, for each $1 \le j \le |T_1| + |T_2| + 1$, we estimate $|X'_j|$. Given j, let s be the $(j-1)$st index in $T_1 \cup T_2$ (if $j = 1$, let $s = 0$) and $s'' + 1$ the jth index. Let Q denote the set of $i \in X_{j-1}$ such that $w\boxminus_i, w'\boxplus_i$ are edges of H_s for each $w \in S_1$ and $w' \in S_2$. We have $|Q| = |X_{j-1}| \pm 2L$. We apply Claim 42.3, with input $W_1 = S_1, W_2 = S_2$, $U = Q$, and s, s''. The result is that with probability at least $1 - \exp\big(-n^{0.4}\big)$, either we have

$$\begin{aligned}|X'_j| &= \big(1 \pm \tfrac{1}{2}\alpha_{s^*}\big)\big(|X_{j-1}| \pm 2L\big)\Big(\tfrac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\Big)^{|S_1\cap V_\boxminus|}\Big(\tfrac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\Big)^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}\Big(\tfrac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\Big)^{|S_2\cap V_\boxplus|}\\ &= \big(1 \pm \alpha_{s^*}\big)|X_{j-1}|\Big(\tfrac{d_{\boxminus\boxminus;s''}}{d_{\boxminus\boxminus;s}}\Big)^{|S_1\cap V_\boxminus|}\Big(\tfrac{d_{\boxminus\boxplus;s''}}{d_{\boxminus\boxplus;s}}\Big)^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}\Big(\tfrac{d_{\boxplus\boxplus;s''}}{d_{\boxplus\boxplus;s}}\Big)^{|S_2\cap V_\boxplus|}\end{aligned}$$

or there is some stage s' between s and s'' inclusive such that either the conditions of Lemma 40 are not met, or its low-probability event occurs.

Similarly, we can estimate $|X_j|$ for each $1 \le j \le |T_1| + |T_2|$. Given j, let s be the jth index in $T_1 \cup T_2$. If s is in T_1, let $a = \boxminus$ and $Q = \{\boxminus_i : i \in X'_j\}$; if s is in T_2, let $a = \boxplus$ and $Q = \{\boxplus_i : i \in X'_j\}$. We apply Lemma 40 part $(PE2)$ with input a and $V' = Q$; recall that by choice of X, we have $Q \cap U_s = \emptyset$. The result is that with probability at least $1 - \exp\big(-n^{0.3}\big)$ either we have

$$|X_j| = \big(1 \pm C\alpha_{s^*}\big)|X'_j|\frac{|V_a \setminus (U_s \cup \operatorname{im}\phi'_s)|}{|V_a \setminus U_s|}$$

or the conditions of Lemma 40 are not met, or its low-probability event occurs.

If all these equations hold, we obtain precisely the desired $(PP5)$, by choice of C'. To see this, observe that simply substituting each equation into the next, we are left with the main term of $(PP5)$, multiplied by an error term of the form $\big(1 \pm C\alpha_{s^*}\big)^{2|T_1|+2|T_2|+1}$, multiplied by a collection of terms of the form $\frac{d_{ab;s}}{d_{ab;s-1}}$ where $s \in T_1 \cup T_2$ and $a, b \in \{\boxminus, \boxplus\}$. Each of these last terms is of size $1 \pm 2\nu^{-1}n^{-1}$, and there are at most $2L(|T_1|+|T_2|)$ of them, so that the product of all these terms is $1 \pm \varepsilon$. Since the final estimate is of size at least $\nu^{5L}n$ this justifies the inductive claim on set sizes. So the probability we do not obtain $(PP5)$ is by the union bound at most $\exp\big(-n^{0.2}\big)$ as required.

Finally, we prove $(PP6)$. Given $S_a \subseteq V_a$ with $|S_a| \le L$ for each $a \in \{\boxminus, \boxplus\}$, and a set $Y \subseteq [s^*]$ such that $(S_\boxminus \cup S_\boxplus) \cap U_s = \emptyset$ for each $s \in Y$, with $|Y| \ge \nu n$, we apply Corollary 25 to estimate the number of $s \in Y$ such that $(S_\boxminus \cup S_\boxplus) \cap \operatorname{im} \phi'_s = \emptyset$. We let the good event for Corollary 25 be that at each stage s the conditions of Lemma 40 are met and its low-probability event does not occur. Within this good event, for each $s \in Y$, by $(PE3)$ the probability that $\operatorname{im} \phi'_s \cap (S_\boxminus \cup S_\boxplus) = \emptyset$, conditioned on the embeddings prior to embedding F_s, is

$$(1 \pm C\alpha_{s^*}) \prod_{a \in \{\boxminus, \boxplus\}} \left(\tfrac{|V_a \setminus (U_s \cup \operatorname{im} \phi'_s)|}{|V_a \setminus U_s|}\right)^{|S_a|}.$$

By Corollary 25, we conclude that with probability at least $1 - \exp(-n^{0.2})$ we have

$$\big|\{s \in Y : (S_\boxminus \cup S_\boxplus) \cap \operatorname{im} \phi'_s\}\big| = (1 \pm 2C\alpha_{s^*}) \sum_{s \in Y} \prod_{a \in \{\boxminus, \boxplus\}} \left(\tfrac{|V_a \setminus (U_s \cup \operatorname{im} \phi'_s)|}{|V_a \setminus U_s|}\right)^{|S_a|},$$

as required for $(PP6)$. □

8. Stage A (Proof of Lemma 30)

In this stage, the graphs $(G_s)_{s \in \mathcal{G} \setminus (\mathcal{K} \cup \mathcal{J})} \cup \big(G_s^{\spadesuit}\big)_{s \in \mathcal{K}} \cup \Big(G_s^{\parallel}\Big)_{s \in \mathcal{J}}$ are packed into H, which is very quasirandom. We do this in two steps. In the first step, we use the algorithm *PackingProcess* of [2] to pack the graphs $(G_s)_{s \in \mathcal{G} \setminus (\mathcal{K} \cup \mathcal{J})}$ into H. We let H_0 be the graph of leftover edges from this packing, which we will argue (quoting results of [2]) is still very quasirandom; we will not need any more facts about this part of the packing.

For the second step, we need to give two algorithms, *RandomEmbedding* (defined in [2]) and *PackingProcess'* (a modification of *PackingProcess* from [2]). To conveniently state these, we need some notation and to define some constants.

8.1. Notation, constants and algorithms.

Suppose G is a graph whose vertices are ordered by natural numbers, $V(G) = [\ell]$. Suppose that $i \in V(G)$. We write $\mathsf{N}^-(i) = \mathsf{N}(i) \cap [i-1]$ and $\deg^-(i) = |\mathsf{N}^-(i)|$ for the *left-neighbourhood* and the *left-degree* of i.

Definition 43. *Let G be a graph with vertex set $[v(G)]$, and H be a graph with $v(H) \ge v(G)$. Further, assume $\psi_j \colon [j] \to V(H)$ is a* partial embedding *of G into H for $j \in [v(G)]$, that is, ψ_j is a graph embedding of $G\big[[j]\big]$ into H. Finally, given $t \in [v(G)]$ we say the* candidate set *of t*

(with respect to ψ_j) is

$$C^j_{G\hookrightarrow H}(t) = \mathsf{N}_H\Big(\psi_j\big(\mathsf{N}^-_G(t)\cap[j]\big)\Big)\,.$$

If $\mathsf{N}^-_G(t)\cap[j]=\emptyset$, we obtain the common neighbourhood of the empty set, which is $V(H)$. When $j=t-1$, we call $C^j_{G\hookrightarrow H}(t)$ the final candidate set *of t.*

We now define some constants, which we will need in the following analysis and which are identically defined in [1] and [2] (so that we can quote results from both verbatim).

We shall work with $d_{\mathbf{A}}$ given by (9). Given D and $\delta,\gamma_{\mathbf{A}}>0$, we choose $0<\gamma<d_{\mathbf{A}}$ small enough for two inequalities, which we give after defining the following (copied from [1] and [2], with the exceptions that $\tilde\delta$ and ξ' defined below are there δ and ξ respectively).

Setting 44. *Let $D,n\in\mathbb{N}$ and $\gamma>0$ be given. We define*

$$\begin{aligned}
&\eta=\frac{\gamma^D}{200D}\,,\quad \tilde\delta=\frac{\gamma^{10D}\eta}{10^6D^4}\,,\quad C=40D\exp\big(1000D\tilde\delta^{-2}\gamma^{-2D-10}\big)\,,\quad C'=10^4C\tilde\delta^{-1}\,,\\
&\alpha_x=\frac{\tilde\delta}{10^8CD}\exp\Big(\frac{10^8CD^3\tilde\delta^{-1}(x-2n)}{n}\Big)\qquad \textit{for each } x\in\mathbb{R},\\
&\varepsilon=\alpha_0\tilde\delta^4\gamma^{10D}/1000CD\,,\quad c=D^{-4}\varepsilon^4/100\quad \textit{and}\quad \xi'=\alpha_0/100\,.
\end{aligned}\tag{32}$$

We require γ small enough that $\tilde\delta\le\frac12\delta$. In addition, we need

$$10^6CC'D\tilde\delta^{-1}\delta^{-1}\gamma^{-1}\alpha_{3n/2}\le\gamma_{\mathbf{A}}\,.\tag{33}$$

Since several of the terms on the left hand side of this inequality depend on γ, it may not be obvious that the left hand side indeed tends to zero as $\gamma\to0$. We have

$$\begin{aligned}
10^6CC'D\tilde\delta^{-1}\delta^{-1}\gamma^{-1}\alpha_{3n/2}&=100C\tilde\delta^{-2}\delta^{-1}\gamma^{-1}\exp\Big(-\tfrac12\cdot10^8CD^3\tilde\delta^{-1}\Big)\\
&\le100C^3\tilde\delta^{-3}\delta^{-1}\exp\Big(-\tfrac12\cdot10^8CD^3\tilde\delta^{-1}\Big)\,.
\end{aligned}$$

Now δ is a fixed constant, while both C and $\tilde\delta^{-1}$, and so also $C\tilde\delta^{-1}$, tend to infinity as $\gamma\to0$. Since any exponential in $C\tilde\delta^{-1}$ is eventually much larger than any polynomial in $C\tilde\delta^{-1}$, indeed this quantity tends to 0 as $\gamma\to0$, and hence there is a choice of $\gamma>0$ such that (33) holds.

We briefly state the purpose of a few of these quantities which are immediately important. We will pack D-degenerate graphs on up to $n-\tilde\delta n$ vertices into an n-vertex graph; after completing the packing, at least γn^2 edges will remain unused. When we borrow analysis from [1] and [2], the quantity α_s (or a multiple of it, usually $C\alpha_s$) will upper bound the errors we have made after packing s graphs; we pack in total at most $\frac32n$ graphs, which is why (33) is what we need to ensure all the error terms in our final analysis are small compared to $\gamma_{\mathbf{A}}$.

We next recall *RandomEmbedding* from [2], which we repeat below. This algorithm takes a graph G with $n-\tilde\delta n$ vertices, and (if successful) embeds it into an n-vertex graph H. We will want to use it to embed graphs with at most $n-\delta n<n-\tilde\delta n$ vertices, so we add isolated vertices to each graph we want to embed.

Algorithm 3: *RandomEmbedding*

Input: graphs G and H, with $V(G) = [v(G)]$ and $v(H) = n$
$\psi_0 := \emptyset$;
$t^* := (1 - \tilde{\delta})n$;
for $t = 1$ **to** t^* **do**
  if $C^{t-1}_{G \hookrightarrow H}(t) \setminus \mathrm{im}(\psi_{t-1}) = \emptyset$ **then** halt with failure;
  choose $v \in C^{t-1}_{G \hookrightarrow H}(t) \setminus \mathrm{im}(\psi_{t-1})$ uniformly at random;
  $\psi_t := \psi_{t-1} \cup \{t \hookrightarrow v\}$;
end
return ψ_{t^*}

In [2] a packing is obtained by an algorithm *PackingProcess* which runs *RandomEmbedding* repeatedly, but which does something in addition in order to allow for packing spanning graphs. Nevertheless, the main task of [2] is to analyse the repeated running of *RandomEmbedding*, and this analysis, and similar analysis in [1], is also valid for the following *PackingProcess*' which simply runs *RandomEmbedding* repeatedly.

Algorithm 4: *PackingProcess′*

Input: • graphs $G_1, \dots, G_{s^*}$, with G_s on vertex set $[v(G_s)]$
 • a graph H_0 on vertex set of order n
for $s = 1$ **to** s^* **do**
  run *RandomEmbedding*(G_s,H_{s-1}) to get an embedding ϕ_s of G_s into H_{s-1};
  let H_s be the graph obtained from H_{s-1} by removing the edges of $\phi_s(G_s)$;
end

8.2. **Packing and maintaining quasirandomness.** We first prove that with high probability the process outlined above does return a packing of $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})} \cup \big(G_s^{\spadesuit}\big)_{s\in\mathcal{K}} \cup \Big(G_s^{||}\Big)_{s\in\mathcal{J}}$, and furthermore after embedding each graph, what is left of H remains quasirandom.

For convenience, we will let H_0 be the spanning subgraph of H which we obtain by using [2, Theorem 11] to pack $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}$, and we will renumber $\mathcal{K}\cup\mathcal{J}$ to be the integers $[s^*]$. Note that $s^* \leq \frac{3}{2}n$. This is since $|\mathcal{K}| \leq \big|\bigcup_{s\in\mathcal{K}} |\mathsf{OddVert}_s|\big| \leq (1+2\alpha)n$ and $|\mathcal{J}| \leq \lambda n$.

Proof of $(A.i)$ and $(\alpha_s, 2D+3)$-quasirandomness of H_s. Let ξ' be as defined in Setting 44. We choose $\eta^* > 0$ such that the graphs of $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}$ have at most $e(H) - \eta^* n^2$ edges, and in addition $\eta^* \leq \frac{1}{20D+30}\xi'$. Let $0 < \xi \leq \xi'$ and $c' > 0$ be small enough for [2, Theorem 11] with these constants. The constants Lemma 30 returns are then ξ and $\min(c, c')$.

Given H, which is $(\xi, 2D+3)$-quasirandom, we first run *PackingProcess* to pack $(G_s)_{s\in\mathcal{G}\setminus(\mathcal{K}\cup\mathcal{J})}$ into H. [2, Theorem 11] states that this packing a.a.s. succeeds. Let H_0 be the graph of edges of H not used in this packing. Recall that H_0 is composed of what remains of the 'bulk' and 'reservoir'. Now [2, Lemma 17] states that the remainder H' of the bulk is $\big(\frac{1}{2}\alpha_0, 2D+3\big)$-quasirandom, while the reservoir H'' has maximum degree $\frac{1}{20D+30}\xi' n$. The size of $\mathsf{N}_{H_0}(v_1, \dots, v_\ell)$

is thus within $\frac{\ell}{20D+30}\xi' n$ of $\big|\mathsf{N}_{H'}(v_1,\dots,v_\ell)\big|$. Since H' is $(\frac{1}{2}\xi', 2D+3)$-quasirandom, it follows that H_0 is $(\xi', 2D+3)$-quasirandom.

We now run *PackingProcess'* to pack the graphs $\big(G_s^{\spadesuit}\big)_{s\in\mathcal{K}} \cup \big(G_s^{||}\big)_{s\in\mathcal{J}}$ into H_0. For convenience, we renumber these graphs such that $\mathcal{K}\cup\mathcal{J}=[s^*]$. For much of the analysis, we will not want to distinguish between the graphs $G_s^{\spadesuit}$ and $G_s^{||}$, so we define graphs G'_s as follows. For $s\in\mathcal{K}$, we set $G'_s = G_s^{\spadesuit}$ (and so $v(G'_s)\le(1-\delta)n$), and for $s\in\mathcal{J}$ we set $G'_s = G_s^{||}$ and get $v(G'_s)=(1-\delta-10\sigma_1)n$ for each $s\in\mathcal{J}_1$, $v(G'_s)=(1-\delta-10\lambda)n$ for each $s\in\mathcal{J}_0$, and $v(G'_s)=(1-\delta-6\lambda)n$ for each $s\in\mathcal{J}_2$.

For each s let H_s be the graph obtained from H_{s-1} by removing the edges used by *PackingProcess'* in embedding G'_s. Let $\phi_s^{\mathbf{A}}$ be the embedding of G'_s. Observe that [2, Lemma 18] states that, provided H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom, then *RandomEmbedding* is unlikely to fail in packing G'_s. We claim that the proof of [2, Lemma 17] works (trivially, by simply ignoring each mention of the 'reservoir') to show that indeed it is unlikely that H_{s-1} is not $(\alpha_{s-1}, 2D+3)$-quasirandom, so that taking a union bound (for each choice of s, over the two unlikely events just mentioned) we conclude that *PackingProcess'* a.a.s. succeeds. However, the reader can also verify that H_{s-1} is unlikely to fail $(\alpha_{s-1}, 2D+3)$-quasirandomness from Lemma 50 below (taking T to be the entire common neighbourhood, and using the union bound). □

What remains is to prove that the remaining properties of Lemma 30 a.a.s. hold, which requires a more careful analysis of *PackingProcess*.

8.3. **Behaviour of the random processes.** We now need to quote some lemmas from [1] and [2] which establish some useful properties of *RandomEmbedding* and *PackingProcess*. We also need one additional concept, the cover condition, which we will see holds for a typical run of *RandomEmbedding*.

Definition 45 (cover condition)**.** *Suppose that G and H are two graphs such that H has order n, the vertex set of G is $[n]$, and H has density p. Suppose that numbers $\beta,\varepsilon>0$ and $i\in[n-\varepsilon n]$ are given. For each $d\in\mathbb{N}$ we define*

$$X_{i,d} := \{x\in V(G)\colon i\le x< i+\varepsilon n, |\mathsf{N}^-(x)|=d\}\,.$$

We say that a partial embedding ψ of G into H, which embeds $\mathsf{N}^-(x)$ for each $i\le x<i+\varepsilon n$, satisfies the (ε,β,i)-cover condition *if for each $v\in V(H)$ such that $v\notin\psi_{i+\varepsilon n-1}$, and for each $d\in\mathbb{N}$, if we have*

$$\Big|\big\{x\in X_{i,d} : v\in \mathsf{N}_H\big(\psi(\mathsf{N}^-(x))\big)\big\}\Big| = (1\pm\beta)p^d|X_{i,d}|\pm\varepsilon^2 n\,.$$

Note that a corresponding condition for $d=0$ is trivial, even with zero error parameters.

The following lemma puts together various results from [1] and [2] which describe the typical behaviour of *RandomEmbedding*.

Lemma 46. *Assume Setting 44, and let $\alpha \in [\alpha_0, \alpha_{2n}]$ be arbitrary. For each $t \in \mathbb{R}$ we define*

$$\beta_t = 2\alpha \exp\left(\tfrac{1000D\tilde{\delta}^{-2}\gamma^{-2D-10}t}{n}\right) .$$

Let G be a graph on vertex set $[v(G)]$ with at most $(1-\tilde{\delta})n$ vertices and maximum degree at most $cn/\log n$ such that $\deg^-(x) \le D$ for each $x \in V(G)$, and let H be an $(\alpha, 2D+3)$-quasirandom n-vertex graph with at least $\gamma\binom{n}{2}$ edges. Fix $k \le 2D+3$, a vertex $x \in V(G)$, distinct vertices $v, v', v'' \in V(H)$ and distinct neighbours $u_1, \dots, u_k$ of v.

Then with probability at least $1 - 2n^{-9}$ all of the following good events hold.

(*a*) *When* RandomEmbedding *is run it does not fail and generates a sequence $(\psi_i)_{i\in[v(G)]}$ of partial embeddings of G into H.*

(*b*) *For each $t \in [v(G)]$ the pair $(H, \operatorname{im}\psi_t)$ satisfies the $(\beta_t, 2D+3)$-diet condition.*

(*c*) *The embedding $\psi_{v(G)}$ of G into H satisfies the $(\varepsilon, 20D\beta_{t-\varepsilon n+2}, t-\varepsilon n+2)$-cover condition for each $t \in [\varepsilon n - 1, n - \tilde{\delta}n]$.*

Note that we have $20D\beta_t \le C\alpha$ for each $0 \le t \le n$. We have

$$\mathbf{P}\big[x \hookrightarrow v\big] = (1 \pm 10^4 C\alpha D\tilde{\delta}^{-1})n^{-1} , \tag{34}$$

$$\mathbf{P}\big[x \hookrightarrow v \text{ and } v' \notin \operatorname{im}\psi_{v(G)}\big] = (1 \pm 10^4 C\alpha D\tilde{\delta}^{-1})\big(n - v(G)\big)n^{-2} \qquad \text{and} \tag{35}$$

$$\mathbf{P}\big[x \hookrightarrow v \text{ and } v', v'' \notin \operatorname{im}\psi_{v(G)}\big] = (1 \pm 10^4 C\alpha D\tilde{\delta}^{-1})\big(n - v(G)\big)^2 n^{-3} . \tag{36}$$

Finally, the probability that there is at least one $u_i v$ to which some edge of G is embedded is

$$\big(1 \pm 1000C\alpha\tilde{\delta}^{-1}\big)^{4D+2} p^{-1} n^{-2} \cdot 2ke(G) . \tag{37}$$

□

The part of this lemma showing that (*a*)—(*c*) are likely are part of [1, Lemma 36]. The statement (34) is [1, Lemma 40]. The equations (35) and (36) follow from the method of [2, Lemma 29] but for completeness we give a proof in the following section. Finally (37) is given by [2, Lemma 30].

We should note that the quantities β_t defined in this lemma are needed only for the more precise bounds in (*b*) and (*c*). We will only need these more precise bounds in two places, namely when we state Lemma 51 below and apply the following lemma from [1] in its proof, and when we apply Lemma 51 to prove (*A.ii*). Otherwise, when we use (*b*) and (*c*) we will need only the weaker bound $C\alpha$.

Lemma 47 ([1, Lemma 37])**.** *We assume Setting 44. Given $\alpha_0 \le \alpha \le \alpha_{2n}$, for each $t \in \mathbb{R}$ we define*

$$\beta_t = 2\alpha \exp\left(\tfrac{1000D\tilde{\delta}^{-2}\gamma^{-2D-10}t}{n}\right) .$$

Let G be a graph on vertex set $[v(G)]$ with at most $(1-\tilde{\delta})n$ vertices and maximum degree at most $cn/\log n$ such that $\deg^-(x) \le D$ for each $x \in V(G)$, and let H be an $(\alpha, 2D+3)$-quasirandom n-vertex graph with at least $\gamma\binom{n}{2}$ edges. When we run RandomEmbedding *to embed G into H, suppose that it produces a partial embedding ψ_j such that $(H, \operatorname{im}\psi_j)$ has the $(\beta_j, 2D+3)$-diet* *condition, and let* $T \subseteq V(H) \setminus \operatorname{im} \psi_j$ *with* $|T| \geq \frac{1}{2}\gamma^{2D+3}\tilde{\delta}n$*. Conditioning on* ψ_j*, with probability at least* $1 - 2n^{-2D-19}$*, one of the following occurs.*

(*a*) $\psi_{v(G)}$ *does not have the* $(\varepsilon, 20D\beta_j, j)$*-cover condition, or*

(*b*) $\big|\{x \colon j \leq x < j + \varepsilon n, \psi_{t-1}(x) \in T\}\big| = (1 \pm 40D\beta_j)\frac{|T|\varepsilon n}{n-j}$.

8.4. **Seminovo analysis.** In this subsection we give some more analysis of *PackingProcess*' which is similar to results of [1] and [2]; the proofs are copied and modified appropriately.

We first need a strengthening of [2, Lemma 28], allowing us to estimate accurately the probability that not just one or two but a collection of up to D vertices of H are avoided in any given interval of vertices of G that *RandomEmbedding* embeds. To begin with, we show the statement for intervals of length εn, a modification of [2, Lemma 27].

Lemma 48. *Assume Setting 44. The following holds for any* $\alpha_0 \leq \alpha \leq \alpha_{2n}$ *and all sufficiently large* n*. Suppose that* G *is a graph on* $[v(G)]$ *with* $v(G) \leq (1-\delta)n$ *such that* $\deg^-(x) \leq D$ *for each* $x \in V(G)$*, and* H *is an* $(\alpha, 2D+3)$*-quasirandom graph with* n *vertices and* $p\binom{n}{2}$ *edges, with* $p \geq \gamma$*. Suppose that* $1 \leq k \leq D$ *and* $u_1, \dots, u_k$ *are any distinct vertices of* H*. When* RandomEmbedding *is run to embed* G *into* H*, for any* $1 \leq t' \leq v(G)+1-\varepsilon n$ *the following holds. Suppose the history* $\mathscr{H}_{t'-1}$ *up to and including embedding* $t'-1$ *is such that* $u_1, \dots, u_k \notin \operatorname{im} \psi_{t'-1}$*, the* $(C\alpha, 2D+3)$*-diet condition holds for* $(H, \operatorname{im} \psi_{t'-1})$*, and*

$$\mathbf{P}\big[(H, \psi_{t'+\varepsilon n-1}) \text{ does not satisfy the } (C\alpha, \varepsilon, t')\text{-cover condition}\big|\mathscr{H}_{t'-1}\big] \leq n^{-3}\,.$$

Then we have

$$\mathbf{P}\Big[\big|\{u_1, \dots, u_k\} \cap \operatorname{im} \psi_{t'+\varepsilon n-1}\big| \geq 1\Big|\mathscr{H}_{t'-1}\Big] = (1 \pm 10C\alpha)\tfrac{k\varepsilon n}{n-t'}\,.$$

The proof of this lemma is a straightforward modification of the proof of [2, Lemma 27], which we include for completeness.

Proof. We modify *RandomEmbedding* as follows to obtain *Modified RandomEmbedding*. At the line where *RandomEmbedding* chooses $v \in C^{t-1}_{G \hookrightarrow H}(t) \setminus \operatorname{im}(\psi_{t-1})$ uniformly at random, we choose instead

$$w \in C^{t-1}_{G \hookrightarrow H}(t) \setminus \big(\operatorname{im}(\psi_{t-1}) \setminus \{u_1, \dots, u_k\}\big)$$

uniformly at random and *report* w. If $w \in \operatorname{im}(\psi_{t-1})$, then we choose $v \in C^{t-1}_{G \hookrightarrow H}(t) \setminus \operatorname{im}(\psi_{t-1})$ uniformly at random; if not, we set $v = w$. We then embed t to v and continue as in *RandomEmbedding*.

Observe that the distribution over embeddings created by *Modified RandomEmbedding* is identical to that created by *RandomEmbedding*. The difference is that *Modified RandomEmbedding* in addition creates a string of reported vertices. For each t, let $r(t)$ be the vertex reported by *Modified RandomEmbedding* at time t. Note that $r(t)$ is the vertex to which t is embedded, except when $r(t)$ is one of the u_i and that u_i is in $\operatorname{im}(\phi_{t-1})$. We shall use the following two auxiliary claims.

Let t' and $\mathscr{H}_{t'-1}$ be as in the lemma statement. Define E as the random variable counting the times when some u_i is reported by *Modified RandomEmbedding* in the interval $t' \le x < t' + \varepsilon n$,

$$E = \Big| \big\{ x \in [t', t'+\varepsilon n) \,:\, r(x-1) \in \{u_1, \dots, u_k\} \big\} \Big| \ .$$

The probability that *RandomEmbedding* uses at least one vertex of $\{u_1, \dots, u_k\}$ in the interval $t' \le x < t' + \varepsilon n$, conditioning on $\mathscr{H}_{t'-1}$, is equal to the probability that *Modified RandomEmbedding* reports some vertex of $\{u_1, \dots, u_k\}$ at least once in that interval, which probability is by definition at least

$$\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right) - \sum_{j=2}^{\varepsilon n} \mathbf{P}\big[\text{vertices of } \{u_1, \dots, u_k\} \text{ are reported} \\ \text{at least } j \text{ times in the interval } [t', t'+\varepsilon n) \big| \mathscr{H}_{t'-1}\big] \,.$$

Our first claim estimates $\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right)$.

Claim 48.1. *We have that*

$$\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right) = (1 \pm 4C\alpha)\frac{\varepsilon k n}{n-t'} \pm 8D(D+1)\varepsilon^2\gamma^{-2D}\delta^{-2} \,.$$

Our second claim is that the sum in the expression above is small.

Claim 48.2. *We have that*

$$\sum_{j=2}^{\varepsilon n} \mathbf{P}\big[\textit{vertices of } \{u_1, \dots, u_k\} \textit{ are reported} \\ \textit{at least } j \textit{ times in the interval } [t', t'+\varepsilon n) \big| \mathscr{H}_{t'-1}\big] \le 8k^2\varepsilon^2\gamma^{-2D}\delta^{-2} \,.$$

By choice of ε, we have $16D(D+1)\varepsilon^2\gamma^{-2D}\delta^{-2} < C\alpha\varepsilon\delta$. Thus the two claims give Lemma 48. We now prove the auxiliary Claims 48.1 and 48.2.

Proof of Claim 48.1. Note that since the $(C\alpha, 2D+3)$-diet condition holds for $(H, \operatorname{im}\psi_{t'-1})$, for each $t' \le x < t' + \varepsilon n$, setting $S = \psi_{x-1}(\mathsf{N}^-(x))$, we have

$$\begin{aligned} \left|C^{x-1}(x) \setminus \operatorname{im}\psi_{x-1}\right| \pm 2 &= \left|\mathsf{N}_H(S) \setminus \operatorname{im}\psi_{t'-1}\right| \pm \varepsilon n \pm k \\ &= (1 \pm C\alpha)p^{|\mathsf{N}^-(x)|}(n-t') \pm \varepsilon n \pm k \qquad (38) \\ &= (1 \pm 2C\alpha)p^{|\mathsf{N}^-(x)|}(n-t') \,. \end{aligned}$$

We first give a simple upper bound on $\mathbf{P}\big[\left|\{u_1, \dots, u_k\} \cap \operatorname{im}\psi_{t'+\varepsilon n-1}\right| \ge 1 \,\big|\, \mathscr{H}_{t'-1}\big]$. When we embed any one vertex x with $t' \le x \le t' + \varepsilon n - 1$, the probability of embedding x to $\{u_1, \dots, u_k\}$ is at most $k\left|C^{x-1}(x) \setminus \operatorname{im}\psi_{x-1}\right|^{-1}$. Using (38) and summing over the εn choices of x, we have

$$\mathbf{P}\big[\left|\{u_1, \dots, u_k\} \cap \operatorname{im}\psi_{t'+\varepsilon n-1}\right| \ge 1 \,\big|\, \mathscr{H}_{t'-1}\big] \le \frac{k\varepsilon n}{\frac{1}{2}p^D\delta n} \le 2k\varepsilon\gamma^{-D}\delta^{-1} \,.$$

By linearity of expectation, we have

$$\begin{aligned}\mathbf{E}\Big[E \mid \mathscr{H}_{t'-1}\Big] &= \sum_{x=t'}^{t'+\varepsilon n-1} \mathbf{P}\big[\text{a vertex of } \{u_1,\dots,u_k\} \text{ is reported at time } x \big| \mathscr{H}_{t'-1}\big] \\ &= \sum_{x=t'}^{t'+\varepsilon n-1} \sum_{j=1}^{k} \mathbf{P}\big[u_j \text{ is reported at time } x \big| \mathscr{H}_{t'-1}\big] \\ &= \sum_{x=t'}^{t'+\varepsilon n-1} \sum_{j=1}^{k} \mathbf{E}\left(\frac{\mathbb{1}\{u_j \in C^{x-1}(x)\}}{|C^{x-1}(x) \setminus (\operatorname{im}\psi_{x-1} \setminus \{u_1,\dots,u_k\})|} \,\Big| \mathscr{H}_{t'-1}\right) \\ &= \sum_{x=t'}^{t'+\varepsilon n-1} \sum_{j=1}^{k} \mathbf{E}\left(\frac{\mathbb{1}\{u_j \in C^{x-1}(x)\}}{|C^{x-1}(x) \setminus \operatorname{im}\psi_{x-1}| \pm k} \,\Big| \mathscr{H}_{t'-1}\right) . \end{aligned} \tag{39}$$

Using (38), we get

$$\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right) = \sum_{x=t'}^{t'+\varepsilon n-1} \sum_{j=1}^{k} \frac{\mathbf{P}\big[u_j \in C^{x-1}(x) \big| \mathscr{H}_{t'-1}\big]}{(1 \pm 2C\alpha) p^{|\mathsf{N}^-(x)|}(n-t')} .$$

Splitting this sum up according to $|\mathsf{N}^-(x)|$, and again using linearity of expectation, we have

$$\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right) = \sum_{j=1}^{k}\sum_{d=0}^{D} \frac{\mathbf{E}\left(|\{x \in X_{t',d} : u_j \in C^{x-1}(x)\}| \big| \mathscr{H}_{t'-1}\right)}{(1 \pm 2C\alpha)p^d (n-t')} .$$

Now fix $0 \le d \le D$ and each $j \in [k]$. If the $(\varepsilon, C\alpha, t')$-cover condition holds, and if $v \notin \operatorname{im}\psi_{t'+\varepsilon n-1}$, we have $\big|\{x \in X_{t,d} : v \in C^{x-1}(x)\}\big| = \big(1 \pm C\alpha\big)p^d |X_{t,d}| \pm \varepsilon^2 n$. If the $(\varepsilon, C\alpha, t')$-cover condition fails, or if $v \in \operatorname{im}\psi_{t+\varepsilon n-1}$ (which occur with total probability $n^{-3} + 2k\varepsilon\gamma^{-D}\delta^{-1}$), we have $0 \le \big|\{x \in X_{t,d} : v \in C^{x-1}(x)\}\big| \le \varepsilon n$. In particular, in this case we have

$$\big|\{x \in X_{t,d} : v \in C^{x-1}(x)\}\big| = \big(1 \pm C\alpha\big)p^d |X_{t,d} \pm \varepsilon^2 n \pm \varepsilon n .$$

Putting these together, we get

$$\begin{aligned}\mathbf{E}\left(|\{x \in X_{t',d} : u_j \in C^{x-1}(x)\}| \big| \mathscr{H}_{t'-1}\right) &= \big((1 \pm C\alpha)p^d |X_{t',d}| \pm \varepsilon^2 n\big) \pm \big(n^{-3} + 2k\varepsilon\gamma^{-D}\delta^{-1}\big) \cdot \varepsilon n \\ &= (1 \pm C\alpha)p^d |X_{t',d}| \pm 4k\gamma^{-D}\delta^{-1}\varepsilon^2 n .\end{aligned}$$

Substituting this in, we have

$$\begin{aligned}\mathbf{E}\left(E \mid \mathscr{H}_{t'-1}\right) &= \sum_{j=1}^{k}\sum_{d=0}^{D} \frac{(1 \pm C\alpha)p^d |X_{t',d}| \pm 4k\gamma^{-D}\delta^{-1}\varepsilon^2 n}{(1 \pm 2C\alpha)p^d (n-t')} \\ &= (1 \pm 4C\alpha)\tfrac{\varepsilon k n}{n-t'} \pm 8D(D+1)\varepsilon^2\gamma^{-2D}\delta^{-2} ,\end{aligned}$$

where the last equality uses $k \le D$, $p \ge \gamma$ and $n - t' \ge \delta n$. □

Proof of Claim 48.2. Since the $(C\alpha, 2D+3)$-diet condition holds for $(H, \operatorname{im}\psi_{t'-1})$, since $p \ge \gamma$, and since $n - t' \ge \delta n$, for each $x \in [t', t' + \varepsilon n)$, when we embed x we report a uniform random vertex from a set of size at least $\frac{1}{2}\gamma^D \delta n$. The probability of reporting one of $u_1, \dots, u_k$ when we embed x is thus at most $2k\gamma^{-D}\delta^{-1}n^{-1}$, conditioning on $\mathscr{H}_{t'-1}$ and any embedding of the vertices $[t',x)$. Since the conditional probabilities multiply, the probability that at each of a given j-set of vertices in $[t',t'+\varepsilon n)$ we report a vertex of $\{u_1,\dots,u_k\}$ is at most $2^jk^j\gamma^{-jD}\delta^{-j}n^{-j}$. Taking the union bound over choices of j-sets, we have

$$\sum_{j=2}^{\varepsilon n}\mathbf{P}\big[\text{vertices of } \{u_1,\dots,u_k\} \text{ are reported at least } j \text{ times in the interval } [t',t'+\varepsilon n)\big|\mathscr{H}_{t'-1}\big]$$
$$\le\sum_{j=2}^{\varepsilon n}\binom{\varepsilon n}{j}2^jk^j\gamma^{-jD}\delta^{-j}n^{-j}\le\sum_{j=2}^{\varepsilon n}\big(2k\varepsilon\gamma^{-D}\delta^{-1}\big)^j\le\tfrac{4k^2\varepsilon^2\gamma^{-2D}\delta^{-2}}{1-2k\varepsilon\gamma^{-D}\delta^{-1}}\le 8k^2\varepsilon^2\gamma^{-2D}\delta^{-2}\,,$$

where we use the bound $\binom{\epsilon n}{j}\le(\epsilon n)^j$ and sum the resulting geometric series. □

This completes the proof of Lemma 48. □

We now deduce a similar result for intervals of any length starting at a given time $t_0\ge 0$, again following [2, Lemma 28].

Lemma 49. *Assume Setting 44. Then the following holds for any $\alpha_0\le\alpha\le\alpha_{2n}$ and all sufficiently large n. Suppose that G is a graph on $[v(G)]$ with $v(G)\le(1-\delta)n$ such that $\deg^-(x)\le D$ for each $x\in V(G)$, and H is an $(\alpha,2D+3)$-quasirandom graph with n vertices and $p\binom{n}{2}$ edges, with $p\ge\gamma$. Let $0\le t_0<t_1\le v(G)$ and let $k\in[D]$. Let $\mathscr{L}$ be a history ensemble of* RandomEmbedding *up to time t_0, and suppose that $\mathbf{P}[\mathscr{L}]\ge n^{-4}$. Then for any distinct vertices $u_1,\dots,u_k\in V(H)$ such that $u_1,\dots,u_k\notin\operatorname{im}\psi_{t_0}$ we have*

$$\mathbf{P}[u_1,\dots,u_k\notin\operatorname{im}\psi_{t_1}|\mathscr{L}]=(1\pm 100CD\alpha\delta^{-1})\big(\tfrac{n-1-t_1}{n-t_0}\big)^k\,.$$

Proof. We divide the interval $(t_0,t_1]$ into $j:=\lceil(t_1-t_0)/\varepsilon n\rceil$ intervals, all but the last of length εn. Let $\mathscr{L}_0:=\mathscr{L}$. Let, for each $1\le i<j$, the set $\mathscr{L}_i$ be the embedding histories up to time $t_0+i\varepsilon n$ of *RandomEmbedding* which extend histories in $\mathscr{L}_{i-1}$ and are such that $u_1,\dots,u_k\notin\psi_{i\varepsilon n}$. Let $\mathscr{L}_j$ be the embedding histories up to time t_1 extending those in $\mathscr{L}_{j-1}$ such that $u_1,\dots,u_k\notin\psi_{t_1}$. Thus we have

$$\mathbf{P}[u_1,\dots,u_k\notin\operatorname{im}\psi_{t_1}]=\mathbf{P}[\mathscr{L}_j]\,.$$

Finally, for each $1\le i\le j$, let the set $\mathscr{L}'_{i-1}$ consist of all histories in $\mathscr{L}_{i-1}$ such that the $(C\alpha,2D+3)$-diet condition holds for $(H,\operatorname{im}\psi_{t_0+(i-1)\varepsilon n})$ and the probability that the $(\varepsilon,C\alpha,t_0+1+(i-1)\varepsilon n)$-cover condition fails, conditioned on $\psi_{t_0+(i-1)\varepsilon n}$, is at most n^{-3}. In other words, $\mathscr{L}'_i$ is the subset of $\mathscr{L}_i$ consisting of typical histories, satisfying the conditions of Lemma 48.

We now determine $\mathbf{P}[\mathscr{L}_j]$, and in particular we show inductively that $\mathbf{P}[\mathscr{L}_i]>n^{-5}$ for each i. Observe that for any time t, the probability (not conditioned on any embedding) that either the $(C\alpha,2D+3)$-diet condition fails for $(H,\operatorname{im}\psi_i)$ for some $i\le t$ or that the $(\varepsilon,C\alpha,t+1)$-cover condition has probability greater than n^{-3} of failing, is at most $2n^{-6}$ by Lemma 46. In other words, for each i we have $\mathbf{P}[\mathscr{L}_i \setminus \mathscr{L}_i'] \le 2n^{-6}$. Thus by Lemma 48 we have

$$\begin{aligned}\mathbf{P}[\mathscr{L}_i] &= \big(1-(1\pm 10C\alpha)\tfrac{\varepsilon kn}{n-(i-1)\varepsilon n}\big)\mathbf{P}[\mathscr{L}_{i-1}'] \pm 2n^{-6}\\ &= \big(1-(1\pm 10C\alpha)\tfrac{\varepsilon n}{n-(i-1)\varepsilon kn}\big)\big(\mathbf{P}[\mathscr{L}_{i-1}] \pm 2n^{-6}\big) \pm 2n^{-6}\\ &= \big(1-(1\pm 20C\alpha)\tfrac{\varepsilon n}{n-(i-1)\varepsilon kn}\big)\mathbf{P}[\mathscr{L}_{i-1}]\,,\end{aligned}$$

where the final equality uses the lower bound $\mathbf{P}[\mathscr{L}_{i-1}] \ge n^{-5}$. Similarly, we have $\mathbf{P}[\mathscr{L}_j] = \big(1\pm(1+20C\alpha)\tfrac{\varepsilon kn}{n-t_1}\big)\mathbf{P}[\mathscr{L}_{j-1}]$.

Putting these observations together, we can compute $\mathbf{P}[\mathscr{L}_j]$:

$$\mathbf{P}[\mathscr{L}_j] = \mathbf{P}[\mathscr{L}]\big(1\pm(1+20C\alpha)\tfrac{\varepsilon kn}{n-t_1}\big)\prod_{i=1}^{j-1}\Big(1-(1\pm 20C\alpha)\tfrac{\varepsilon kn}{n-t_0-(i-1)\varepsilon n}\Big)\,.$$

Observe that the approximation $\log(1+x) = x \pm x^2$ is valid for all sufficiently small x. In particular, since $n-(i-1)\varepsilon n \ge n-t_1 \ge \delta n$ and by choice of ε, for each i we have

$$\log\Big(1-(1\pm 20C\alpha)\tfrac{\varepsilon kn}{n-t_0-(i-1)\varepsilon n}\Big) = -(1\pm 30C\alpha)\tfrac{\varepsilon kn}{n-t_0-(i-1)\varepsilon n}\,.$$

Thus we obtain

$$\begin{aligned}\log\mathbf{P}[\mathscr{L}_j] &= \log\mathbf{P}[\mathscr{L}] \pm (1+30C\alpha)\tfrac{\varepsilon kn}{n-t_1} - \sum_{i=1}^{j-1}(1\pm 30C\alpha)\tfrac{\varepsilon kn}{n-t_0-(i-1)\varepsilon n}\\ &= \log\mathbf{P}[\mathscr{L}] \pm 2\delta^{-1}\varepsilon - (1\pm 40C\alpha)\int_{x=t_0}^{t_0+(j-1)\varepsilon n}\tfrac{k}{n-t_0-x}\,\mathrm{d}x\\ &= \log\mathbf{P}[\mathscr{L}] \pm 2\delta^{-1}\varepsilon - (1\pm 50C\alpha)k\big(\log(n-t_0)-\log(n-1-t_1)\big)\\ &= \log\mathbf{P}[\mathscr{L}] + \log\big(\tfrac{n-1-t_1}{n-t_0}\big)^k \pm 2\delta^{-1}\varepsilon \pm 50Ck\alpha\log\delta^{-1}\,, \qquad (40)\end{aligned}$$

where we use $t_1 \le n-\delta n$, and we justify that the integral and sum are close by observing that for each i in the summation, if $t_0+(i-1)\varepsilon n \le t_0+x \le t_0+i\varepsilon n$ then we have

$$\tfrac{1}{n-t_0-i\varepsilon n} \le \tfrac{1}{n-t_0-x} \le \tfrac{1}{n-t_0-(i-1)\varepsilon n} \le (1+\alpha)\tfrac{1}{n-t_0-i\varepsilon n}\,,$$

where the final inequality uses $n-t_0-i\varepsilon n \le n-t_1 \le \delta n$ and the choice of ε. By choice of ε and since $k \le D$, this gives the lemma statement. Furthermore, (40), and the fact $t_1 \le n-\delta n$, imply that $\mathbf{P}[\mathscr{L}_j] \ge n^{-5}$. Since the $\mathscr{L}_i$ form a decreasing sequence of events the same bound holds for each $\mathscr{L}_i$. □

Now, we prove (35) and (36), by copying the proof of [2, Lemma 29] and modifying it very slightly (simply by removing the vertex y and references to it).

Proof of (35) *and* (36). Let $x \in V(G)$ and three distinct vertices $v, v', v'' \in V(H)$ be given. We begin with (35). Let $z_1, \dots, z_k$ be the vertices of $\mathsf{N}^-(x)$ in increasing order. Define time intervals using $z_1, \dots, z_k, x$ as separators: $I_0 = [1, z_1-1]$, $I_1 = [z_1+1, z_2-1]$, $\dots$, $I_k = [z_k+1, x-1]$, $I_{k+1} = [x+1, v(G)]$.

We now define a nested collection of events, the first being the trivial (always satisfied) event and the last being the event $\{x \hookrightarrow v, v' \notin \operatorname{im}\psi_{v(G)}\}$, whose probability we wish to estimate. These events are simply that we have not yet (by given increasing times in *RandomEmbedding*) made it impossible to have $\{x \hookrightarrow v, v' \notin \operatorname{im}\psi_{v(G)}\}$. We will see that we can estimate accurately the probability of each successive event, conditioned on its predecessor.

Let $\mathscr{L}'_{-1}$ be the trivial (always satisfied) event. If $\mathscr{L}'_{i-1}$ is defined, we let $\mathscr{L}_i$ be the event that $\mathscr{L}'_{i-1}$ holds intersected with the event that

(A1) (if $i \le k$:) no vertex of G in the interval I_i is mapped to v or v', or

(A2) (if $i = k+1$:) no vertex of G in the interval I_{k+1} is mapped to v'.

In other words, $\mathscr{L}_i$ is the event that we have not covered v or v' in the interval I_i. It turns out that we do not need to know anything else about the embeddings in the interval I_i.

If $\mathscr{L}_i$ is defined, we let $\mathscr{L}'_i$ be that event that $\mathscr{L}_i$ holds and that

(B1) (if $i < k$:) we have the event $z_{i+1} \hookrightarrow \mathsf{N}_H(v) \setminus \{v'\}$,

(B2) (if $i = k$:) we have the event $x \hookrightarrow v$.

Again, in order for $\{x \hookrightarrow v, v' \notin \operatorname{im}\psi_{v(G)}\}$ to occur we obviously need that a neighbour of x is embedded to a neighbour of v and so on, hence the above conditions.

By definition, we have $\mathscr{L}_{k+1} = \{x \hookrightarrow v, v' \notin \operatorname{im}\psi_{v(G)}\}$. Since we have $\mathscr{L}'_i \subseteq \mathscr{L}_i \subseteq \mathscr{L}'_{i-1}$ for each i and $\mathscr{L}'_{-1}$ is the sure event, we see

$$\mathbf{P}\left[x \hookrightarrow v, v' \notin \operatorname{im}\psi_{v(G)}\right] = \frac{\mathbf{P}[\mathscr{L}_0]}{\mathbf{P}[\mathscr{L}'_{-1}]} \cdot \prod_{i=0}^{k} \frac{\mathbf{P}[\mathscr{L}'_i]}{\mathbf{P}[\mathscr{L}_i]} \cdot \frac{\mathbf{P}[\mathscr{L}_{i+1}]}{\mathbf{P}[\mathscr{L}'_i]} \tag{41}$$

$$= \mathbf{P}\left[\mathscr{L}_0 \mid \mathscr{L}'_{-1}\right] \prod_{i=0}^{k} \mathbf{P}\left[\mathscr{L}'_i \mid \mathscr{L}_i\right] \mathbf{P}\left[\mathscr{L}_{i+1} \mid \mathscr{L}'_i\right] \,. \tag{42}$$

Thus, we need to estimate the factors in (42). This is done in the two claims below. In each claim we assume $\mathbf{P}[\mathscr{L}'_i], \mathbf{P}[\mathscr{L}_i] > n^{-4}$. This assumption is justified, using an implicit induction, since the smallest of all the events we consider is $\mathscr{L}_{k+1}$, whose probability according to the following (46) is bigger than n^{-4}.

Claim 49.1. *We have*

$$\prod_{i=0}^{k+1} \mathbf{P}\left[\mathscr{L}_i \mid \mathscr{L}'_{i-1}\right] = (1 \pm 200C\alpha\tilde{\delta}^{-1})^{2k+4} \cdot \frac{(n-x)(n-v(G))}{n^2} \,.$$

Proof. By definition of (A1), for each $i = 0, \dots, k$, we have

$$\mathbf{P}\left[\mathscr{L}_i \mid \mathscr{L}'_{i-1}\right] = (1 \pm 200C\alpha\tilde{\delta}^{-1}) \cdot \frac{(n-1-\max(I_i))^2}{(n-\min(I_i)+1)^2} \tag{43}$$

by the 2-vertex case of Lemma 49, with $\mathscr{L} = \mathscr{L}'_{i-1}$. Note that looking at two consecutive indices i and $i+1$ in (43) we have cancellation of the former nominator and the latter denominator,

$n-1-\max(I_i)=n-\min(I_{i+1})+1$. Thus,

$$\prod_{i=0}^{k}\mathbf{P}\left[\mathscr{L}_i\mid\mathscr{L}'_{i-1}\right]=(1\pm 200C\alpha\tilde{\delta}^{-1})^{2k+2}\cdot\frac{(n-x)^2}{n^2}\;. \tag{44}$$

To express $\mathbf{P}\left[\mathscr{L}_{k+1}\mid\mathscr{L}'_k\right]$, by definition of (A2) we have to repeat the above replacing the 2-vertex case of Lemma 49 with the 1-vertex case. We get that

$$\mathbf{P}\left[\mathscr{L}_{k+1}\mid\mathscr{L}'_k\right]=(1\pm 200C\alpha\tilde{\delta}^{-1})^{2}\cdot\frac{n-v(G)}{n-x}\;. \tag{45}$$

Putting (44) and (45) together, we get the statement of the claim. □

Claim 49.2. *We have*

$$\prod_{i=0}^{k}\mathbf{P}\left[\mathscr{L}'_i\mid\mathscr{L}_i\right]=(1\pm 100C\alpha)^{k+1}\cdot\frac{1}{n+1-x}\;.$$

Proof. Suppose that we have embedded up to vertex $\max(I_i)$, and that $\mathscr{L}_i$ holds. The probability of the event $\mathscr{L}'_i$ depends on which of the cases (B1) and (B2) applies. When $\mathscr{L}'_i$ is defined using (B1) then the probability $\mathbf{P}[\mathscr{L}'_i|\mathscr{L}_i]$ is equal to $\mathbf{P}[\{z_{i+1}\hookrightarrow \mathsf{N}_H(v)\setminus\{v'\}\}|\mathscr{L}_i]$. Let $X:=\mathsf{N}_H\big(\psi(\mathsf{N}^-_G(z_{i+1}))\big)\setminus\operatorname{im}\psi_{z_{i+1}-1}$ be the set of vertices in H to which we could embed z_{i+1}, given the embedding of all vertices before z_{i+1}. Suppose that the $(C\alpha,2D+3)$-diet condition holds for $(H,\operatorname{im}\psi_{z_{i+1}-1})$. Then we have

$$\begin{aligned}\mathbf{P}\left[z_{i+1}\hookrightarrow \mathsf{N}_H(v)\setminus\{v'\}|\mathscr{L}_i\right]&=\frac{|(\mathsf{N}_H(v)\setminus\{v'\})\cap X|}{|X|}=\frac{|\mathsf{N}_H(v)\cap X|\pm 2}{|X|}\\&=\frac{(1\pm C\alpha)p^{1+\deg^-(z_{i+1})}(n-(z_{i+1}-1))\pm 2}{(1\pm C\alpha)p^{\deg^-(z_{i+1})}(n-(z_{i+1}-1))}=(1\pm 4C\alpha)p\;,\end{aligned}$$

where the last line uses the $(C\alpha,2D+3)$-diet condition for $(H,\operatorname{im}\psi_{z_{i+1}-1})$ twice, in the denominator with the set $\psi(\mathsf{N}^-(z_{i+1}))$ and in the numerator with the set $\{v\}\cup\psi(\mathsf{N}^-(z_{i+1}))$. Recall that we assume the event $\mathscr{L}_i$, and so we have $v\notin\operatorname{im}\psi_{z_{i+1}-1}$. Therefore, the set $\{v\}\cup\psi(\mathsf{N}^-_G(z_{i+1}))$ has indeed size $1+\deg^-(z_{i+1})$.

Let us now deal with the term $\mathbf{P}\left[\mathscr{L}'_k\mid\mathscr{L}_k\right]$, which corresponds to (B2). Suppose that $\mathscr{L}_k$ holds. In particular, $\mathsf{N}^-(x)$ is embedded to $\mathsf{N}_H(v)$. Suppose first that the $(C\alpha,2D+3)$-diet condition for $(H,\operatorname{im}\psi_{x-1})$ holds. With this, conditioning on the embedding up to time $x-1$, the probability of embedding x to v is $(1\pm 2C\alpha)p^{-\deg^-(x)}\frac{1}{n+1-x}$.

Thus, letting $\mathcal{F}$ be the event that the $(C\alpha,2D+3)$-diet condition fails for $(H,\operatorname{im}\psi_t)$ for some $t\in[v(G)]$ we have

$$\prod_{i=0}^{k}\mathbf{P}\left[\mathscr{L}'_i\mid\mathscr{L}_i\right]=\Big(\big((1\pm 4C\alpha)p\big)^k\cdot(1\pm 2C\alpha)p^{-\deg^-(x)}\tfrac{1}{n+1-x}\Big)\pm\mathbf{P}[\mathcal{F}]\;,$$

We have $\mathbf{P}[\mathcal{F}] \leq 2n^{-9}$ by Lemma 46. Thus we obtain

$$\prod_{i=0}^{k} \mathbf{P}\left[\mathscr{L}_i' \mid \mathscr{L}_i\right] = (1 \pm 4C\alpha)^{k+1} \cdot \tfrac{1}{n+1-x} \pm 2n^{-9} \,,$$

and the claim follows. □

Plugging Claims 49.1 and 49.2 into (42), we get

$$\mathbf{P}\left[x \hookrightarrow v, v' \notin \operatorname{im} \psi_{v(G)}\right] = (1 \pm 500C\alpha\tilde{\delta}^{-1})^{3D+3} \cdot \tfrac{(n-x)(n-v(G))}{n^2(n-x+1)} \,, \tag{46}$$

and (35) follows.

For (36), we use the same approach. The only difference is that we define events

(A'1) (if $i \leq k$:) no vertex of G in the interval I_i is mapped to v, v' or v'', or

(A'2) (if $i = k+1$:) no vertex of G in the interval I_{k+1} is mapped to v' or v''.

and if $\mathscr{L}_i$ is defined, we let $\mathscr{L}_i'$ be that event that $\mathscr{L}_i$ holds and that

(B'1) (if $i < k$:) we have the event $z_{i+1} \hookrightarrow \mathsf{N}_H(v) \setminus \{v', v''\}$,

(B'2) (if $i = k$:) we have the event $x \hookrightarrow v$.

The calculations are almost identical: in particular the calculations and results for the B' events are verbatim the same, while for the A' events we use respectively the 3-vertex and 2-vertex cases of Lemma 49 rather than the 2- and 1-vertex cases; but the calculations go through without further trouble. We omit the details. □

We need a slight strengthening of [1, Lemma 43]. The idea of this lemma is that we pause *PackingProcess* after it has packed s graphs, and fix a set of vertices T in the common neighbourhood of some $v_1, \dots, v_k$ in the graph H_s. We then allow the process to continue up to some point s', and of course the common neighbourhood of $v_1, \dots, v_k$ shrinks; this lemma states that its intersection with T shrinks proportionally. The difference to [1, Lemma 43] is that we look at the common neighbourhood of a set of vertices rather than one vertex, and we give a different bound on the set size, but the proof is very similar.

Lemma 50. *Assume Setting 44 and let $s, s' \in \mathcal{J} \cup \mathcal{K}$ with $s < s'$. Consider the following experiment. Suppose $v_1, \dots, v_k$ are fixed vertices of H_0, with $k \leq 2D+3$. Run* PackingProcess' *with input $(G'_{s''})_{s'' \in \mathcal{J} \cup \mathcal{K}}$ and H_0 up to and including the embedding of G'_s. Then fix $T \subseteq \mathsf{N}_{H_s}(v_1, \dots, v_k)$ with $|T| \geq \frac{1}{2}\gamma^D \delta^D n$, and continue* PackingProcess' *to perform the embedding of $G'_{s+1}, \dots, G'_{s'}$.*

The probability that PackingProcess' *fails before embedding $G'_{s'}$, or H_i fails to be $(\alpha_i, 2D+3)$-quasirandom for some $1 \leq i \leq s'$, or we have*

$$\left|T \cap \mathsf{N}_{H_{s'}}(v)\right| = (1 \pm d_{\mathbf{A}}^{-1}\alpha_{s'})\left(\tfrac{p_{s'}}{p_s}\right)^k |T| \,,$$

is at least $1 - n^{-C}$.

To prove this, we follow the proof of [1, Lemma 43], using (37) in place of [1, Lemma 39].

Proof of Lemma 50. For $s \leq i \leq s'$, we define the event $\mathcal{E}_i$ that *PackingProcess* does not fail before embedding G'_i, and H_j is $(\alpha_j, 2D+3)$-quasirandom for each $1 \leq j \leq i$, and $|T \cap \mathsf{N}_{H_j}(v_1, \dots, v_k)| = (1 \pm d_{\mathbf{A}}^{-1}\alpha_j)\big(\frac{p_j}{p_s}\big)^k|T|$ for each $s \leq j \leq i$. If the event in the lemma statement fails to occur, then there must exist some $s \leq i < s'$ such that $\mathcal{E}_i$ occurs and

$$|T \cap \mathsf{N}_{H_{i+1}}(v_1, \dots, v_k)| \neq (1 \pm d_{\mathbf{A}}^{-1}\alpha_{i+1})\big(\tfrac{p_{i+1}}{p_s}\big)^k|T| \,.$$

It suffices to show that each of these bad events occurs with probability at most n^{-C-1}, since then the union bound over the at most $\frac{3}{2}n$ choices of i gives the lemma statement. This is an estimate we can obtain using Corollary 25. We now fix $s \leq i < s'$ and prove the desired estimate.

Suppose $s \leq j \leq i$, and let $Y_j := |\mathsf{N}_{H_j}(v_1, \dots, v_k) \cap T \setminus \mathsf{N}_{H_{j+1}}(v_1, \dots, v_k)|$ count the number of stars with leaves $v_1, \dots, v_k$ and centre in T, at least one edge of which is used for the embedding of G'_{j+1}. Then we have $\big|T \cap \mathsf{N}_{H_{i+1}}(v_1, \dots, v_k)\big| = |T| - \sum_{j=s}^{i} Y_j$, and what we want to do is argue that the sum of random variables is concentrated. To that end, suppose $\mathscr{H}$ is a history of *PackingProcess*' up to time j such that H_j is $(\alpha_j, 2D+3)$-quasirandom and $|T \cap \mathsf{N}_{H_j}(v_1, \dots, v_k)| = (1 \pm d_{\mathbf{A}}^{-1}\alpha_j)\big(\frac{p_j}{p_s}\big)|T|$. Then we have

$$\mathbf{E}\big[Y_j \,\big|\, \mathscr{H}\big] = \big(1 \pm 1000C\alpha_j\tilde{\delta}^{-1}\big)^{4D+2} p_j^{-1} n^{-2} \cdot 2ke(G'_{j+1}) \cdot \big(1 \pm d_{\mathbf{A}}^{-1}\alpha_j\big)\big(\tfrac{p_j}{p_s}\big)^k|T| \,,$$

where we use linearity of expectation: the first factor is (37) the probability that a given star with leaves $v_1, \dots, v_k$ and centre in T in H_j is used in the embedding of G'_{j+1}, and the second factor is the number of such edges. Simplifying, we obtain

$$\mathbf{E}\big[Y_j \,\big|\, \mathscr{H}\big] = \frac{2kp_j^{k-1}e(G'_{j+1})|T|}{p_s^k n(n-1)} \pm \frac{10^5\tilde{\delta}^{-1}CD^2|T|}{p_s n}\alpha_j \,,$$

where for the error term we use the upper bound $e(G'_{j+1}) \leq Dn$, the fact $p_j/p_s \leq 1$, and $\tilde{\delta}^{-1} > d_{\mathbf{A}}^{-1}$. Let

$$\tilde{\mu} := \sum_{j=s}^{i} \frac{2kp_j^{k-1}e(G'_{j+1})|T|}{p_s^k n(n-1)} \quad \textit{and} \quad \tilde{\nu} := \sum_{j=s}^{i} \frac{10^5\tilde{\delta}^{-1}CD^2|T|}{p_s n}\alpha_j$$

and observe that $\tilde{\mu} \leq |T| \leq n$ and $\tilde{\nu} \leq \frac{10^5\tilde{\delta}^{-1}CD^2|T|}{p_s}\alpha_i < \frac{|T|}{10^3}$ since $p_s \geq d_{\mathbf{A}}$ and by the definition of α_j.

We trivially have $0 \leq Y_j \leq k\Delta(G'_{j+1}) \leq kcn/\log n$. So what Corollary 25(*b*), with $\tilde{\varrho} = \varepsilon n$, gives us is that

$$\mathbf{P}\left[\mathcal{E}_i \text{ and } \sum_{j=s}^{i} Y_i \neq \tilde{\mu} \pm (\tilde{\nu} + \varepsilon n)\right] < 2\exp\Big(-\tfrac{\varepsilon^2 n^2}{4kcn^2/\log n}\Big) < n^{-C-1} \,,$$

where we use the upper bound $\tilde{\mu} + \tilde{\nu} + \tilde{\varrho} \leq 2n$ for the first inequality and the choice of c as well as $\varepsilon < \frac{1}{C}$ for the second. This is the probability bound we wanted. We now simply need to show that if

$$\sum_{j=s}^{i} Y_i = \tilde{\mu} \pm (\tilde{\nu} + \varepsilon n)$$

then we have

$$|T \cap \mathsf{N}_{H_{i+1}}(v)| = (1 \pm d_{\mathbf{A}}^{-1}\alpha_{i+1})\big(\tfrac{p_{i+1}}{p_s}\big)^k |T| \,.$$

In order to estimate $\tilde{\mu}$, observe that $e(G'_{j+1}) = (p_j - p_{j+1})\binom{n}{2}$, and that for every $x, h \in [0,1]$ we have $(x+h)^k - x^k = kh(x+h)^{k-1} \pm 2^k h^2$. Using the latter equality with $x = p_{j+1}$ and $h = p_j - p_{j+1}$, and using $(p_j - p_{j-1})\binom{n}{2} \le Dn$, we see

$$kp_j^{k-1} e(G'_{j+1}) = \big(p_j^k - p_{j+1}^k\big)\binom{n}{2} \pm 2^{k+2} D^2 \,.$$

Using this we see

$$|T| - \tilde{\mu} = |T|\Big(1 - \sum_{j=s}^{i} \big(p_j^k - p_{j+1}^k\big)p_s^{-k} \pm \frac{2^{k+2}D^2}{\binom{n}{2}}\Big) = \tfrac{p_{i+1}^k}{p_s^k}|T| \pm 2^{k+4}D^2 \,,$$

since $i+1 \le 2n$. So what remains is to argue $\tilde{\nu} + \varepsilon n + 2^{k+4}D^2 < d_{\mathbf{A}}^{-1}\alpha_{i+1}\frac{p_{i+1}}{p_s}|T|$. Since $\alpha_j = \frac{\tilde{\delta}}{10^8 CD}\exp\Big(\frac{10^8 CD^3\tilde{\delta}^{-1}(j-2n)}{n}\Big)$ is increasing in j, we have

$$\begin{aligned} \sum_{j=s}^{i} \alpha_j &\le \int_s^{i+1} \alpha_j \,\mathrm{d}j \le \int_{-\infty}^{i+1} \alpha_j \,\mathrm{d}j \\ &= \Big[\frac{\tilde{\delta}}{10^8 CD} \cdot \frac{n}{10^8 CD^3 \tilde{\delta}^{-1}} \cdot \exp\Big(\frac{10^8 CD^3 \tilde{\delta}^{-1}(j-2n)}{n}\Big)\Big]_{j=-\infty}^{i+1} = \frac{\tilde{\delta} n}{10^8 CD^3}\alpha_{i+1} \,. \end{aligned} \tag{47}$$

It follows that

$$\tilde{\nu} + \varepsilon n + 2^{k+4}D^2 \le \frac{10^5\tilde{\delta}^{-1}CD^2|T|}{p_s n} \cdot \frac{\tilde{\delta}n}{10^8 CD^3}\alpha_{i+1} + \varepsilon n + 2^{k+4}D^2 \le \tfrac{\alpha_{i+1}}{1000D} \cdot \tfrac{1}{p_s}|T| + \varepsilon n + 2^{k+4}D^2 \,.$$

Finally, since $p_{i+1}, p \ge d_{\mathbf{A}}$, by choice of ε, since $d_{\mathbf{A}} \le p_{i+1}$, because $|T| \ge \frac{1}{2}\delta^D\gamma^D$, and because n is sufficiently large, we conclude $\tilde{\nu} + \varepsilon n + 2^{k+4}D^2 \le d_{\mathbf{A}}^{-1}\alpha_{i+1}\frac{p_{i+1}}{p_s}|T|$ as desired. □

The next lemma, which is almost the same as [1, Lemma 44], shows that in a run of *RandomEmbedding* (considered as step s in *PackingProcess'*) the image of $V(G'_s)$ covers a predictable amount of any previously given reasonably large vertex set S. For completeness, we copy the proof from [1] and make the appropriate small modifications.

Lemma 51. *Assume Setting 44 and let $s \in \mathcal{J} \cup \mathcal{K}$. Run* PackingProcess*' with input $(G'_{s''})_{s'' \in \mathcal{J}\cup\mathcal{K}}$ and H_0 up to just before the embedding of G'_s. Suppose $v(G'_s) \le (1-\delta)n$. We define for each $t \in \mathbb{R}$*

$$\beta_t = 2\alpha_{s-1}\exp\Big(\tfrac{1000D\tilde{\delta}^{-2}\gamma^{-2D-10}t}{n}\Big) \,.$$

Then fix any $S \subseteq V(H_{s-1})$ with $|S| \ge \frac{1}{2}\gamma^D\delta^D n$, and let PackingProcess*' perform the embedding of G'_s. With probability at least $1 - n^{-2D-18}$, either H_{s-1} is not $(\alpha_{s-1}, 2D+3)$-quasirandom, or* RandomEmbedding *fails to construct the embedding of G'_s, or the embedding of G'_s fails to have* *the $(\varepsilon, 20D\beta_t, t)$-cover condition for some $1 \le t \le v(G'_s)+1-\varepsilon n$, or for some $1 \le t \le v(G'_s)$ the pair $\big(H_{s-1}, \phi_s([t])\big)$ does not have the $(\beta_t, 2D+3)$-diet condition, or we have*

$$|S \setminus \operatorname{im}\phi_s^{\mathbf{A}}| = (1 \pm C'\alpha_s)\tfrac{n-v(G'_s)}{n}|S|\,.$$

We will refer to the first four of the events mentioned above as 'bad events', and the fifth (the equation) as the good event.

Proof. Fix $s \in \mathcal{J} \cup \mathcal{K}$, and condition on H_{s-1}. If H_{s-1} is not $(\alpha_{s-1}, 2D+3)$-quasirandom, or the embedding of G'_s fails, or we do not have the $(\varepsilon, C\alpha_i, i)$-cover condition, then the bad event of this lemma cannot occur. So it is enough to show that the probability that none of these bad events occurs but $|S \setminus \operatorname{im}\phi_s^{\mathbf{A}}| \neq (1 \pm C'\alpha_s)\mu|S|$, conditioned on H_{s-1}, is at most n^{-2D-18}. This is what we will now do, so we suppose that H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom. Consider the run of *RandomEmbedding* which embeds G'_s.

Define $S_0 = S$, and for $i = 1, \dots, \tau$ with $\tau = \lceil \frac{v(G'_s)}{\varepsilon n} \rceil$ set $S_i = S_{i-1} \setminus \operatorname{im}\psi_{i\varepsilon n}$. Since $S_\tau \subseteq S \setminus \operatorname{im}\phi_s^{\mathbf{A}} \subseteq S_{\tau-1}$, it is enough to show both $|S_{\tau-1}|$ and $|S_\tau|$ are likely to be in the claimed range. Recall that $v(G'_s) \le (1-\delta)n < (1-\tilde{\delta}-\varepsilon)n$, so that *RandomEmbedding* does embed vertices (namely isolated vertices added to G'_s) long enough to create S_τ. Since $|S_{\tau-1}|$ and $|S_\tau|$ differ by at most εn, we will focus on estimating $|S_\tau|$.

Given $0 \le j \le \tau - 1$, either one of the bad events of the lemma occurs, or $\phi_{j\varepsilon n}$ exists and $(H_{s-1}, \operatorname{im}\phi_{j\varepsilon n})$ has the $(\beta_{j\varepsilon n}, 2D+3)$-diet condition. Lemma 47, with input $T = S_{j\varepsilon n} = S \setminus \operatorname{im}\psi_{j\varepsilon n}$, then states that conditioning on $\phi_{j\varepsilon n}$, with probability at least $1 - 2n^{-2D-19}$ either $\psi_{v(G)}$ does not have the $(\varepsilon, 20D\beta_j, j)$-cover condition, or we have

$$\big|\big\{x:\ j\varepsilon n \le x < (j+1)\varepsilon n:\ \psi_{(j+1)\varepsilon n}(x) \in S \setminus \operatorname{im}\psi_{j\varepsilon n}\big\}\big| = (1 \pm 40D\beta_{j\varepsilon n})\frac{|S \setminus \operatorname{im}\psi_{j\varepsilon n}|\varepsilon n}{n - j\varepsilon n}\,. \tag{48}$$

In particular, since the failure of the cover condition is one of the bad events of the lemma, taking the union bound over the at most τ choices of j, with probability at least $1-2\tau n^{-2D-19} \ge 1-n^{-2D-18}$, either one of the bad events of the lemma occurs or we have (48) for each $0 \le j < \tau$. Let us suppose that the latter is the case. Then we have for each $1 \le j \le \tau$

$$|S_{j\varepsilon n}| = |S_{(j-1)\varepsilon n}|\Big(1 - \frac{(1 \pm 40D\beta_{(j-1)\varepsilon n})\varepsilon}{1-(j-1)\varepsilon}\Big)\,,$$

and hence

$$|S_\tau| = |S|\prod_{i=1}^{\tau}\Big(1 - \frac{(1 \pm 40D\beta_{(i-1)\varepsilon n})\varepsilon}{1-(i-1)\varepsilon}\Big)\,.$$

In order to evaluate this product, observe that

$$1 - \frac{(1 \pm 40D\beta_{i\varepsilon n})\varepsilon}{1-i\varepsilon} = \frac{1-(i+1)\varepsilon}{1-i\varepsilon} \pm \frac{40D\beta_{i\varepsilon n}\varepsilon}{1-i\varepsilon} = \frac{1-i\varepsilon-\varepsilon}{1-i\varepsilon}\Big(1 \pm \frac{40D\beta_{i\varepsilon n}\varepsilon}{1-(i+1)\varepsilon}\Big)\,,$$

and therefore

$$|S_\tau| = |S|\prod_{i=0}^{\tau-1}\frac{1-i\varepsilon-\varepsilon}{1-i\varepsilon}\cdot\Big(1 \pm \frac{40D\beta_{i\varepsilon n}\varepsilon}{1-(i+1)\varepsilon}\Big) = |S|(1-\tau\varepsilon)\prod_{i=0}^{\tau-1}\Big(1 \pm \frac{40D\beta_{i\varepsilon n}\varepsilon}{1-(i+1)\varepsilon}\Big)\,.$$

By the definition of τ we have $\frac{n-v(G'_s)}{\varepsilon n} \le \tau \le \frac{n-v(G'_s)}{\varepsilon n}+1$ and hence $(1-\tau\varepsilon) = \frac{n-v(G'_s)}{n}(1\pm\frac{\varepsilon}{\delta})$. Moreover, we obtain that

$$\begin{aligned}\sum_{i=0}^{\tau-1}\frac{40D\beta_{i\varepsilon n}\varepsilon}{1-(i+1)\varepsilon} &\le \frac{40D\varepsilon}{1-\tau\varepsilon}\sum_{i=0}^{\tau-1}\beta_{i\varepsilon n} \le \frac{80D\varepsilon n}{n-v(G'_s)}\sum_{i=0}^{\tau-1}\beta_{i\varepsilon n}\\ &\le \frac{80D}{\delta n}\int_0^{\tau}\varepsilon n\beta_{i\varepsilon n}\,\mathrm{d}i \le \frac{80D}{\delta n}\int_0^{\tau\varepsilon n}\beta_x\,\mathrm{d}x\\ &\le \frac{80D}{\delta\cdot 1000D\tilde{\delta}^{-2}\gamma^{-2D-10}}\beta_{\tau\varepsilon n}\\ &\le \frac{80D}{\delta\cdot 1000D\tilde{\delta}^{-2}\gamma^{-2D-10}}\beta_{(1-\delta+\varepsilon)n} \le \frac{1}{2}C'\alpha_{s-1}\end{aligned}$$

since $\beta_{(1-\delta+\varepsilon)n} = 2\alpha_{s-1}\exp(1000D\tilde{\delta}^{-2}\gamma^{-2D-10}(1-\delta+\varepsilon))$ and

$$C' = 10^4\cdot\frac{40D}{\tilde{\delta}}\exp(1000D\tilde{\delta}^{-2}\gamma^{-2D-10})\,.$$

So, since $\prod_i(1\pm x_i) = 1\pm 2\sum_i x_i$ as long as $\sum_i x_i < \frac{1}{100}$ and since $\frac{1}{2}C'\alpha_s < \frac{1}{100}$, we get

$$\begin{aligned}|S_\tau| &= |S|(1-\tau\varepsilon)\Big(1\pm 2\sum_{i=0}^{\tau-1}\frac{40D\beta_{i\varepsilon n}\varepsilon}{1-(i+1)\varepsilon}\Big) = |S|(1-\tau\varepsilon)\Big(1\pm\frac{80D\varepsilon}{1-\tau\varepsilon}\sum_{i=0}^{\tau-1}\beta_{i\varepsilon n}\Big)\\ &= |S|\Big(1-\tau\varepsilon\pm 80D\varepsilon\sum_{i=0}^{\tau-1}\beta_{i\varepsilon n}\Big) = |S|\tfrac{n-v(G'_s)}{n}\Big(1\pm\frac{\varepsilon}{\delta}\pm\frac{80D\varepsilon}{\delta}\sum_{i=0}^{\tau-1}\beta_{i\varepsilon n}\Big)\\ &= |S|\tfrac{n-v(G'_s)}{n}\Big(1\pm\frac{1}{2}\alpha_s\pm\frac{1}{2}C'\alpha_s\Big)\,,\end{aligned}$$

where for the last equation we use that $\varepsilon \le \alpha_0\tilde{\delta}^2\gamma \le \frac{1}{2}\alpha_s\delta$. It follows that

$$|S\setminus\operatorname{im}\phi'_s| = |S_\tau|\pm\varepsilon n = |S|\tfrac{n-v(G'_s)}{n}\big(1\pm\tfrac{1}{2}\alpha_s\pm\tfrac{1}{2}C'\alpha_s\big)\pm\varepsilon n = \big(1\pm C'\alpha_s\big)\tfrac{n-v(G'_s)}{n}|S|\,,$$

as desired. □

8.5. **Proofs of the remaining properties of Lemma 30.** We are now in a position to prove $(A.vi)$. The idea is that first we will show the equation holds with $H_{\mathbf{A}}$ replaced by H_{s-1} and $d_{\mathbf{A}}$ replaced with the density d_{s-1} of H_{s-1}, and then Lemma 50 tells us it is likely that $(A.vi)$ holds. For the first part, we argue (using the diet condition) that each path-end vertex has about a d_s chance of being embedded to the neighbourhood of v, conditional on the previous embedding history, and then Corollary 25 gives the conclusion we want.

Proof of $(A.vi)$. Fix $v\in V(H)$ and s. Suppose that *PackingProcess*' has not failed before creating H_{s-1}, and furthermore H_{s-1} is $(\alpha_{s-1},2D+3)$-quasirandom. Let d_{s-1} be the density of H_{s-1}.

Let $X = \bigcup_{P\in\mathsf{SpecPaths}_s}\{\mathsf{leftpath}_0(P),\mathsf{rightpath}_0(P)\}$. Let $\mathscr{H}_{x-1}$ denote the history of *RandomEmbedding* embedding G'_s into H_{s-1}, up to but not including the embedding of x; let ψ_{x-1} denote the partial embedding of G'_s at this time.

Suppose that $\mathscr{H}_{x-1}$ is such that $(H_{s-1}, \operatorname{im}\psi_{x-1})$ satisfies the $(C\alpha_{s-1}, 2D+3)$-diet condition, and furthermore that $v \notin \operatorname{im}\psi_{x-1}$. Then we embed x uniformly to $\mathsf{N}_{H_{s-1}}(y_1,\dots,y_\ell) \setminus \operatorname{im}\psi_{x-1}$, where the y_i are the back-neighbours of x in G'_s. This is a set of size $(1 \pm C\alpha_{s-1})(n+1-x)d_{s-1}^\ell$ by the diet condition. Again by the diet condition, and since v is not one of the y_i (it is not in $\operatorname{im}\psi_{x-1}$), the number of these vertices which are in $\mathsf{N}_{H_{s-1}}(v)$ is $(1 \pm C\alpha_{s-1})(n+1-x)d_{s-1}^{\ell+1}$. So the probability that x is embedded to $\mathsf{N}_{H_{s-1}}(v)$, conditioned on $\mathscr{H}_{x-1}$, is $(1 \pm 3C\alpha_{s-1})d_{s-1}$.

We now apply Corollary 25(b) to estimate the total number of vertices of X embedded to $\mathsf{N}_{H_{s-1}}(v)$. We take $\tilde\eta = 3C\alpha_{s-1}$ and $\tilde\mu = |X|d_{s-1}$, $R = 1$, and $\mathcal{E}$ the event that each $\mathscr{H}_{x-1}$ satisfies the diet condition and $v \notin \operatorname{im}\phi_s^{\mathbf{A}}$. The conclusion is that with probability at least $1 - 2\exp\big(-|X|d_{s-1}\cdot 9C^2\alpha_{s-1}^2/4\big) \ge 1 - n^{-10}$, we have $\bar{\mathcal{E}}$ or

$$\big|\big\{x \in X : \phi_s^{\mathbf{A}}(x) \in \mathsf{N}_{H_{s-1}}(v)\big\}\big| = (1 \pm 6C\alpha_{s-1})|X|d_{s-1}\,.$$

By Lemma 46, the probability that $v \notin \operatorname{im}\phi_s^{\mathbf{A}}$ and we are not in $\mathcal{E}$ is at most $2n^{-9}$. So with probability at least $1 - 3n^{-9}$, either $v \in \operatorname{im}\phi_s^{\mathbf{A}}$ or

$$\big|\big\{x \in X : \phi_s^{\mathbf{A}}(x) \in \mathsf{N}_{H_{s-1}}(v)\big\}\big| = (1 \pm 6C\alpha_{s-1})|X|d_{s-1}\,.$$

We condition on this good event. Now if $v \notin \operatorname{im}\phi_s^{\mathbf{A}}$ then we have $\mathsf{N}_{H_{s-1}}(v) = \mathsf{N}_{H_s}(v)$, and furthermore $d_{s-1}\binom{n}{2} = d_s\binom{n}{2} \pm Dn$, so if $v \notin \operatorname{im}\phi_s^{\mathbf{A}}$ we have

$$\big|\big\{x \in X : \phi_s^{\mathbf{A}}(x) \in \mathsf{N}_{H_s}(v)\big\}\big| = (1 \pm 10C\alpha_{s-1})|X|d_s\,.$$

We now apply Lemma 50, with T the subset of $\mathsf{N}_{H_s}(v)$ lying in X, and s' the final stage of *PackingProcess*'. The conclusion is that, with probability at least $1 - n^{-C}$, we have

$$\big|\big\{x \in X : \phi_s^{\mathbf{A}}(x) \in \mathsf{N}_{H_{\mathbf{A}}}(v)\big\}\big| = (1 \pm \gamma^{-1}\alpha_{s'})(1 \pm 10C\alpha_{s-1})|X|d_{\mathbf{A}}\,,$$

which by (33) is what we need for $(A.vi)$. Taking the union bound over choices of s and v and the various bad events, we conclude that a.a.s. $(A.vi)$ holds. □

Putting Lemmas 50 and 51 together, we can show that $(A.ii)$ holds with very high probability for any given S and T, and so by the union bound holds with high probability.

Proof of $(A.ii)$. Let $S \subseteq V(H)$ and $T \subseteq \mathcal{G}$, with $|S|, |T| \le D$, be given. Let $T = \{t_1,\dots,t_{|T|}\}$, and for convenience let $t_0 = 0$ and $t_{|T|+1} = |\mathcal{J}| + |\mathcal{K}| + 1$. We aim to show that the following holds for each $0 \le i \le |T|$:

$$\Big|\mathsf{N}_{H_{t_i}}(S) \setminus \bigcup_{j=1}^{i} \operatorname{im}^{\mathbf{A}}(j)\Big| = \big(1 \pm 2(i+1)(d_{\mathbf{A}}^{-1} + C')\alpha_{t_i}\big)\Big(\tfrac{e(H_{t_i})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=0}^{i}\big(1 - \tfrac{\operatorname{im}^{\mathbf{A}}(j)}{n}\big)\Big)\,. \tag{49}$$

Observe that the $i = 0$ case of (49) is true because H_0 is $(\alpha_0, 2D+3)$-quasirandom. Observe furthermore that for any i, the set on the left hand side of (49) is fixed before *PackingProcess*' begins (if $i = 0$) or immediately after completing the embedding of G'_{t_i} (if $i \ge 1$).

It follows that for each $0 \le i \le |T|$, we have the setup for Lemma 50, which tells us that with probability at least $1 - n^{-C}$ either *PackingProcess*' fails before embedding $G'_{t_{i+1}-1}$, or H_j fails to be $(\alpha_j, 2D+3)$-quasirandom for some $1 \le j \le t_{i+1}-1$ (these are the bad events of Lemma 50), or we have the good event

$$\begin{aligned}
&\Big|\mathsf{N}_{H_{t_{i+1}-1}}(S) \setminus \bigcup_{j=1}^{i} \mathrm{im}^{\mathbf{A}}(j)\Big| \\
&= (1 \pm d_{\mathbf{A}}^{-1}\alpha_{t_{i+1}-1})\big(1 \pm 2(i+1)(d_{\mathbf{A}}^{-1}+C')\alpha_{t_i}\big)\Big(\tfrac{e(H_{t_{i+1}-1})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=1}^{i}\big(1-\tfrac{\mathrm{im}^{\mathbf{A}}(j)}{n}\big)\Big) \\
&= \big(1 \pm 2(i+1)(d_{\mathbf{A}}^{-1}+C')\alpha_{t_i} \pm \tfrac{3}{2}d_{\mathbf{A}}^{-1}\alpha_{t_{i+1}-1}\big)\Big(\tfrac{e(H_{t_{i+1}-1})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=0}^{i}\big(1-\tfrac{\mathrm{im}^{\mathbf{A}}(j)}{n}\big)\Big).
\end{aligned}$$

Observe that if $i = |T|$, by (33) this is the desired statement of $(A.ii)$. If $0 \le i \le |T|-1$, then we have the setup for Lemma 51, analysing the embedding of $G'_{t_{i+1}}$, which tells us that with probability at least $1-n^{-2D-18}$, either one of the following bad events (as listed in Lemma 51) occurs, or the final good event occurs. The bad events are: $H_{t_{i+1}-1}$ is not $(\alpha_{t_{i+1}-1}, 2D+3)$-quasirandom, or *RandomEmbedding* fails to construct the embedding of $G'_{t_{i+1}}$, or $\big(H_{t_{i+1}-1}, \phi^{\mathbf{A}}_{t_{i+1}}([j])\big)$ does not have the $(\beta_j, 2D+3)$-diet condition for some $1 \le j \le v(G'_{t_{i+1}})$, or $\phi^{\mathbf{A}}_{t_{i+1}}$ fails to have the $(\varepsilon, 20D\beta_j, j)$-cover condition for some $1 \le j \le v(G'_{t_{i+1}})+1-\varepsilon n$. Here $\beta_j = \beta_j\big(\alpha_{t_{i+1}-1}\big)$ is as defined in Lemma 51. The good event is that we have

$$\begin{aligned}
&\Big|\mathsf{N}_{H_{t_{i+1}-1}}(S) \setminus \bigcup_{j=1}^{i+1} \mathrm{im}^{\mathbf{A}}(j)\Big| \\
&= (1 \pm C'\alpha_{t_{i+1}})\big(1 \pm 2(i+1)(d_{\mathbf{A}}^{-1}+C')\alpha_{t_i} \pm \tfrac{3}{2}d_{\mathbf{A}}^{-1}\alpha_{t_{i+1}-1}\big)\Big(\tfrac{e(H_{t_{i+1}-1})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=1}^{i+1}\big(1-\tfrac{\mathrm{im}^{\mathbf{A}}(j)}{n}\big)\Big) \\
&= \big(1 \pm 2(i+1)(d_{\mathbf{A}}^{-1}+C')\alpha_{t_i} \pm \tfrac{3}{2}d_{\mathbf{A}}^{-1}\alpha_{t_{i+1}-1} \pm \tfrac{3}{2}C'\alpha_{t_{i+1}}\big)\Big(\tfrac{e(H_{t_{i+1}-1})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=1}^{i+1}\big(1-\tfrac{\mathrm{im}^{\mathbf{A}}(j)}{n}\big)\Big).
\end{aligned}$$

Finally, observe that at most $\frac{cn}{\log n}$ edges are removed from any vertex of S when $G'_{t_{i+1}}$ is embedded, so we get

$$\begin{aligned}
\Big|\mathsf{N}_{H_{t_{i+1}}}(S) \setminus \bigcup_{j=1}^{i+1} \mathrm{im}^{\mathbf{A}}(j)\Big| &= \Big|\mathsf{N}_{H_{t_{i+1}-1}}(S) \setminus \bigcup_{j=1}^{i+1} \mathrm{im}^{\mathbf{A}}(j)\Big| \pm |S|\Delta \\
&= \big(1 \pm 2(i+2)(d_{\mathbf{A}}^{-1}+C')\alpha_{t_{i+1}}\big)\Big(\tfrac{e(H_{t_{i+1}-1})}{\binom{n}{2}}\Big)^i n\Big(\prod_{j=1}^{i+1}\big(1-\tfrac{\mathrm{im}^{\mathbf{A}}(j)}{n}\big)\Big)
\end{aligned}$$

since α_j is increasing with j and since n is sufficiently large. For $0 \le i \le |T|-1$, this shows that with probability at least $1-n^{-C}+n^{-2D-18}$, if (49) holds for i then it holds for $i+1$ or one of the bad events of Lemma 50 or 51 occurs. Putting these together, we conclude that either a bad event of Lemma 50 or 51 occurs, or $(A.ii)$ holds for the given S and T with probability at least $1-(|T|+1)n^{-C}-|T|n^{-2D-18}$, and so taking the union bound over the at most $(2n)^{|S|+|T|}$ choices of S and T, we conclude that either one of the bad events of Lemma 50 or 51 occurs, or $(A.ii)$ holds, with probability at least $1-n^{-15}$. Now we already established in Section 8.2 that with high probability H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom for each $s \in \mathcal{K} \cup \mathcal{J}$, and Lemma 46 (a)—(b) state that the remaining bad events with high probability do not occur. So we conclude that with high probability $(A.ii)$ holds, as desired. □

For the two $\mathsf{OddVert}$ conditions, we do not need to obtain accurate estimates and will not try to do so. The first condition, $(A.iv)$, states that no vertex v of $H_{\mathbf{A}}$ can be the embedded neighbour of too many vertices of $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$. We separate this estimate into two parts: the contribution due to vertices of some G_s which are adjacent to at most $n^{0.8}$ vertices in $\mathsf{OddVert}_s$, and the rest. The contribution of the first class is not too large by Corollary 25 (the expected contribution is in fact constant, and by definition no one vertex embedding contributes more than $n^{0.8}$), and we will show that it is very unlikely that 20 or more vertices of the second class (of which there are at most $2n^{0.2}$)are embedded to v; since any one vertex contributes at most $\Delta(G_s) \le \frac{cn}{\log n}$ we obtain the claimed bound.

Proof of $(A.iv)$. Let $\mathcal{E}$ be the event that *PackingProcess* does not fail and that for each s the graph H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom and for each time in the embedding of G'_s by *RandomEmbedding* we have the $(C\alpha_{s-1}, 2D+3)$-diet condition. Note that $\mathcal{E}$ a.a.s. occurs.

Fix $v \in V(H_0)$. For any given $s \in \mathcal{K}$ and $y \in G'_s$, let $\mathscr{H}_y$ denote the history of *PackingProcess* up to immediately before embedding y. Suppose that $\mathscr{H}_y$ is in $\mathcal{E}$. The probability that we embed y to v, conditioning on $\mathscr{H}_y$, is at most $2d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}$, since we embed y to a set of size at least $\frac{1}{2}d_{\mathbf{A}}^{D}\delta n$ by the diet condition. Let $\omega(y)$ denote the number of neighbours of y in $\mathsf{OddVert}_s$. Then for $(A.iv)$ we want to show

$$\sum_y \omega(y)\mathbb{1}_{y\to v} \le \tfrac{20cn}{\log n}\,.$$

We split this sum into two parts. Let S denote the vertices y such that $\omega(y) \le n^{0.8}$. Note that $\sum_{y\in S}\omega(y) \le 2Dn$. So Corollary 25 tells us that the probability that $\mathcal{E}$ occurs and

$$\sum_{y\in S} \omega(y)\mathbb{1}_{y\to v} \ge \frac{4Dn}{d_{\mathbf{A}}^{D}\delta n} + n^{0.9}$$

is at most $2\exp\big(-\frac{n^{1.8}}{4n^{0.8}n^{0.9}}\big) \le \exp\big(-n^{0.05}\big)$.

Now consider the vertices of $\bar{S}$, i.e. those y such that $\omega(y) > n^{0.8}$. There are at most $2Dn^{0.2}$ such vertices. Since conditional probabilities multiply, the probability that $\mathcal{E}$ occurs and 20 or more of the vertices of $\bar{S}$ are mapped to v is at most

$$\binom{2Dn^{0.2}}{20}\big(2d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}\big)^{20} \le n^{-10}\,.$$

We conclude that with probability at least $1-n^{-9}$, either we are not in $\mathcal{E}$ or

$$\sum_y \omega(y)\mathbb{1}_{y\to v} \le \frac{4Dn}{d_{\mathbf{A}}^{D}\delta n} + n^{0.9} + 19\tfrac{cn}{\log n} \le \tfrac{20cn}{\log n}\,.$$

Taking the union bound over choices of v, and since as observed $\mathcal{E}$ a.a.s. occurs, we conclude that a.a.s. we have $(A.iv)$. $\square$

For $(A.v)$ we have to work a bit harder. Fix a vertex $v \in V(H_0)$. In order for a vertex $x \in \mathsf{OddVert}_s$ to contribute to the sum in $(A.v)$, two things have to happen. First, we have to embed $\mathsf{N}_{G_s}(x)$ to vertices of $\mathsf{N}_{H_{s-1}}(v)$ and not embed any vertex of G'_s to v (we say we create x covering v). Second, we have to not use any of the edges from v to the embedded neighbours of x in the remaining stages of *PackingProcess* (we say we lose x covering v if this happens). It turns out to be rather inconvenient to deal with these two things together.

So what we do is first show that we create a reasonably large number of $\mathsf{OddVert}$ covering v in total. We do this by considering the embedding of each G'_s one after another. It is enough to estimate, for each $s \in \mathcal{K}$ and $x \in \mathsf{OddVert}_s$, the probability that $\mathsf{N}_{G_s}(x)$ is embedded to $\mathsf{N}_{H_{s-1}}(v)$ and $v \notin \operatorname{im}\phi_s$, conditioned on the history of *PackingProcess* up to the point where H_{s-1} is defined; this is an easy consequence of the diet condition. Then Corollary 25 gives what we want, because the contribution made by embedding any one G'_s is at most $|\mathsf{OddVert}_s| \leq \frac{cn}{\log n}$. We should note that it is at this point where we need this bound on $|\mathsf{OddVert}_s|$. But for our approach this bound is necessary: if $|\mathcal{K}| \leq \frac{1}{10}\log n$, it is likely that there would be a vertex v in the image of all the ϕ_s for $s \in \mathcal{K}$, and hence $(A.v)$ would fail.

We then try to argue separately that we do not lose too many of these $\mathsf{OddVert}$ as edges are removed to form $H_{\mathbf{A}}$. For this argument, we do not work G'_s by G'_s, but rather consider separately each vertex embedding that causes an edge to be removed at v. There are two types of such vertex embedding: for each $s \in \mathcal{K} \cup \mathcal{J}$ the vertex embedded to v (if there is one) removes at most D edges, and thereafter each neighbour removes one edge. In both cases, by $(A.iv)$ the number of $\mathsf{OddVert}$ we lose is at most $\frac{20Dcn}{\log n}$, which is small enough for Corollary 25 to give useful probability bounds. For this sketch we ignore the first type, which turns out not to cause much trouble. It is not too hard to see that each time we remove an edge at v, we can give an upper bound on the number of $\mathsf{OddVert}$ covering v we lose which is proportional to the current number of $\mathsf{OddVert}$ covering v.

We separate two cases. First, the number of $\mathsf{OddVert}$ covering v never reaches ζn (where $\zeta > 0$ is some not-too-small constant). In this case, we can use Corollary 25 and the diet condition to show that the total number of $\mathsf{OddVert}$ covering v that we lose is small. But this gives us a contradiction: at the end of the process we have less than ζn $\mathsf{OddVert}$ covering v, and we lost in total few $\mathsf{OddVert}$, yet we created many $\mathsf{OddVert}$ covering v.

Second, there is some time in this process where there are ζn $\mathsf{OddVert}$ covering v. We argue (using Corollary 25 and the diet condition) that (ignoring any creation of new $\mathsf{OddVert}$ covering v) the number of edges we have to remove at v in order to lose $\frac{\zeta n}{2}$ $\mathsf{OddVert}$ is $\Theta(n)$. Once we removed half of the $\mathsf{OddVert}$, we lose $\mathsf{OddVert}$ at half the rate, so it takes as many edges to remove the next $\frac{\zeta n}{4}$ $\mathsf{OddVert}$, and so on. Since we remove less than n edges at v (because H_0 is an n-vertex graph) this argument necessarily stops after a constant number of iterations, and this gives us $(A.v)$.

Proof of (A.v). Let $\mathcal{E}$ be the event that *PackingProcess* does not fail, that for each s the graph H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom, and that at each time in using *RandomEmbedding* to embed G'_s the $(C\alpha_{s-1}, 2D+3)$-diet condition holds. Note that $\mathcal{E}$ a.a.s. occurs.

Fix $v \in V(H_0)$, $s \in \mathcal{K}$ and $x \in \mathsf{OddVert}_s$. Suppose that H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom. We aim to find a lower bound for the probability that $\mathsf{N}_{G_s}(x)$ is mapped by *RandomEmbedding* to $\mathsf{N}_{H_{s-1}}(v)$ and furthermore $v \notin \operatorname{im} \phi_s$. Observe that *RandomEmbedding* embeds in total less than n vertices of G'_s. Consider the following coupling with *RandomEmbedding*. At each time *RandomEmbedding* embeds a vertex y of G'_s which is not a neighbour of x, we sample from a distribution on $\{0,1\}$ with probability $2d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}$ of getting 1, such that if $y \to v$ we get 1. At each time *RandomEmbedding* embeds a vertex of G'_s which is a neighbour of x, we sample from a distribution on $\{0,1\}$ with $\frac{1}{2}d_{\mathbf{A}}$ probability of getting a zero, such that if $y \not\to \mathsf{N}_{H_{s-1}}(v)$ we get 1. We claim that if at each step we have the $(C\alpha_{s-1}, 2D+3)$-diet condition, then this coupling succeeds. Furthermore, if in the coupled independent Bernoulli random variables we never get 1, then necessarily we create x covering v. The probability of none of these Bernoulli random variables taking the value 1 is at most

$$\big(1 - 2d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}\big)^n \cdot \big(\tfrac{1}{2}d_{\mathbf{A}}\big)^D \le \exp\big(-4d_{\mathbf{A}}^{-D}\delta^{-1}\big)2^{-D}d_{\mathbf{A}}^D \le \exp\big(-8Dd_{\mathbf{A}}^{-D}\delta^{-1}\big)\,,$$

and the probability of the coupling failing — that is, of the diet condition at some point not holding — is at most $3n^{-9}$, so the probability that *RandomEmbedding* creates x covering v is at least $\exp\big(-10Dd_{\mathbf{A}}^{-D}\delta^{-1}\big)$.

By linearity of expectation, the expected number of vertices of $\mathsf{OddVert}_s$ which *RandomEmbedding* create covering v, conditioning on H_{s-1}, is at least $|\mathsf{OddVert}_s|\exp\big(-10Dd_{\mathbf{A}}^{-D}\delta^{-1}\big)$. Summing this over all $s \in \mathcal{K}$ we obtain at least $n\exp\big(-10Dd_{\mathbf{A}}^{-D}\delta^{-1}\big)$, since there are in total at least n $\mathsf{OddVert}$. Finally Corollary 25 tells us that, since no $\mathsf{OddVert}_s$ has size more than $\frac{cn}{\log n}$, with probability at least n^{-C} either $\mathcal{E}$ does not occur, or in total we create at least

$$\tfrac{1}{2}n\exp\big(-10Dd_{\mathbf{A}}^{-D}\delta^{-1}\big) \tag{50}$$

$\mathsf{OddVert}$ covering v. This completes the first step of the sketch.

We now need to show that we do not lose too many of the $\mathsf{OddVert}$ covering v we know exist. To that end, given H_{s-1}, let $X_{s-1} \subseteq \bigcup_{s' \in \mathcal{K}} \mathsf{OddVert}_{s'}$ be the set of $x \in \mathsf{OddVert}_{s'}$ with $s' \le s-1$ such that $\mathsf{N}_{G_{s'}}(x)$ is embedded to $\mathsf{N}_{H_{s-1}}(v)$ and $v \notin \operatorname{im}\phi_{s'}$. Let

$$\omega_{s-1}(u) := \sum_{x \in X_{s-1}} \mathbb{1}_{\phi_{s'}^{-1}(u)x \in E(G_{s'})}$$

where given x in the summation we set s' such that $x \in G_{s'}$. Thus removing the edge vu from H_{s-1} removes exactly $\omega(u)$ vertices from X_{s-1}; removing several edges vu_i removes at most $\sum_i \omega(u_i)$ vertices from X_{s-1}. Furthermore we have

$$|X_{s-1}| \le \sum_{u \in \mathsf{N}_{H_{s-1}}(u)} \omega(u) \le D|X_{s-1}|\,,$$

since each vertex of X_{s-1} has between one and D embedded neighbours. Consider the entire running of *PackingProcess* as it embeds each graph vertex by vertex. There are two possible*loss events*, i.e. ways that we can use edges at v: for each G'_s, the embedding of a vertex x (if it exists; if not we still count the embedding of G'_s as a loss event) of G'_s to v can use edges, and the subsequent times when neighbours of x are embedded. In either case, we use at most D edges at v and hence the maximum change is by $(A.iv)$ at most $\frac{20Dn}{\log n}$. We need to estimate the expected number of $\mathsf{OddVert}$ lost in each case. Note that if a non-zero number of $\mathsf{OddVert}$ are lost at v, then necessarily a vertex of G'_s is embedded to v and so no vertex of $\mathsf{OddVert}_s$ covers v.

We begin with estimating the effect of the vertex of G'_s embedded to v, conditioned on H_{s-1}. This is at most the effect of the entire embedding of G'_s. By linearity of expectation, this is

$$\sum_{u\in \mathsf{N}_{H_{s-1}}(v)} \omega(u)\mathbf{P}\big[uv \text{ used in embedding } G'_s\big|H_{s-1}\big]\,,$$

which we can estimate by (37) with $k=1$; we see that this conditional expectation is at most

$$2D|X_{s-1}|\tfrac{2e(G'_s)}{d_{\mathbf{A}}n^2} \leq 4D^2|X_{s-1}|n^{-1}\,.$$

Suppose that a vertex x of G'_s has been embedded to v; let y be a neighbour of x not yet embedded. The expectation of $\omega\big(\phi(u)\big)$, conditioned on the embedding up to immediately before embedding u and assuming this embedding satisfies the $(C\alpha_{s-1},2D{+}3)$-diet condition, is at most $\frac{D|X_{s-1}|}{\frac{1}{2}d_{\mathbf{A}}^D\delta n}$, since we embed y to a set of size at least the denominator which contains at most all vertices with positive weight.

In either case, the expected effect is at most

$$4D^2d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}|X_{s-1}|\,. \tag{51}$$

We want to estimate the sum of these conditional expectations in order to apply Corollary 25. The total number of loss events is at most $|\mathcal{K}|+|\mathcal{J}|+n\leq 3n$, since if we have removed n edges from v certainly $\mathcal{E}$ fails (the remaining graph is no longer quasirandom).

We now separate two cases. First, suppose that there is no s such that

$$|X_s|\geq \zeta n := \tfrac{1}{100}D^{-2}d_{\mathbf{A}}^D\delta\exp\big(-10Dd_{\mathbf{A}}^{-D}\delta^{-1}\big)n\,.$$

Then the sum of conditional expectations is at most $12D^2d_{\mathbf{A}}^{-D}\delta^{-1}\zeta n$. By Corollary 25, with probability at least n^{-C} either we are not in $\mathcal{E}$ or the total number of $\mathsf{OddVert}$ lost is at most $24D^2d_{\mathbf{A}}^{-D}\delta^{-1}\zeta n$. However, if this event occurs, after *PackingProcess* runs we have at most ζn leaves covering v, and have lost at most $24D^2d_{\mathbf{A}}^{-D}\delta^{-1}\zeta n$, so at most $25D^2d_{\mathbf{A}}^{-D}\delta^{-1}\zeta n$ leaves covering v can have been created. As we just showed in (50), the probability of $\mathcal{E}$ occurring and this few leaves being created is at most n^{-C}. So the probability of $\mathcal{E}$ occurring and being in this case is at most $2n^{-C}$.

Second, suppose that at some stage s' we have $|X_{s'}|\geq\zeta n$. We (slightly abusing notation) at this point ignore any new creation of $\mathsf{OddVert}$ covering v, and write X_s and $\omega_s(u)$ for $s\geq s'$

for the corresponding sets and functions. Note that (51) still upper bounds the expected loss of members of the (redefined) X_{s-1}.

Consider following *PackingProcess* from the point at which $X_{s'}$ is created (i.e. when $H_{s'}$ is defined) onwards. We *pause* for each $\ell \in \mathbb{N}$ when for the first time a vertex is embedded and we have lost at least $\frac{4^\ell-2}{4^\ell}|X_{s'}|$ of the $X_{s'}$ covering v. We claim that it is likely that we pause less than $25D^2 d_{\mathbf{A}}^{-D}\delta^{-1}$ times, so that in particular before the final vertex embedding which causes us to lose vertices of $X_{s'}$ covering v, there are at least $2\cdot 4^{-25D^2 d_{\mathbf{A}}^{-D}\delta^{-1}}|X_{s'}|$ vertices of $X_{s'}$ covering v; since that final vertex embedding causes us to lose at most $\frac{20Dn}{\log n}$ of the $X_{s'}$ covering v, when *PackingProcess* finishes, still at least $4^{-25D^2 d_{\mathbf{A}}^{-D}\delta^{-1}}|X_{s'}|$ vertices cover v, which is what we want to show.

Let us now prove this claim. At the start of the interval where we have paused $\ell-1$ times (when we begin to embed $G'_{s'+1}$, if $\ell=0$) we have between $4^{1-\ell}|X_{s'}|$ and $2\cdot 4^{1-\ell}|X_{s'}|$ vertices of $X_{s'}$ covering v. So assuming $\mathcal{E}$ holds, in this interval, at each loss event, the expected number of vertices of $X_{s'}$ covering v we lose, conditioned on the history immediately before the loss event, is by (51) at most

$$4D^2 d_{\mathbf{A}}^{-D}\delta^{-1}n^{-1}\cdot 2\cdot 4^{1-\ell}|X_{s'}|\,.$$

Suppose the interval where we have paused $\ell-1$ times has less than $\frac{1}{8}D^{-2}d_{\mathbf{A}}^{D}\delta n$ loss events. Then by Corollary 25 the probability that we are in $\mathcal{E}$ and lose in total $2\cdot 4^{-\ell}|X_{s'}|$ vertices covering v is at most n^{-C}. So within $\mathcal{E}$ the probability that this interval has less than $\frac{1}{8}D^{-2}d_{\mathbf{A}}^{D}\delta n$ loss events is at most n^{-C}. Taking the union bound over ℓ, the probability that we are in $\mathcal{E}$ and any interval has less than $\frac{1}{8}D^{-2}d_{\mathbf{A}}^{D}\delta n$ loss events is at most n^{1-C}. There are in total at most $|\mathcal{J}|+|\mathcal{K}|+n\le 3n$ loss events, so in this likely event there are less than $25D^2 d_{\mathbf{A}}^{-D}\delta^{-1}$ intervals, which is what we wanted to show. □

To prove $(A.iii)$ we use Lemma 49 to estimate the probability that the embedding of a given G_s by *RandomEmbedding* uses a vertex of S. Given this estimate, $(A.iii)$ follows easily from Corollary 25 and the union bound.

Proof of $(A.iii)$. Let $\mathcal{E}$ be the event that *PackingProcess* succeeds and for each s the graph H_{s-1} is $(\alpha_{s-1},2D+3)$-quasirandom.

Given $S_1,S_2\subseteq V(H_0)$ with $1\le|S_i|\le D$, $i=1,2$, and $s\in\mathcal{J}$, we condition on H_{s-1} and assume we are in $\mathcal{E}$. By Lemma 49 and the inclusion-exclusion principle the probability that $\operatorname{im}\phi_s^{\mathbf{A}}$ is disjoint from S_1 and includes S_2 is

$$(1\pm 200CD^2\alpha_{s-1}\delta^{-1})\big(\tfrac{n-1-v(G'_s)}{n}\big)^{|S_1|}\cdot\sum_{k=0}^{|S_2|}\binom{|S_2|}{k}=\big(1\pm\tfrac{1}{8}\gamma_{\mathbf{A}}\big)\cdot\big(\tfrac{n-v(G'_s)}{n}\big)^{|S_1|}\cdot\big(\tfrac{v(G'_s)}{n}\big)^{|S_2|}\,.$$

It follows that

$$\sum_{s\in\mathcal{J}_0}\mathbf{P}\big[\operatorname{im}\phi_s^{\mathbf{A}}\cap S_1=\emptyset, S_2\subseteq\operatorname{im}\phi_s^{\mathbf{A}}\big|H_{s-1}\big]=\big(1\pm\tfrac{1}{8}\gamma_{\mathbf{A}}\big)|\mathcal{J}_0|\cdot\big(\delta+10\lambda\big)^{|S_1|}\cdot\big(1-\delta-10\lambda\big)^{|S_2|}\,,$$

since each G'_s with $s \in \mathcal{J}_0$ has $(1-\delta-10\lambda)n$ vertices. We obtain similar estimates for $\mathcal{J}_1$ and $\mathcal{J}_2$; then by Corollary 25 and the union bound over $\{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2\}$ and choices of S_1 and S_2, $(A.iii)$ follows. □

To prove $(A.vii)$, given v, we need to estimate the probability that the embedding of a single G'_s with $s \in \mathcal{J}$ puts a $\mathsf{SpecPaths}$ end-vertex on v. Given such an estimate, Corollary 25 then tells us that with high probability we have all the three desired equalities for each v. To estimate the probability of embedding some path end-vertex of G_s to v, it is enough (since the events are disjoint) to estimate the probability of a specific path end-vertex $x \in G_s$ being embedded to v, which is provided by (34). The proof of $(A.viii)$ is essentially the same, except that (34) is replaced with (35), and we prove both statements together.

Proof of $(A.vii)$ *and* $(A.viii)$. Fix $u, v \in V(H)$, and let $\mathcal{E}$ be the event that *PackingProcess* succeeds and for each s the graph H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom.

We first prove $(A.vii)$. For a given $s \in \mathcal{J}$, let $X_s := \{\mathsf{leftpath}_0(P), \mathsf{rightpath}_0(P) : P \in \mathsf{SpecPaths}_s\}$. Note that $|X_s| = 2|\mathsf{SpecPaths}_s|$. Conditioning on H_{s-1}, and assuming we are in $\mathcal{E}$, for any $x \in X_s$ we have by (34)

$$\mathbf{P}\big[x \hookrightarrow v\big] = \big(1 \pm 10^4 C\alpha_{s-1} D\tilde{\delta}^{-1}\big)n^{-1}\,.$$

Since these events are disjoint, the conditional probability that some vertex of X_s is mapped to v is

$$\big(1 \pm 10^4 C\alpha_{s-1} D\tilde{\delta}^{-1}\big)\cdot 2|X_s|n^{-1} = \big(2 \pm \tfrac{1}{8}\gamma_{\mathbf{A}}\big)|X_s|n^{-1}\,.$$

For each $\mathcal{J}_i$ the sum of these conditional probabilities is easy to compute. For $\mathcal{J}_1$, we have $|\mathcal{J}_1|$ summands, in each of which $|X_s| = \sigma_1 n$. For $\mathcal{J}_0$ and $\mathcal{J}_2$, we have $|X_s| = \lambda n$. Applying Corollary 25, we see that with high probability either $\mathcal{E}$ does not occur, or we have $(A.vii)$. The former is unlikely, so with high probability we have $(A.vii)$.

The proof of $(A.viii)$ is almost identical, so we only mention the difference. For (a)–(c), we use (35) to obtain the probability that the embedding of G'_s maps a given $x \in X_s$ to v and also $u \notin \operatorname{im}\phi_s^{\mathbf{A}}$, and we note that $\big(n - v(G'_s)\big)n^{-1} = \delta + 10\sigma_1$ for $s \in \mathcal{J}_1$, $\big(n - v(G'_s)\big)n^{-1} = \delta + 10\lambda$ for $s \in \mathcal{J}_0$, and $\big(n - v(G'_s)\big)n^{-1} = \delta + 6\lambda$ for $s \in \mathcal{J}_2$. For (d), we use (36) to obtain the probability that the embedding of G'_s maps a given $x \in X_s$ to v and also $u, u' \notin \operatorname{im}\phi_s^{\mathbf{A}}$. □

Finally we prove $(A.ix)$. Here we do not need an accurate estimate. Given v_1, v_3, we find an upper bound for the probability of the embedding of a given G'_s contributing to the count for this pair, and apply Corollary 25 to obtain the desired result.

Proof of $(A.ix)$. Fix $v_1, v_2 \in V(H)$. Let $\mathcal{P} = \bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}_s$; we aim to count the number of paths in $\mathcal{P}$ whose ends are embedded to $\{v_1, v_2\}$.

Observe that the embedding of any given G'_s for $s \in \mathcal{J}$ contributes at most one to this count, so it is enough to find a good upper bound on the probability that a given G'_s with $s \in \mathcal{J}$ does contribute one. Consider the embedding of G'_s into H_{s-1} by *RandomEmbedding*. Suppose that at some point a vertex x, which is the first end of a path $P \in \mathcal{P}$, is embedded to one of $\{v_1, v_2\}$ and the other of $\{v_1, v_2\}$ is not yet used in the embedding. Then the embedding of G'_s contributes only if the other end y of P is embedded to the other of $\{v_1, v_2\}$. If the $(C\alpha_{s-1}, 2D+3)$-diet condition holds at the time when y is embedded, then y is embedded to a set of size at least $\frac{1}{2}d_{\mathbf{A}}^D\delta n$, so the probability of its being embedded to the other of $\{v_1, v_2\}$ is at most $\frac{2}{d_{\mathbf{A}}^D\delta n}$ in this case. If H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom, then by Lemma 46 the probability that the diet condition fails is at most $2n^{-9}$.

So we conclude that if H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom, the probability that G_s embeds the ends of a path of $\mathcal{P}$ to $\{v_1, v_2\}$ is at most $\frac{3}{d_{\mathbf{A}}^D\delta n}$. Let $\mathcal{E}$ be the event that H_{s-1} is $(\alpha_{s-1}, 2D+3)$-quasirandom for each s; then Corollary 25(b), with $\tilde{\mu} = \tilde{\nu} = 3\gamma^{-D}\delta^{-1}$ and $\tilde{\varrho} = n^{0.2}$, says that the probability that $\mathcal{E}$ occurs and yet more than $2n^{0.2}$ paths of $\mathcal{P}$ have their ends embedded to $\{v_1, v_2\}$, is at most $\exp\big(-n^{0.1}\big)$.

Taking the union bound over choices of v_1 and v_2, a.a.s. either $\mathcal{E}$ fails or we have $(A.ix)$. Since we proved that $\mathcal{E}$ a.a.s. does not fail, a.a.s. we have $(A.ix)$. □

9. Stage B (Proof of Lemma 31)

The main purpose of this stage is to embed $(\mathsf{OddVert}_s)_{s\in\mathcal{K}}$ so that the parities of the degrees of the vertices of the host graph are prepared for later stages of the embedding process. This key parity condition is $(B.iii)$. We begin by finding a pairing of vertices in $H_{\mathbf{A}}$ with some vertices of $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ such that embedding any collection of the paired $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ to their paired vertices of $H_{\mathbf{B}}$ gives a valid packing. Recall that each graph $G_s^{\spadesuit}$ with $s \in \mathcal{K}$ has by (5) $(1-\delta)n - |\mathsf{OddVert}_s|$ vertices. Since $|\mathsf{OddVert}_s| \le \frac{cn}{\log n}$ by Definition 9(e), this means $v(G_s^{\spadesuit}) = \big(1-\delta+o(1)\big)n$ and the $o(1)$ will always absorb into the error term (5) in the following arguments.

Lemma 52. *There is an injective map $\pi : V(H_{\mathbf{A}}) \to \bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ such that if $\pi(v) = x \in V(G_s)$, then $v \notin \mathrm{im}^{\mathbf{A}}(s)$, the pair $v\phi_s^{\mathbf{A}}(y)$ is an edge of $H_{\mathbf{A}}$ for each $y \in \mathsf{N}_{G_s}(x)$, and no edge of $H_{\mathbf{A}}$ is obtained twice in this way (i.e. no embedded neighbour of $\pi^{-1}\big(\phi_s^{\mathbf{A}}(y)\big)$ is embedded to v).*

Proof. Let $\overrightarrow{H}$ be a uniform random orientation of $E(H_{\mathbf{A}})$. Given any $\ell \le 2D+3$ and vertices $v_1, \dots, v_\ell$ of $H_{\mathbf{A}}$, and any subset T of $\mathsf{N}_{H_{\mathbf{A}}}(v_1, \dots, v_\ell)$ such that $|T| \ge \gamma_{\mathbf{A}} n$, the expected number of vertices of T which are out-neighbours of each of $v_1, \dots, v_\ell$ in $\overrightarrow{H}$ is $2^{-\ell}|T|$, and by Chernoff's inequality the probability that the actual number is not $(1 \pm \gamma_{\mathbf{A}})2^{-\ell}|T|$ is at most $\exp\big(-n^{0.1}\big)$. In particular with high probability, by the union bound, for each of the polynomially many sets defined by $(A.ii)$ with ℓ vertices, a $(1 \pm \gamma_{\mathbf{A}})2^{-\ell}$-fraction are out-neighbours of each of the ℓ vertices in $\overrightarrow{H}$. We will need two specific cases of this: if $x \in \mathsf{OddVert}_s$, then

$$\big|\mathsf{N}_{\overrightarrow{H}}^{out}\big(\phi_s^{\mathbf{A}}(\mathsf{N}_{G_s}(x))\big) \setminus \mathrm{im}^{\mathbf{A}}(s)\big| = (1 \pm 3\gamma_{\mathbf{A}})2^{-D_{\mathrm{odd}}} d_{\mathbf{A}}^{D_{\mathrm{odd}}}\delta n\,, \tag{52}$$

and if in addition $y \in \mathsf{OddVert}_{s'}$, with $s' \neq s$, is such that $\phi_s^{\mathbf{A}}(\mathsf{N}_{G_s}(x))$ and $\phi_{s'}^{\mathbf{A}}(\mathsf{N}_{G_{s'}}(y))$ are disjoint, we have

$$\big|\mathsf{N}_{\overrightarrow{H}}^{out}\big(\phi_s^{\mathbf{A}}(\mathsf{N}_{G_s}(x)) \cup \phi_{s'}^{\mathbf{A}}(\mathsf{N}_{G_{s'}}(y))\big) \setminus \big(\mathrm{im}^{\mathbf{A}}(s) \cup \mathrm{im}^{\mathbf{A}}(s')\big)\big| = (1 \pm 3\gamma_{\mathbf{A}})2^{-2D_{\mathrm{odd}}} d_{\mathbf{A}}^{2D_{\mathrm{odd}}}\delta^2 n\,. \tag{53}$$

Furthermore, from $(A.v)$, with high probability for each $v \in V(\overrightarrow{H})$ there are at least

$$2^{-D_{\mathrm{odd}}-1} \cdot 4^{-50D^2 d_{\mathbf{A}}^{-D} \delta^{-1}} n \tag{54}$$

vertices of $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ which we could map to v using only edges going to v and without destroying the packing (i.e. v is not in the image of that particular graph).

We draw an auxiliary bipartite graph F whose parts are $V(\overrightarrow{H})$ and $\bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$. We put an edge from $x \in \mathsf{OddVert}_s$ to $v \in V(\overrightarrow{H})$ if and only if $v \notin \mathrm{im}^{\mathbf{A}}(s)$ and v is an out-neighbour of $\phi_s^{\mathbf{A}}(y)$ for each $y \in \mathsf{N}_{G_s}(x)$.

Claim 52.1. *There exists a matching in F covering $V(\overrightarrow{H})$.*

Such a matching gives the desired π; the orientation of $\overrightarrow{H}$ ensures that no edge is used twice.

Proof of Claim 52.1. We verify the defect Hall condition for F. To this end, consider an arbitrary nonempty set $T \subseteq V(\overrightarrow{H})$. Let $S \subseteq \bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s$ be the vertices of adjacent to at least one member of T.

Firstly, applying (54) to an arbitrary vertex of T, we see that $|S| \geq 2^{-D_{\mathrm{odd}}-1} \cdot 4^{-50D^2 d_{\mathbf{A}}^{-D} \delta^{-1}} n$. In particular, the only cases for which it remains to prove Hall's condition are when

$$|T| > 2^{-D_{\mathrm{odd}}-1} \cdot 4^{-50D^2 d_{\mathbf{A}}^{-D} \delta^{-1}} n \,. \tag{55}$$

We claim that in such a case, we actually have in this case we have

$$|S| \geq \Big| \bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s \Big| - 20\gamma_{\mathbf{A}} n \,. \tag{56}$$

This obviously verifies Hall's condition since

$$|S| \geq \Big| \bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s \Big| - 20\gamma_{\mathbf{A}} n > n \geq |T| \,.$$

So, suppose for a contradiction that (56) fails, and for the set $S^* := \bigcup_{s\in\mathcal{K}} \mathsf{OddVert}_s \setminus S$ we have

$$|S^*| \geq 20\gamma_{\mathbf{A}} n \,. \tag{57}$$

Let $T^* := V(H) \setminus T$. Let us count the number N of triples (u, v, u') such that $u, u' \in S^*$ and $v \in T^*$ so that $uv, u'v \in E(F)$.

By (52), we have that the average degree of a vertex in T^* to S^* is

$$\frac{e_F(T^*, S^*)}{|T^*|} = \frac{\sum_{x\in S^*} \deg_F(x)}{|T^*|} \geq \frac{(1-3\gamma_{\mathbf{A}}) d_{\mathbf{A}}^{D_{\mathrm{odd}}} 2^{-D_{\mathrm{odd}}} \delta n |S^*|}{|T^*|} \,.$$

Hence by Jensen's inequality the average number of pairs of F-neighbours a vertex $t \in T^*$ has in S^* is at least

$$\frac{|S^*|^2}{|T^*|^2} (1-3\gamma_{\mathbf{A}})^2 d_{\mathbf{A}}^{2D_{\mathrm{odd}}} 2^{-2D_{\mathrm{odd}}} \delta^2 n^2 \,.$$

It follows that

$$N \geq \frac{|S^*|^2}{|T^*|}(1-3\gamma_{\mathbf{A}})^2 d_{\mathbf{A}}^{2D_{\mathrm{odd}}} 2^{-2D_{\mathrm{odd}}}\delta^2 n^2 \,. \tag{58}$$

On the other hand, we bound N from above by counting starting from pairs (u,u') of (not necessarily distinct) vertices of S^*. Let $\boldsymbol{\iota}(u)$ denote the index such that $u \in V(G_{\boldsymbol{\iota}(u)})$. We say that (u,u') is of *Type 1* if $\boldsymbol{\iota}(u)=\boldsymbol{\iota}(u')$. We say that (u,u') is of *Type 2* if $\boldsymbol{\iota}(u)\neq\boldsymbol{\iota}(u')$ and the embeddings of the neighbourhoods of u and u' are not disjoint; in particular this happens if $\boldsymbol{\iota}(u)=\boldsymbol{\iota}(u')$. The remaining pairs (u,u') are *Type 3*. To bound the number M_1 of pairs of Type 1, we have

$$M_1 = \sum_{x\in S^*} |S^* \cap \mathsf{OddVert}_{\boldsymbol{\iota}(x)}| \overset{\mathrm{D9}(e)}{\leq} |S^*|\cdot \tfrac{cn}{\log n}\,. \tag{59}$$

To bound the number M_2 of pairs of Type 2, we consider an arbitrary vertex $x\in S^*$. By $(A.iv)$, we know that for each $v\in\phi^{\mathbf{A}}_{\boldsymbol{\iota}(x)}(\mathsf{N}_{G_{\boldsymbol{\iota}(x)}}(x))$ there there are at most $\frac{20cn}{\log n}$ vertices $x'\in S^*$ whose embedded neighbourhood touches v. That means that there are at most $\frac{20D_{\mathrm{odd}}cn}{\log n}$ vertices $x'\in S^*$ whose embedded neighbourhood is not disjoint with the embedded neighbourhood of x. Hence,

$$M_2 \leq \frac{20D_{\mathrm{odd}}cn}{\log n}\cdot|S^*|\,. \tag{60}$$

For the number M_3 of pairs of Type 3, we use the trivial bound

$$M_3 \leq |S^*|^2\,. \tag{61}$$

We now need to bound the number of ways a given pair (u,u') can be extended to a triple (u,v,u') as above. We shall use the trivial bound that the number of such extensions is at most $|T^*|$ for pairs of Type 1 and 2. For pairs of Type 3, we use (53) and see that the number of the extensions is at most

$$(1+3\gamma_{\mathbf{A}})d_{\mathbf{A}}^{2D_{\mathrm{odd}}}2^{-2D_{\mathrm{odd}}}\delta^2 n\,.$$

Using (59), (60) and (61), we obtain

$$\begin{aligned} N &\leq M_1\cdot|T^*| + M_2\cdot|T^*| + M_3\cdot(1+3\gamma_{\mathbf{A}})d_{\mathbf{A}}^{2D_{\mathrm{odd}}}2^{-2D_{\mathrm{odd}}}\delta^2 n \\ &\leq |S^*|^2(1+4\gamma_{\mathbf{A}})d_{\mathbf{A}}^{2D_{\mathrm{odd}}}2^{-2D_{\mathrm{odd}}}\delta^2 n\,, \end{aligned} \tag{62}$$

where the argument that the contribution from Type 1 and 2 is negligible compared to Type 3 uses (57).

Comparing (58) and (62), we get

$$n(1-3\gamma_{\mathbf{A}})^2 \leq (1+4\gamma_{\mathbf{A}})|T^*|\,,$$

which is a contradiction to (55).

This verifies the defect Hall condition. □

The existence of the map π follows by the discussion above. □

Now let

$$S := \left\{ v \in V(H_{\mathbf{A}}) : \deg_{H_{\mathbf{A}}}(v) \not\equiv \mathrm{PathTerm}(v) + \mathrm{OddOut}(v) \mod 2 \right\}.$$

Observe that $X := \bigcup_{s \in \mathcal{K}} \mathsf{OddVert}_s \setminus \pi(S)$ is a set of size at least $\gamma_{\mathbf{A}} n$. Let us now show that $|X|$ is even. Indeed, writing $=$ for equality and $\equiv$ for equality modulo 2, we have

$$\begin{aligned} |X| &\equiv \sum_{s \in \mathcal{K}} |\mathsf{OddVert}_s| + \sum_{v \in V(H)} \big(\deg_{H_{\mathbf{A}}}(v) - \mathrm{OddOut}(v) - \mathrm{PathTerm}(v)\big) \\ &= \sum_{v \in V(H)} \big(\deg_{H_{\mathbf{A}}}(v) - \mathrm{PathTerm}(v)\big) \\ &\equiv 2e(H_{\mathbf{A}}) - 2\left|\bigcup_{s \in \mathcal{J}} \mathsf{SpecPaths}_s\right| \equiv 0\,. \end{aligned}$$

We draw a graph F' on vertex set X, putting an edge between $u, v \in X$ if $\boldsymbol{\iota}(u) \neq \boldsymbol{\iota}(v)$ and their embedded neighbours are disjoint. By $(A.iv)$ and Definition 9(e), every vertex of X is adjacent to all but at most $(D_{\mathrm{odd}} + 1)cn$ others. In particular F' has minimum degree at least $|X|/2$, so by Dirac's theorem F' contains a perfect matching, say M.

We now construct the $\phi_s^{\mathbf{B}}$ for $s \in \mathcal{K}$. First, for each $x \in S$, suppose $x \in G_s$ and extend $\phi_s^{\mathbf{A}}$ by mapping x to $\pi^{-1}(x)$. Let H' be the graph of edges remaining from $H_{\mathbf{A}}$ (removing the edges used by this embedding). Having embedded each $x \in S$, we greedily embed pairs of vertices of X as given by M. Suppose $xy \in M$, with $x \in G_s$ and $y \in G_{s'}$. Recall that $s \neq s'$. We choose $v \in V(H)$ which is not in $\mathrm{im}^{\mathbf{A}}(s) \cup \mathrm{im}^{\mathbf{A}}(s')$, and which was not used to embed any vertex of $\mathsf{OddVert}_s \cup \mathsf{OddVert}_{s'}$, and which is a common neighbour in H' of the vertices $\phi_s^{\mathbf{A}}(\mathsf{N}_{G_s}(x)) \cup \phi_{s'}^{\mathbf{A}}(\mathsf{N}_{G_{s'}}(y))$, and which was not chosen more than cn times in this process. We embed both x and y to v, and remove from H' the edges used by these embeddings. We claim this process succeeds, and defines the required embeddings $\phi_s^{\mathbf{B}}$ for $s \in \mathcal{K}$ and remaining graph $H_{\mathbf{B}}$. Note that if the process succeeds, then we automatically have $(B.i)$, $(B.ii)$.

Observe that in this construction, there are two ways we use edges at a given vertex v. We use an edge at v for each vertex of $\bigcup_s \mathsf{OddVert}_s$ for which v is an embedded neighbour, and we use edges at v whenever we embed a vertex of $\bigcup_s \mathsf{OddVert}_s$ to v. By $(A.iv)$, we use at most $\frac{20cn}{\log n}$ edges of the first type. Since each choice of v to embed one or two $\mathsf{OddVert}$ uses at most $2D_{\mathrm{odd}}$ edges at v, and since we choose v at most cn times, there are at most $6Dcn$ edges of the second type. So in total the degree of v decreases by at most $8Dcn$ from $H_{\mathbf{A}}$ to $H_{\mathbf{B}}$, hence obtaining $(B.iv)$. Now, the only way in which our construction can fail is that at some point we need to embed some matched pair $xy \in M$ and all the vertices to which we can embed them have already been used cn or more times. By $(A.ii)$, the set of vertices to which we could embed both x and y before the start of this stage has size at least $(1 - \gamma_{\mathbf{A}}) d_{\mathbf{A}}^{2D} \delta^2 n$. Of these vertices, at most $\frac{2cn}{\log n}$ may have been used for embedding the $\mathsf{OddVert}_s$ and $\mathsf{OddVert}_{s'}$ (where $x \in G_s$ and $y \in G_{s'}$) and hence have become unavailable. A further at most $16D^2cn$ may have become unavailable since an edge from one of the at most $2D$ embedded neighbours of x or y has been used. However this leaves at least cn vertices; they cannot all have been used cn times since the size of $\bigcup_s \mathsf{OddVert}_s$ is at most $2n$. We conclude that the construction process succeeds.

Let us now turn to the key property $(B.iii)$. Given a vertex $v \in V(H_{\mathbf{A}})$, the each edge vw used in Stage B was of the following two types:

(a) for some $x \in \mathsf{OddVert}_s$, and some $y \in \mathsf{N}_{G_s}(x)$, y was embedded on v (in Stage A) and x was embedded on w (in Stage B),

(b) for some $x \in \mathsf{OddVert}_s$, and some $y \in \mathsf{N}_{G_s}(x)$, y was embedded on w (in Stage A) and x was embedded on v (in Stage B).

The number of edges as in (a) is $\mathrm{OddOut}(v)$. As for edges as in (b) in the case $v \in S$, we recall when we placed $\pi(v)$ we used exactly D_{odd} many such edges, which is odd (c.f. Definition 9). Additional edges as in (b) come exclusively by placing pairs $\{x, y\} \subseteq \bigcup_s \mathsf{OddVert}_s$ (here, $xy \in M$) on v; placement of each such pair uses exactly $2D_{\mathrm{odd}}$ edges, which is even. In the case $v \notin S$, the edges incident with v considered in (b) come only from placing pairs, and hence their total number is even. Put together, we see that if $v \in S$ the total number of edges incident to v used in Stage B is equal modulo 2 to $\mathrm{OddOut}(v) + 1$, while if $v \notin S$, it is equal modulo 2 to $\mathrm{OddOut}(v)$. This establishes $(B.iii)$.

Observe that $(B.v)$ follows from $(A.ii)$ and $(B.iv)$.

10. Stage C (Proof of Lemma 33)

If n is odd, we pick an arbitrary vertex of H to be $\boxdot$. We then need to embed all the paths in $\bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}_s$ which are anchored at $\boxdot$ (of which there can be at most one per $s \in \mathcal{J}$) and in addition some more paths in order that all the edges at $\boxdot$ are used, and exactly one path from each $\mathsf{SpecPaths}_s$ for $s \in \mathcal{J}_0$ is embedded.

Let us first provide an overview of the proof. Set

$$\mathcal{J}_{\boxdot} := \left\{ s \in \mathcal{J} \ : \ \exists P \in \mathsf{SpecPaths}_s, \{\phi_s^{\mathbf{B}}(\mathsf{leftpath}_0(P)), \phi_s^{\mathbf{B}}(\mathsf{rightpath}_0(P)\} \cap \{\boxdot\} \neq \emptyset \right\} .$$

Observe that for each $s \in \mathcal{J}_{\boxdot}$, there is exactly one such path, which we denote by $P_s^{\boxdot}$. We need to use $|\mathcal{J}_{\boxdot}|$ edges leaving $\boxdot$ to embed these paths, after which we are left with $\deg_{H_{\mathbf{B}}}(\boxdot) - |\mathcal{J}_{\boxdot}|$ edges remaining to cover. By $(B.iii)$, $q := \frac{\deg_{H_{\mathbf{B}}}(\boxdot) - |\mathcal{J}_{\boxdot}|}{2}$ is an integer. By $(A.ii)$, with $S = \{\boxdot\}$ and $T = \emptyset$, we have $\deg_{H_{\mathbf{A}}}(\boxdot) = (1 \pm \gamma_{\mathbf{A}})d_{\mathbf{A}}n$ and by $(B.iv)$ we therefore have

$$\deg_{H_{\mathbf{B}}}(\boxdot) = (1 \pm 2\gamma_{\mathbf{A}})d_{\mathbf{A}}n \overset{(10)}{=} (14 \pm 0.2)\lambda^2 n .$$

By $(A.vii)$ we then have

$$|\mathcal{J}_{\boxdot}| = (2 \pm \gamma_{\mathbf{A}})\big(\lambda|\mathcal{J}_0| + \sigma_1|\mathcal{J}_1| + \lambda|\mathcal{J}_2|\big) = (2 \pm 0.1)\lambda^2 n \text{ , and} \tag{63}$$

$$q = (6 \pm 0.3)\lambda^2 n . \tag{64}$$

The only part of this embedding where we need to be careful is when we embed the edges at $\boxdot$. To do this, we use a matching argument. Recall that since in Stage B we embed only edges of graphs G_s for $s \in \mathcal{K}$, we have for each $s \in \mathcal{J}$ that $\mathrm{im}^{\mathbf{A}}(s) = \mathrm{im}^{\mathbf{B}}(s)$. We define an auxiliary balanced bipartite graph B. One part of B is $X := \{P_s^{\boxdot} : s \in \mathcal{J}_{\boxdot}\} \cup \{x_i, y_i : i \in [q]\}$, where x_i and y_i are auxiliary elements. The other part of B is $\mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$. We draw an edge in this graph from each $P_s^{\boxdot}$ to each $u \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$ such that $u \notin \mathrm{im}^{\mathbf{B}}(s)$. To define the remaining edges, we work as follows. We define

$$Z := \left\{s \in \mathcal{J} \setminus \mathcal{J}_{\boxdot} : \boxdot \notin \mathrm{im}^{\mathbf{B}}(s)\right\} = \left\{s \in \mathcal{J} : \boxdot \notin \mathrm{im}^{\mathbf{B}}(s)\right\} .$$

From $(A.iii)$ with $S_1 = \{\boxdot\}$ and $S_2 = \emptyset$, we see that

$$(1 - \gamma_{\mathbf{A}})\delta|\mathcal{J}| \leq |Z| \leq (1 + \gamma_{\mathbf{A}})\delta|\mathcal{J}| . \tag{65}$$

In particular, $|Z| \geq q$. Hence, we can choose uniformly at random distinct indices $s_1, \dots, s_q$, and for each $i \in [q]$ we draw an edge from each of x_i and y_i to each $u \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$ such that $u \notin \mathrm{im}^{\mathbf{B}}(s_i)$. The point of this graph is the following: a perfect matching in B tells us how to embed the first edge of each $P_s^{\boxdot}$ (and we will be able to greedily finish off the embedding) and for each $i \in [q]$, two edges which will be in the middle of a path of $\mathsf{SpecPaths}_{s_i}$ (we will be able to choose which path, and complete the embedding greedily). See Figure 5.

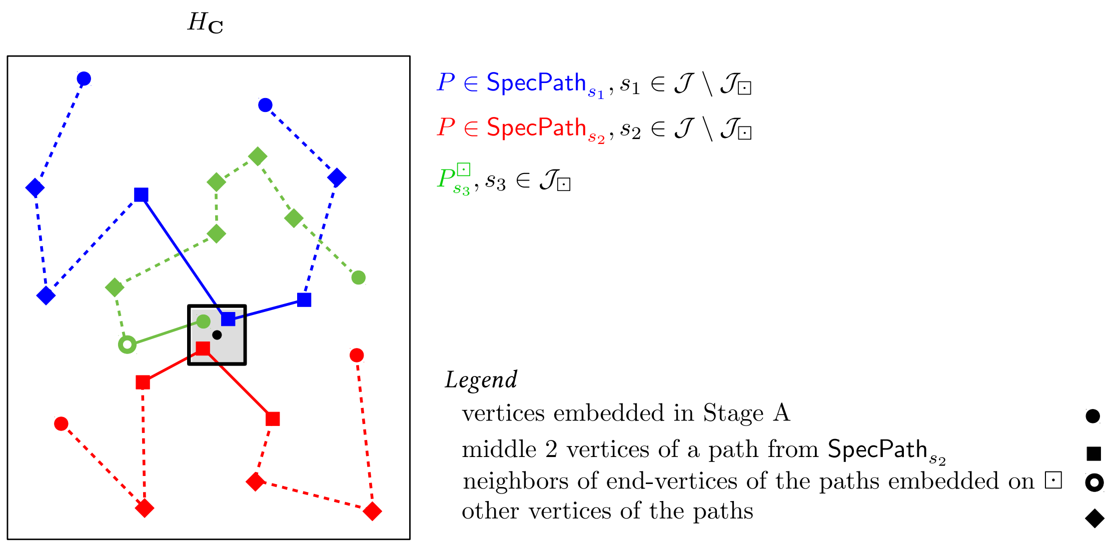

FIGURE 5. Embedding in Stage C. Vertex $\boxdot$ enlarged for clarity.

Claim 52.2. *With positive probability, there is a perfect matching M in B.*

Proof. Since B is balanced, the desired property is equivalent to the existence of a matching covering X. To that end, we shall use Hall's condition. Let us note some degree and codegree properties of B. For any vertex a in X, by $(A.ii)$, with $S = \{\boxdot\}$ and T being either s where $a = P_s^{\boxdot}$ or s_i where $a \in \{x_i, y_i\}$, and by $(B.iv)$, we see that

$$\deg_B(a) = (1 \pm \gamma_{\mathbf{A}})d_{\mathbf{A}}n\big(1 - \tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big) \pm 8Dcn = (1 \pm 2\gamma_{\mathbf{A}})d_{\mathbf{A}}n\big(1 - \tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big) ,$$

where the first term comes from $(A.ii)$ and the second term uses $(B.iv)$ to estimate the number of these edges removed in Stage B.

Similarly, given $a, a' \in X$ which are not of the form x_i, y_i for any $i \in [q]$, again using $(A.ii)$ with $S = \boxdot$ and this time with $T = \{s, s'\}$ where s is the index corresponding as above to a, and s' (which is by assumption not s) is the index corresponding to a', we see that

$$\deg_B(\{a, a'\}) = (1 \pm 2\gamma_{\mathbf{A}})d_{\mathbf{A}}n\big(1 - \tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1 - \tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)\,. \tag{66}$$

Finally, given $v \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$, using $(A.iii)$ with $S_1 = \{\boxdot, v\}$ and $S_2 = \emptyset$, we see that for the number f of indices $s \in \mathcal{J}$ such that neither $\boxdot$ nor v is in $\mathrm{im}^{\mathbf{B}}(s)$ we have $f \geq (1 - \gamma_{\mathbf{A}})\delta^2|\mathcal{J}|$. Hence the number of indices s_i with $v \notin \mathrm{im}^{\mathbf{B}}(s_i)$ has hypergeometric distribution with parameters $|Z|$, f, q. Hence, the expectation of this random variable is

$$\frac{fq}{|Z|} \overset{(64),(65)}{\geq} \frac{(1-\gamma_{\mathbf{A}})\delta^2|\mathcal{J}| \cdot 5\lambda^2 n}{(1+\gamma_{\mathbf{A}})\delta|\mathcal{J}|} \geq 40\gamma_{\mathbf{A}}n\,.$$

Fact 23 tells us that the probability that we choose less than $30\gamma_{\mathbf{A}}n$ of these indices among $s_1, \ldots, s_q$ is less than n^{-2}. In particular, taking the union bound over the at most n choices of v, we see $\deg_B(v) \geq 30\gamma_{\mathbf{A}}n$ holds for all $v \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$ with probability at least $1 - n^{-1}$. Suppose that this likely event occurs.

Now given any non-empty $X' \subseteq X$, we verify Hall's condition. We shall distinguish three cases based on the size of $|X'|$.

If $|X'| < \frac{1}{2}d_{\mathbf{A}}\delta n$, then choosing any $x \in X'$ (which can be either in $\{P_s^{\boxdot} : s \in \mathcal{J}_{\boxdot}\}$ or in $\{x_i, y_i : i \in [q]\}$) we have by $(A.ii)$ (and by $(B.iv)$) that $\deg_B(x) > |X'|$. This verifies Hall's condition for X'.

If $|X'| > (1-20\gamma_{\mathbf{A}})d_{\mathbf{A}}n$, then we claim that the joint neighborhood of X' in B is the entire part $\mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$, which will obviously verify Hall's condition for X'. Indeed, suppose that $v \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$ is an arbitrary vertex. By $(A.iii)$, there are at least $(1 - \gamma_{\mathbf{A}})\delta|\mathcal{J}|$ indices $\mathcal{Q}_v := \{s \in \mathcal{J} : v \notin \mathrm{im}^{\mathbf{B}}(s)\}$. So, the claim will follow if $(\mathcal{J}_{\boxdot} \cup \{s_1, \ldots, s_q\}) \cap \mathcal{Q}_v \neq \emptyset$, which we will show next. Indeed, otherwise we would have

$$|\mathcal{J}| \geq |\mathcal{J}_{\boxdot}| + q - |X \setminus X'| + |\mathcal{Q}_v| \overset{(63),(64)}{\geq} 7.6\lambda n - 40\gamma_{\mathbf{A}}d_{\mathbf{A}}n + (1-\gamma_{\mathbf{A}})\delta|\mathcal{J}| \overset{(2)}{>} |\mathcal{J}|\,,$$

a contradiction.

Suppose now X' is any set with $\frac{1}{2}d_{\mathbf{A}}\delta n \leq |X'| \leq (1-20\gamma_{\mathbf{A}})d_{\mathbf{A}}n$. Let the joint neighbourhood of X' in B be Y. We count the total number of triples (a, a', y) with $a, a' \in X'$ and $y \in Y$ such that ay and $a'y$ are edges of B. On the one hand, this is equal to

$$\sum_{a,a' \in X'} \deg_B(\{a, a'\})\,.$$

We can use the upper bound (66) for all but at most $3|X'|$ pairs a, a'. On the other, it is also equal to

$$\sum_{y\in Y}\deg_B(y)^2 \geq \tfrac{1}{|Y|}\Big(\sum_{y\in Y}\deg_B(y)\Big)^2 \geq \tfrac{1}{|Y|}\Big(\sum_{a\in X'}\deg_B(a)\Big)^2 = \tfrac{1}{|Y|}\sum_{a,a'\in X'}\deg_B(a)\deg_B(a')$$

by Jensen's inequality. Putting these two together, and for convenience writing s and s' respectively for the indices corresponding to a and a' in summations, we have

$$\sum_{a,a'\in X'}(1+2\gamma_{\mathbf{A}})d_{\mathbf{A}}n\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)+3|X'|n$$
$$\geq \tfrac{1}{|Y|}\sum_{a,a'\in X'}(1-2\gamma_{\mathbf{A}})^2d_{\mathbf{A}}^2n^2\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)\,.$$

Now the $3|X'|n$ term on the first line is, because $|X'|$ is sufficiently large, small compared to the main term; in particular we have

$$\sum_{a,a'\in X'}(1+3\gamma_{\mathbf{A}})d_{\mathbf{A}}n\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)$$
$$\geq \tfrac{1}{|Y|}\sum_{a,a'\in X'}(1-2\gamma_{\mathbf{A}})^2d_{\mathbf{A}}^2n^2\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)\,.$$

Now most of the terms cancel (in particular the sum over $\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s)|}{n}\big)\big(1-\tfrac{|\mathrm{im}^{\mathbf{A}}(s')|}{n}\big)$ is identical on both sides) and we get $(1+3\gamma_{\mathbf{A}}) \geq \tfrac{1}{|Y|}(1-2\gamma_{\mathbf{A}})^2d_{\mathbf{A}}n$. In particular, we have $|Y| > |X'|$ as required. This completes the verification of Hall's condition. □

So, suppose that our choice of indices $s_1, \dots, s_q$ yields a perfect matching in Claim 52.2. We now explain how to use this perfect matching M to embed complete paths. The condition we need to maintain is that we do not use any vertex other than $\boxdot$ more than $n^{0.5}$ times in these embeddings. We assign to each vertex v in $V(H)\setminus\{\boxdot\}$ a number of uses, which initially is equal to the number of paths in $\{P_s^{\boxdot} : s \in \mathcal{J}_{\boxdot}\}$ which are anchored at v (i.e. their endvertex not mapped by $\phi_s^{\mathbf{A}}$ to $\boxdot$ is mapped to v), plus one if $v \in \mathsf{N}_{H_{\mathbf{B}}}(\boxdot)$. By $(A.ix)$, the initial number of uses of any given v is at most $n^{0.3}+1$. Whenever we use a vertex in the following greedy embeddings, we add one to its number of uses. When a vertex has number of uses $n^{0.5}$, we mark it as forbidden; we claim that at no stage do we have more than $20n^{0.5}$ forbidden vertices. Indeed, initially summing the number of uses over all vertices v we have at most n (since at most $n/2$ paths are anchored at $\boxdot$ by $(A.vii)$, and since the number of 'plus ones' is $\deg_{H_{\mathbf{B}}}(\boxdot) \leq n/2$). The total number of paths we embed greedily is at most n (since $|\mathcal{J}| \leq n$ and we embed at most one path of any given graph), and all these paths have at most 11 vertices, hence the total number of uses cannot exceed $20n$.

To begin with, we choose for each $i \in [q]$ a path $P_{s_i}^{\boxdot}$, and for each $s \in \mathcal{J}_0 \setminus \mathcal{J}_{\boxdot}$ such that $s \neq s_i$ for $i \in [q]$ a path $P_s^{\boxdot}$. We choose these greedily, in each step choosing a path in G_s whose endvertices are both mapped by $\phi_s^{\mathbf{A}}$ to vertices which are not forbidden; we then add one to the number of uses of each endvertex and update the set of forbidden vertices. Since each $\mathsf{SpecPaths}_s$ contains at least $\sigma_1 n$ paths, of which at most $40n^{0.5}$ are forbidden, this is always possible.

To begin with, we choose greedily embeddings of the paths $P_s^{\boxdot}$ for $s \in \mathcal{J}_{\boxdot}$. For $s \in \mathcal{J}_{\boxdot}$, we start with the embedding $\phi_s^{\mathbf{A}}$, and extend it to $\phi_s^{\mathbf{C}-}$ by mapping the vertex u_2 of $P_s^{\boxdot}$ adjacent to $\big(\phi_s^{\mathbf{A}}\big)^{-1}(\boxdot)$ to the vertex v_2 paired in the matching M with $P_s^{\boxdot}$. Let $v' = \phi_s^{\mathbf{A}}(u')$ where u' is the other endvertex of $P_s^{\boxdot}$. We then for each $i \geq 3$ in succession map the ith unembedded vertex u_i of $P_s^{\boxdot}$ to a vertex v_i in $\mathsf{N}_{H_{\mathbf{B}}}(v_{i-1}, v') \setminus \mathrm{im}^{\mathbf{A}}(s)$. We choose v_i which is not forbidden, which is not one of the at most 10 previous vertices used for G_s in Stage C, and such that the edges $v_{i-1}v_i$ and $v_{i-1}v'$ have not been used for any previous embedding in Stage C. By construction, the result is an embedding of $P_s^{\boxdot}$ if each step is possible. We claim that each step is indeed possible: by $(A.ii)$, with $S = \{v_{i-1}, v'\}$ and $T = \{s\}$, there are at least $(1-\gamma_{\mathbf{A}})(d_{\mathbf{A}})^2\delta n$ vertices which are adjacent in $H_{\mathbf{A}}$ to both v_{i-1} and v' and which are not in $\mathrm{im}^{\mathbf{A}}(s)$. At most $16Dcn$ of these vertices are no longer in the common neighbourhood in $H_{\mathbf{B}}$ by $(B.iv)$; at most $20n^{0.5}$ are forbidden; at most 10 were previously used in Stage C for G_s; and at most $2n^{0.5}$ of these vertices cannot be used because the edge to v_{i-1} or v' was previously used in Stage C. Thus there are at least $(1-2\gamma_{\mathbf{A}})(d_{\mathbf{A}})^2\delta n$ valid choices.

Next, we embed the paths $P_{s_i}^{\boxdot}$ for $i \in [q]$. We use the same strategy, except that for each i we embed separately a path from the M-neighbour of x_i to one endvertex of $P_{s_i}^{\boxdot}$, and a path from the M-neighbour of y_i to the other endvertex of $P_{s_i}^{\boxdot}$, choosing these two paths with the same number of internal vertices up to an error at most one such that the total number of vertices is $|P_{s_i}^{\boxdot}|$. Note that this means each path has at least one internal vertex, and hence by the estimates above this too is always possible.

Finally, we embed the remaining paths $P_s^{\boxdot}$ (for which we have $s \in \mathcal{J}_0$). We use the same strategy again, and the same estimates show that this is possible.

If n is odd, we have now a collection of embeddings $\phi_s^{\mathbf{C}-}$ for some $s \in \mathcal{J}$. Whether n is odd or even, if we have not yet defined $\phi_s^{\mathbf{C}-}$ we let it be equal to $\phi_s^{\mathbf{A}}$, and we let $H_{\mathbf{C}^-}$ be the subgraph of $H_{\mathbf{B}}$ obtained by removing all the edges used in these embeddings, and the vertex $\boxdot$. For each $s \in \mathcal{J}$, we let $\mathsf{SpecPaths}_s^{**}$ be the set of paths in $\mathsf{SpecPaths}_s$ which are not embedded by $\phi_s^{\mathbf{C}-}$.

The critical properties we should note are the following. We have $|\mathsf{SpecPaths}_s^{**}| \geq |\mathsf{SpecPaths}_s| - 1$. The density $d_{\mathbf{C}^-}$ of $H_{\mathbf{C}^-}$ is $d_{\mathbf{A}} \pm O(n^{-1})$. By $(B.iv)$ and by construction of the embeddings above, for every vertex $v \in V(H_{\mathbf{C}^-})$ we have $\deg_{H_{\mathbf{C}^-}}(v) \geq \deg_{H_{\mathbf{A}}}(v) - 10Dcn$. For each $s \in \mathcal{J}$, we have $\big|\mathrm{im}^{\mathbf{A}}(s) \setminus \mathrm{im}^{\mathbf{C}^-}(s)\big| \leq 20$. Finally, for each $v \in V(H_{\mathbf{C}^-})$, the number of $s \in \mathcal{J}$ such that $v \in \mathrm{im}^{\mathbf{C}^-}(s) \setminus \mathrm{im}^{\mathbf{A}}(s)$ is at most $10Dcn$. To see that the last of these is true, note that if v is used in Stage B or C^- by a given graph G_s, then we also use at least one edge at v to embed an edge of G_s in one of these stages, and we use at most $10Dcn$ edges at v in these two stages. What we obtain from these observations is the following, which we will use repeatedly in the rest of this stage.

$(*)$ For each condition from Lemma 30, the same statement holds after Stage C^-, replacing $\gamma_{\mathbf{A}}$ with $2\gamma_{\mathbf{A}}$.

We next label $V(H_{\mathbf{C}^-})=\{\boxminus_i,\boxplus_i : i\in[\lfloor n/2\rfloor]\}$. Consider such a labelling chosen uniformly at random. We claim that this gives a quasirandom setup.

Lemma 53. *With positive probability, the uniform random labelling $V(H_{\mathbf{C}^-})=\{\boxminus_i,\boxplus_i : i\in[\lfloor n/2\rfloor]\}$, together with $\mathcal{J}_i$ for $i=0,1,2$, with path-forest $F_s^{\mathbf{C}-}$ consisting of the paths of $\mathsf{SpecPaths}_s^{**}$ for each $s\in\mathcal{J}$, with used sets $U_s^{\mathbf{C}-}=\operatorname{im}\phi_s^{\mathbf{C}-}$, with anchor sets $A_s^{\mathbf{C}-}$ given by $\phi_s^{\mathbf{C}-}$ applied to the leaves of $\mathsf{SpecPaths}_s^{**}$ and with $I_s^{\mathbf{C}-}=\{j\in[\lfloor n/2\rfloor] : \boxminus_j,\boxplus_j\not\in\operatorname{im}^{\mathbf{C}^-}(s)\}$, is a $(10\gamma_{\mathbf{A}},L_{\mathbf{C}},d_{\mathbf{C}^-},d_{\mathbf{C}^-})$-quasirandom setup. Furthermore, for each $u\in V(H_{\mathbf{C}^-})$ and $j\in[\lfloor n/2\rfloor]$ such that $u\not\in\{\boxminus_j,\boxplus_j\}$ we have*

$$\left|\{s\in\mathcal{J}_0 : u\in A_s^{\mathbf{C}-}, j\in I_s^{\mathbf{C}-}\}\right|=(1\pm2\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot2\lambda(\delta+10\lambda)^2\,,$$

and for each $s\in\mathcal{J}_0$ we have $|I_s^{\mathbf{C}-}|=(1\pm10\gamma_{\mathbf{A}})\frac{n}{2}\big(1-\frac{|U_s^{\mathbf{C}-}|}{n}\big)^2$.

Proof. We split the proof of this lemma into the following claims.

The following claim takes care of $(Quasi\,3)$.

Claim 53.1. *For every $v\in V(H_{\mathbf{C}^-})$, $a\in\{\boxminus,\boxplus\}$ and $i\in\{1,2\}$, let $X_{v,V_a,\mathcal{J}_i}$ be a random variable counting the quantity from $(Quasi\,3)$. Then $\mathbf{E}[X_{v,V_a,\mathcal{J}_i}]=(1\pm2\gamma_{\mathbf{A}})\frac{1}{2n}\sum_{s\in\mathcal{J}_i}|A_s^{\mathbf{C}-}|$.*

Further, $\mathbf{P}\left[|X_{v,V_a,\mathcal{J}_i}|-\mathbf{E}[X_{v,V_a,\mathcal{J}_i}]|>n^{0.9}\right]<\exp(-n^{0.2})$.

Proof. Given v, a and i, recall that by $(A.vii)$ and $(*)$, we have that the number of $P\in\bigcup_{s\in\mathcal{J}_i}\mathsf{SpecPaths}_s^{**}$ for which $v\in\{\phi_s^{\mathbf{C}-}(\mathsf{leftpath}_0(P)),\phi_s^{\mathbf{C}-}(\mathsf{rightpath}_0(P))\}$ is equal to

$$(1\pm\gamma_{\mathbf{A}})\frac{1}{n}\sum_{s\in\mathcal{J}_i}|A_s^{\mathbf{C}-}|\,.$$

Each such path is counted in $X_{v,V_a,\mathcal{J}_i}$ if and only if the unique vertex of

$$\Big\{\phi_s^{\mathbf{C}-}\big(\mathsf{leftpath}_0(P)\big),\phi_s^{\mathbf{C}-}\big(\mathsf{rightpath}_0(P)\big)\Big\}\setminus\{v\}$$

ends up in the set V_a; the probability of this is $\frac{1}{2}$. Hence, the statement about the expectation follows.

Let us now turn to the statement about the concentration. To this end, we model the random splitting in a particular way. We consider a space $\Omega=[0,1]^{V(H_{\mathbf{C}^-})}$ (equipped with the Lebesgue measure). Suppose that $\vec{x}\in\Omega$. Suppose further that all the coordinates of $\vec{x}$ are distinct; this requirement creates an exceptional set which is null. Then based on $\vec{x}$ we can define $V_\boxminus$ as those vertices $v\in V(H_{\mathbf{C}^-})$ for which $\vec{x}(v)$ is among the $\lfloor\frac{n}{2}\rfloor$ smallest values of $\{\vec{x}(u)\}_{u\in V(H_{\mathbf{C}^-})}$ and $V_\boxplus$ as those vertices $v\in V(H_{\mathbf{C}^-})$ for which $\vec{x}(v)$ is among the $\lfloor\frac{n}{2}\rfloor$ largest values of $\{\vec{x}(u)\}_{u\in V(H_{\mathbf{C}^-})}$. Taking $\vec{x}$ at random yields the right distribution of $V_\boxminus$ and $V_\boxplus$. This way, we can think of $X_{v,V_a,\mathcal{J}_i}$ as a random variable on Ω. Observe that when we change a variable $\vec{x}\in\Omega$ in one coordinate, $X_{v,V_a,\mathcal{J}_i}(\vec{x})$ changes by at most $n^{0.3}$ by $(A.ix)$. Hence, Lemma 24 gives that $\mathbf{P}\left[|X_{v,V_a,\mathcal{J}_i}|-\mathbf{E}[X_{v,V_a,\mathcal{J}_i}]>n^{0.9}\right]<2\exp\left(-\frac{2(n^{0.9})^2}{(n^{0.3})^2n}\right)<\exp(-n^{0.2})$. □

The following claim takes care of $(Quasi\,5)$ and $(Quasi\,2)$.

Claim 53.2. *For all $s \in \mathcal{J}$, for all $v \notin \mathrm{im}^{\mathbf{C}-}(s)$ and for all $a, b \in \{\boxminus, \boxplus\}$, let $A_s^{\mathbf{C}-,b}$ denote the subset of $u \in A_s^{\mathbf{C}-}$ such that the path of $\mathsf{SpecPaths}_s^{**}$ with leaves $\left(\phi_s^{\mathbf{C}-}\right)^{-1}(u)$ and y has $\phi_s^{\mathbf{C}-}(y) \in V_b$. With probability at least $1 - \exp(-n^{0.1})$ we have that*

$$|V_a \cap A_s^{\mathbf{C}-,b} \cap \mathsf{N}_H(v)| = (1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-} \frac{|A_s^{\mathbf{C}-}|}{4} \quad \text{and} \quad |V_a \cap A_s^{\mathbf{C}-,b}| = (1 \pm 3\gamma_{\mathbf{A}}) \frac{|A_s^{\mathbf{C}-}|}{4}\,.$$

Proof. By $(A.vi)$ and $(*)$, $|A_s^{\mathbf{C}-} \cap \mathsf{N}_{H_{\mathbf{C}^-}}(v)| = (1 \pm 2\gamma_{\mathbf{A}}) d_{\mathbf{C}-} |A_s^{\mathbf{C}-}|$. For each $x \in A_s^{\mathbf{C}-} \cap \mathsf{N}_{H_{\mathbf{C}^-}}(v)$, the probability that $x \in A_s^{\mathbf{C}-,b} \cap V_a$ is $(1 \pm c)\frac{1}{4}$, so the expectation of $|A_s^{\mathbf{C}-,b} \cap \mathsf{N}_{H_{\mathbf{C}^-}}(v) \cap V_a|$ is $(1 \pm 3\gamma_{\mathbf{A}}) d_{\mathbf{C}-} |A_s^{\mathbf{C}-}|/4$. Similarly $|V_a \cap A_s^{\mathbf{C}-}|$ has expectation $(1 \pm \gamma_{\mathbf{A}})\frac{1}{4}|A_s^{\mathbf{C}-}|$. The concentration follows as in Claim 53.1. In particular, observe that a change of one coordinate in a variable $\vec{x} \in \Omega$ changes $|V_a \cap A_s^{\mathbf{C}-,b} \cap \mathsf{N}_H(v)|$ or $|V_a \cap A_s^{\mathbf{C}-,b}|$ by at most 2. □

The following claim is a first step towards $(Quasi\,1)$.

Claim 53.3. *Let $a \in \{\boxminus, \boxplus\}$ and $S \subseteq V(H_{\mathbf{C}^-})$, and $T \subseteq \mathcal{J}_0$ be such that $|S| \leq L_{\mathbf{C}}$ and $|T| \leq L_{\mathbf{C}}$. Then the following holds with probability at least $1 - \exp(-n^{0.55})$.*

$$\left| V_a \cap \mathsf{N}_{H_{\mathbf{C}^-}}(S) \setminus \bigcup_{s \in T} U_s^{\mathbf{C}-} \right| = (1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S|} \prod_{s \in T} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right) \cdot \frac{n}{2}\,. \tag{67}$$

Proof. We think of the random labelling described above as follows. Firstly, we split $V(H_{\mathbf{C}^-})$ into $V_\boxminus$, $V_\boxplus$ (where $|V_\boxminus| = |V_\boxplus| = \lfloor \frac{n}{2} \rfloor$) at random. Secondly, we consider random labellings $\{\boxminus_i\}_{i \in [\lfloor \frac{n}{2} \rfloor]}$ of $V_\boxminus$ and $\{\boxplus_i\}_{i \in [\lfloor \frac{n}{2} \rfloor]}$ of $V_\boxplus$. Note that after the first stage (without considering the random labelling), we are already able to determine (67). By $(A.ii)$ and $(*)$ we have

$$\left| \mathsf{N}_{H_{\mathbf{C}^-}}(S) \setminus \bigcup_{s \in T} U_s^{\mathbf{C}-} \right| = (1 \pm 2\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S|} n \prod_{s \in T} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right)\,.$$

Since each vertex of $\mathsf{N}_{H_{\mathbf{C}^-}}(S) \setminus \bigcup_{s \in T} U_s^{\mathbf{C}-}$ has probability $\frac{1}{2}$ of being in V_a, we obtain

$$\mathbf{E}\left[\left| V_a \cap \mathsf{N}_{H_{\mathbf{C}^-}}(S) \setminus \bigcup_{s \in T} U_s^{\mathbf{C}-} \right| \right] = (1 \pm 2\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S|} \frac{n}{2} \prod_{s \in T} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right)\,.$$

Furthermore, the random variable $|V_a \cap \mathsf{N}_{H_{\mathbf{C}^-}}(S) \setminus \bigcup_{s \in T} U_s^{\mathbf{C}-}|$ is exponentially concentrated by Fact 23. □

We can now finish establishing $(Quasi\,1)$, and our bounds on $|I_s^{\mathbf{C}-}|$ for $s \in \mathcal{J}_0$. Recall that for $s \in \mathcal{J}_0$, the set $I_s^{\mathbf{C}-}$ is the set of $i \in [\lfloor n/2 \rfloor]$ such that $\boxminus_i, \boxplus_i \notin U_s^{\mathbf{C}-}$.

Claim 53.4. *Let $S_1, S_2 \subseteq V_\boxminus \cup V_\boxplus$, and $T_1, T_2, T_3 \subseteq \mathcal{J}$ be as in Definition 28. Then with probability at least $1 - \exp(-n^{0.5})$ we have that the set*

$$X := \left\{ j \in [\lfloor \tfrac{n}{2} \rfloor] : \boxminus_j \in \mathsf{N}_{H_{\mathbf{C}^-}}(S_1) \setminus \bigcup_{s \in T_1 \cup T_3} U_s^{\mathbf{C}-}, \boxplus_j \in \mathsf{N}_{H_{\mathbf{C}^-}}(S_2) \setminus \bigcup_{s \in T_2 \cup T_3} U_s^{\mathbf{C}-} \right\}$$

has size

$$(1 \pm 10\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S_1|+|S_2|} \prod_{s \in T_1 \cup T_2} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right) \cdot \prod_{s \in T_3} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right)^2 \cdot \frac{n}{2} \, .$$

Proof. We consider the same random experiment as in Claim 53.3. With probability at least $1 - \exp(-n^{0.5})$, $V(H_{\mathbf{C}^-})$ is split between $V_\boxminus$ and $V_\boxplus$ so that (67) holds simultaneously for all choices of $a \in \{\boxminus, \boxplus\}$, S, and T.

Suppose now that S_1, S_2, T_1, T_2, T_3 are given. We expose the labellings $\{\boxminus_j\}_{j \in [\lfloor \frac{n}{2} \rfloor]}$ and $\{\boxplus_j\}_{j \in [\lfloor \frac{n}{2} \rfloor]}$. Consider the sets

$$Y_\boxminus := \left\{ j \in [\lfloor \tfrac{n}{2} \rfloor] : \boxminus_j \in \mathsf{N}_{H_{\mathbf{C}^-}}(S_1) \setminus \bigcup_{s \in T_1 \cup T_3} U_s^{\mathbf{C}-} \right\} \text{ and}$$

$$Y_\boxplus := \left\{ j \in [\lfloor \tfrac{n}{2} \rfloor] : \boxplus_j \in \mathsf{N}_{H_{\mathbf{C}^-}}(S_2) \setminus \bigcup_{s \in T_2 \cup T_3} U_s^{\mathbf{C}-} \right\} \, .$$

We have that $X = Y_\boxminus \cap Y_\boxplus$. Thus, $|X|$ has hypergeometric distribution with parameters $(\lfloor \frac{n}{2} \rfloor, |Y_\boxminus|, |Y_\boxplus|)$. Note that the sizes of $Y_\boxminus$ and $Y_\boxplus$ do not depend on the exposed labelling but only on the sets $V_\boxminus$ and $V_\boxplus$. We apply (67) with $a = \boxminus$, $S := S_1$, $T := T_1 \cup T_3$ and get

$$|Y_\boxminus| = (1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S_1|} \cdot \frac{n}{2} \prod_{s \in T_1 \cup T_3} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right) \, .$$

Similarly,

$$|Y_\boxplus| = (1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-}^{|S_2|} \cdot \frac{n}{2} \prod_{s \in T_2 \cup T_3} \left(1 - \frac{|U_s^{\mathbf{C}-})|}{n}\right) \, .$$

Now, Fact 23 yields the statement. □

Next, we turn to verifying (*Quasi* 4).

Claim 53.5. *Suppose that $s \in \mathcal{J}_0$ and $v \in V(H_{\mathbf{C}^-}) \setminus U_s^{\mathbf{C}-}$. Then with probability at least $1 - \exp(n^{0.5})$, we have*

$$\left|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{C}^-}}(v) : i \in I_s^{\mathbf{C}-}\}\right|, \left|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{C}^-}}(v) : i \in I_s^{\mathbf{C}-}\}\right| = (1 \pm 10\gamma_{\mathbf{A}}) d_{\mathbf{C}^-} |I_s^{\mathbf{C}-}|$$

Proof. We prove the statement for the $\boxplus_i$; that for the $\boxminus_i$ is symmetric. By Claim 53.3, after revealing the sets $V_\boxminus$ and $V_\boxplus$, we have $(1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-} \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right) \frac{n}{2}$ vertices in $\mathsf{N}_{H_{\mathbf{C}^-}}(v) \cap (V_\boxplus \setminus U_s^{\mathbf{C}-})$, and $(1 \pm 4\gamma_{\mathbf{A}}) \left(1 - \frac{|U_s^{\mathbf{C}-}|}{n}\right) \frac{n}{2}$ vertices in $V_\boxminus \setminus U_s^{\mathbf{C}-}$. We then consider the uniform random labelling of the $\boxminus_i$ and $\boxplus_i$ within $V_\boxminus$ and $V_\boxplus$, as in Claim 53.4. We have i in the desired set if and only if $\boxminus_i \in V_\boxminus \setminus U_s^{\mathbf{C}-}$ and $\boxplus_i \in \mathsf{N}_{H_{\mathbf{C}^-}}(v) \cap (V_\boxplus \setminus U_s^{\mathbf{C}-})$, so the size $\left|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{C}^-}}(v) : i \in I_s^{\mathbf{C}-}\}\right|$ has hypergeometric distribution with parameters

$$\left(\lfloor \tfrac{n}{2} \rfloor, (1 \pm 4\gamma_{\mathbf{A}}) \left(1 - \tfrac{|U_s^{\mathbf{C}-}|}{n}\right) \tfrac{n}{2}, (1 \pm 4\gamma_{\mathbf{A}}) d_{\mathbf{C}^-} \left(1 - \tfrac{|U_s^{\mathbf{C}-}|}{n}\right) \tfrac{n}{2}\right) .$$

As in Claim 53.4, we conclude by Fact 23 that the lemma statement holds. □

For $(Quasi6)$, given $S_1, S_2 \subseteq V(H_{\mathbf{C}^-})$ disjoint, and additionally a set $T \subseteq [\lfloor n/2 \rfloor]$ such that for each $j \in T$ we have $\boxminus_j, \boxplus_j \notin S_1 \cup S_2$, with each set of size at most $L_{\mathbf{C}}$, we apply $(A.iii)$ and $(*)$. For $i = 1, 2$ we use input S_1, S_2, $\mathcal{J}_i$. For $i = 0$, we use input $S_1 \cup \{\boxminus_j, \boxplus_j : j \in T\}$, S_2 and $\mathcal{J}_0$. This gives precisely the required $(Quasi6)$. For $(Quasi7)$, given $i \in \{0, 1, 2\}$ and $v \in V(H_{\mathbf{C}^-})$, we apply $(A.vii)$ and $(*)$ which gives $(Quasi7)$. For $(Quasi8)$, given $i \in \{0, 1, 2\}$ and $u \neq v \in V(H_{\mathbf{C}^-})$, we apply $(A.viii)$ and $(*)$ which gives $(Quasi8)$.

Finally, we prove the required bounds on $\big|\{s \in \mathcal{J}_0 : u \in A_s^{\mathbf{C}-}, j \in I_s^{\mathbf{C}-}\}$ for $u \in V(H_{\mathbf{C}^-})$ and j an index such that $u \notin \{\boxminus_j, \boxplus_j\}$. By (d) and $(*)$, for any three distinct vertices u, v, v' of $V(H_{\mathbf{C}^-})$, we have

$$\big|\{s \in \mathcal{J}_0 : u \in A_s^{\mathbf{C}-}, v, v' \notin U_s^{\mathbf{C}-}\} = (1 \pm 2\gamma_{\mathbf{A}})|\mathcal{J}_0| \cdot 2\lambda(\delta + 10\lambda)^2 \,.$$

We apply this with $v = \boxminus_j$ and $v' = \boxplus_j$, and note that by definition $j \in I_s^{\mathbf{C}-}$ if and only if $\boxminus_j, \boxplus_j \notin U_s^{\mathbf{C}-}$.

Since the error probabilities in Claims 53.1–53.5 are superpolynomially small and since there are only polynomially many instances of $(Quasi1)$–$(Quasi8)$, we conclude that the random labelling satisfies all of them with positive probability. In particular, there exists one such labelling, completing the proof of Lemma 53. $\square$

To complete the proof of Lemma 33, we need to embed a few more paths, from the graphs G_s with $s \in \mathcal{J}_2$, such that all edges of the form $\boxminus_i\boxplus_i$ are used. We will ensure that we use only few vertices of each G_s, and that we embed in total few edges at any given vertex, in this step. This ensures that we preserve our quasirandom setup with only a little degradation of the parameters.

Lemma 54. *There are sets* $\mathsf{SpecPaths}_s^{\diamond} \subseteq \mathsf{SpecPaths}_s^{**}$, $s \in \mathcal{J}_2$ *and mappings* $(\phi_s^{\diamond})_{s \in \mathcal{J}_2}$ *such that*

- (i) *For each* $s \in \mathcal{J}_2$*, the mapping* $\phi_s^{\diamond}$ *is an embedding of the paths* $\mathsf{SpecPaths}_s^{\diamond}$ *into* $H_{\mathbf{C}^-}$ *that agrees with* $\phi_s^{\mathbf{C}-}$ *on the leaves of* $\mathsf{SpecPaths}_s^{\diamond}$*. For each edge* $uv \in E(H_{\mathbf{C}^-})$*, there is at most one* $\phi_s^{\diamond}$ *which uses* uv*, and we let* $H_{\mathbf{C}}$ *be the graph obtained from* $H_{\mathbf{C}^-}$ *by removing all the edges used by* $(\phi_s^{\diamond})_{s \in \mathcal{J}_2}$.
- (ii) *For each* $i \in \big[\lfloor \frac{n}{2} \rfloor\big]\}$ *we have* $\boxminus_i\boxplus_i \notin E(H_{\mathbf{C}})$.
- (iii) *For each* $v \in V_{\boxminus} \cup V_{\boxplus}$ *we have* $\deg_{H_{\mathbf{C}}}(v) > \deg_{H_{\mathbf{C}^-}}(v) - n^{0.6}$*, and for each* $s \in \mathcal{J}_2$ *we have* $|\mathsf{SpecPaths}_s^{\diamond}| \leq n^{0.5}$.

Proof. First we specify the sets $\mathsf{SpecPaths}_s^{\diamond}, s \in \mathcal{J}_2$. Initially, set $\mathsf{SpecPaths}_s^{\diamond} = \emptyset$ for all $s \in \mathcal{J}_2$. In order to comply with (iii), we need to keep track on how much we map something to a vertex, as any such mapping will decrease its degree by 2. If we embed too many paths anchored at the same vertex, this also will decrease the degree of this vertex too much, and thus should choose the sets $\mathsf{SpecPaths}_s^{\diamond}, s \in \mathcal{J}_2$ accordingly. For each vertex $v \in V(H)$, let us use a counter $c(v)$ initially set to 2 for each $\boxminus_i, \boxplus_i$ such that $\boxminus_i\boxplus_i \in E(H_{\mathbf{C}^-})$ and otherwise to 0. For each paths included in $\mathsf{SpecPaths}_s^{\diamond}, s \in \mathcal{J}_2$ anchored at v, we shall increase $c(v)$ by one and for any internal vertex of a path from $\mathsf{SpecPaths}_s^{\diamond}, s \in \mathcal{J}_2$ mapped to v, we shall rise $c(v)$ by 2. Let $U := \{v \in V(H), c(v) > \sqrt{n}\}$ be the set of vertices that gets dangerously overloaded. Observe that initially $U = \emptyset$. As we shall embed at most $n/2$ paths in this lemma, observe that the size of U will never exceed $2 \cdot 7\lambda n/\sqrt{n} < \sqrt{n}$.

For each $i \in [\frac{n}{2}]$, let $C(i) := \{s \in \mathcal{J}_2, \mathrm{im}\,\phi^{\mathbf{C}-}(s) \cap \{\boxminus_i, \boxplus_i\} = \emptyset\}$. By Lemma 53 and $(Quasi\,6)$, setting $S_1 = \{\boxminus_i, \boxplus_i\}$ and $S_2 := \emptyset$, we have

$$|C(i)| \geq (1 - 10\gamma_{\mathbf{A}})(\delta + 6\lambda)^2 |\mathcal{J}_2| \geq \delta^2 \lambda n \,.$$

For each $i \in [\frac{n}{2}]$ such that $\boxminus_i \boxplus_i \in E(H_{\mathbf{C}^-})$ in succession, pick $s_i \in C(i)$ such that $|\mathsf{SpecPaths}^{\diamond}_{s_i}| < \sqrt{n} - 1$. Observe that the number of $s \in \mathcal{J}_2$ such that $|\mathsf{SpecPaths}^{\diamond}_{s}| \geq \sqrt{n} - 1$ is less than $n/\sqrt{n} = \sqrt{n} < |C(i)|$, so that this is always possible. Take any path $P_i \in \mathsf{SpecPaths}^{**}_{s_i} \setminus \mathsf{SpecPaths}^{\diamond}_{s_i}$ that is not anchored in U. This is possible because $|U|, |\mathsf{SpecPaths}^{\diamond}_{s_i}| \leq \sqrt{n}$ and $\mathsf{SpecPaths}^{**}_{s_i}$ contains at least $n^{0.9}$ paths.

Add P_i to $\mathsf{SpecPaths}^{\diamond}_{s_i}$ and increase the counter $c(v)$ of its anchors by one. Recall that P_i has six internal vertices $x_1, \ldots, x_6$ together with leaves x_0, x_7. We construct an embedding of P_i as follows. We embed $\phi^{\diamond}_{s_i}(x_2) = \boxminus_i$ and $\phi^{\diamond}_{s_i}(x_3) = \boxplus_i$. We choose $\phi^{\diamond}_{s_i}(x_1)$ arbitrarily from $\mathsf{N}_{H_{\mathbf{C}^-}}\big(\phi^{\mathbf{C}-}_{s_i}(x_0), \boxminus_i\big) \setminus (\mathrm{im}\,\phi^{\mathbf{C}-}_{s_i} \cup U)$ such that the edges to $\phi^{\mathbf{C}-}_{s_i}(x_0)$ and $\boxminus_i$ have not previously been used in this lemma and which has not previously been used in this lemma for G_{s_i}. By Lemma 53 and $(Quasi\,1)$, we have

$$\mathsf{N}_{H_{\mathbf{C}^-}}\big(\phi^{\mathbf{C}-}_{s_i}(x_0), \boxminus_i\big) \setminus \mathrm{im}\,\phi^{\mathbf{C}-}_{s_i} \geq (1 - 10\gamma_{\mathbf{A}}) d^2_{\mathbf{C}^-}(\delta + 6\lambda)n \,,$$

of which at most $4n^{0.6}$ vertices are disallowed due to being in U or the edges to $\boxminus_i$ or $\phi^{\mathbf{C}-}_{s_i}(x_0)$, used previously in this lemma, or the vertex used for G_{s_i} in this lemma. Thus the choice of $\phi^{\diamond}_{s_i}(x_1)$ is possible; and we increase $c\big(\phi^{\diamond}_{s_i}(x_1)\big)$ by 2. We now choose similarly for each $j = 4, 5, 6$ in succession a vertex $\phi^{\diamond}_{s_i}(x_j)$ from

$$\mathsf{N}_{H_{\mathbf{C}^-}}\big(\phi^{\diamond}_{s_i}(x_{j-1}), \phi^{\mathbf{C}-}_{s_i}(x_7)\big) \setminus (\mathrm{im}\,\phi^{\mathbf{C}-}_{s_i} \cup U)$$

such that the edges to $\phi^{\diamond}_{s_i}(x_{j-1})$ and $\phi^{\mathbf{C}-}_{s_i}(x_7)$, have not previously been used in this lemma, or the vertex used for G_{s_i} in this lemma. By the same calculation as above, this is always possible. We increase the counter $c\big(\phi^{\diamond}_{s_i}(x_j)\big)$ by 2. □

We now complete the proof of Lemma 33. For each $s \in \mathcal{J}_2$, we set $\mathsf{SpecPaths}^{*}_{s} := \mathsf{SpecPaths}^{**}_{s} \setminus \mathsf{SpecPaths}^{\diamond}_{s}$, and we set $\phi^{\mathbf{C}}_{s}$ the union of $\phi^{\mathbf{C}-}_{s}$ and $\phi^{\diamond}_{s}$; note that these two maps have some common vertices in their domains (the leaves of the $\mathsf{SpecPaths}^{\diamond}_{s}$) but by construction they agree on these vertices. For each $s \in \mathcal{J}_0 \cup \mathcal{J}_1$ we set $\mathsf{SpecPaths}^{*}_{s} = \mathsf{SpecPaths}^{**}_{s}$ and $\phi^{\mathbf{C}}_{s} = \phi^{\mathbf{C}-}_{s}$. Let $d_{\mathbf{C}} = e(H_{\mathbf{C}})/\binom{n}{2}$, and observe that $d_{\mathbf{C}} = d_{\mathbf{C}^-} \pm o(1)$.

We claim that $H_{\mathbf{C}}$ as returned by Lemma 54, with the splitting into $V_{\boxminus} \cup V_{\boxplus}$ from Lemma 53, with for each $s \in \mathcal{J}$ the path-forest $F^{\mathbf{C}}_{s}$ consisting of the paths of $\mathsf{SpecPaths}^{*}_{s}$, with $U^{\mathbf{C}}_{s} = \mathrm{im}\,\phi^{\mathbf{C}}_{s}$, with $A^{\mathbf{C}}_{s}$ the image under $\phi^{\mathbf{C}}_{s}$ of the leaves of $F^{\mathbf{C}}_{s}$ and with the corresponding bijection, and with $I^{\mathbf{C}}_{s}$ the set of $i \in [\lfloor n/2 \rfloor]$ such that $\boxminus_i, \boxplus_i \notin \mathrm{im}\,\phi^{\mathbf{C}}_{s}$, is a $(100\gamma_{\mathbf{A}}, L_{\mathbf{C}}, d_{\mathbf{C}}, d_{\mathbf{C}})$-quasirandom setup.

This follows from Lemma 53, which guarantees a $(10\gamma_{\mathbf{A}}, L_{\mathbf{C}}, d_{\mathbf{C}^-}, d_{\mathbf{C}^-})$-quasirandom setup after stage $\mathbf{C}^-$, together with Lemma 54 which tells us that all the quantities in the definition of a quasirandom setup have changed between these two stages by at most $10L_{\mathbf{C}}n^{0.6}$.

Finally, the size bounds relating to $\mathcal{J}_0$ in Lemma 33 have not changed from after stage $\mathbf{C}^-$ and hence hold by Lemma 53.

11. Stage D (Proof of Lemma 34)

In this stage, we need to accommodate all vertices of P, but the two middle ones for all $s \in \mathcal{J}_0$ and $P \in \mathsf{SpecPaths}^*_s$; this corresponds to finding the placement for the solid lines in Figure 3 (the outermost vertices of P were embedded already in Stage A). Recall that the aim here is that after this stage, what remains to pack of each graph in $\mathcal{J}_0$ will be a path-forest in which each path has 2 unembedded vertices, and the two leaves are embedded to a *terminal pair*.

We do that in two steps. First, in Lemma 55 we decide, for each $P \in \bigcup_{s\in\mathcal{J}_0} \mathsf{SpecPaths}^*_s$, what its terminal pair will be. To that end, given such a P we let y_P^{left} denote the fifth vertex of P, and y_P^{right} the eighth vertex of P. That is, y_P^{left} is an endvertex of $P^{\mathsf{left}} := \mathsf{leftpath}_4(P)$, and y_P^{right} is an endvertex of $P^{\mathsf{right}} := \mathsf{rightpath}_4(P)$. We will choose an index $r_P \in [\lfloor n/2 \rfloor]$ and embed y_P^{left} to $\boxminus_{r_P}$, and y_P^{right} to $\boxplus_{r_P}$. For this to be possible, for each $P \in \mathsf{SpecPaths}^*_s$ of course we need

$$r_P \in I_s^{\mathbf{C}} := \{i \in [\lfloor n/2 \rfloor] \,:\, \boxminus_i, \boxplus_i \notin \mathrm{im}^{\mathbf{C}}(s)\}\,.$$

We let, for each $s \in \mathcal{J}_0$, the set $\mathsf{SpecShortPaths}^*_s$ be the set of subpaths of $\mathsf{SpecPaths}^*_s$ with endvertices y_P^{left} and y_P^{right}. That is, each path in $\mathsf{SpecShortPaths}^*_s$ is the middle three edges (four vertices) of a path of $\mathsf{SpecPaths}^*_s$; this is what will remain to pack from $(G_s)_{s\in\mathcal{J}_0}$ at the conclusion of this stage.

The content of Lemma 55 is then that we can make the assignment of indices r_P and still have a quasirandom setup; in addition, we need one property regarding the set $A_s^{\mathbf{C}}$, which is the set of anchors after Stage C. After this, we use Lemma 42 to argue that we can pack all the P^{left} and P^{right} and again obtain a quasirandom setup, which completes the proof of Lemma 34.

Lemma 55. *There exist indices* $(r_P \in [\lfloor \frac{n}{2} \rfloor])_{s\in\mathcal{J}_0, P\in\mathsf{SpecPaths}^*_s}$ *with the following properties.*

(D⁻i) *For each* $s \in \mathcal{J}_0$*, the indices* $(r_P)_{P\in\mathsf{SpecPaths}^*_s}$ *are distinct elements of* $I_s^{\mathbf{C}}$*.*

(D⁻ii) *For* $s \in \mathcal{J}_1 \cup \mathcal{J}_2$ *we set* $\phi_s^{\mathbf{D}-} = \phi_s^{\mathbf{C}}$*. For* $s \in \mathcal{J}_0$ *we extend* $\phi_s^{\mathbf{C}}$ *to* $\phi_s^{\mathbf{D}-}$ *by setting* $\phi^{\mathbf{D}-}(y_P^{\mathsf{left}}) = \boxminus_{r_P}$ *and* $\phi_s^{\mathbf{D}^-}(y_P^{\mathsf{right}}) = \boxplus_{r_P}$*.*

(D⁻iii) *The graph* $H_{\mathbf{C}}$ *with its labelling as* $V_\boxminus \cup V_\boxplus$*, together with the path-forests* $F_s^{\mathbf{D}-} := \mathsf{SpecPaths}^*_s$ *for* $s \in \mathcal{J}_1 \cup \mathcal{J}_2$ *and* $F_s^{\mathbf{D}-} := \mathsf{SpecShortPaths}^*_s$ *for* $s \in \mathcal{J}_0$*, with used sets* $U_s^{\mathbf{D}-} := \mathrm{im}^{\mathbf{D}-}(s)$ *for each* $s \in \mathcal{J}$*, with anchor sets* $A_s^{\mathbf{D}-}$ *being the image under* $\phi_s^{\mathbf{D}-}$ *of the leaves of* $F_s^{\mathbf{D}-}$ *(and with the corresponding bijection) and with* $I_s := I_s^{\mathbf{D}-}$ *for each* $s \in \mathcal{J}_0$*, gives a* $(100 \cdot 3L_{\mathbf{C}}4^{3L_{\mathbf{C}}}\delta^{2L_{\mathbf{C}}}(\lambda^*)^{L_{\mathbf{C}}}\gamma_{\mathbf{A}}, 2L_{\mathbf{D}}, d_{\mathbf{C}}, d_{\mathbf{C}})$*-quasirandom setup. For convenience, we say this is the* setup after $\mathbf{D}^-$.

(D-*iv*) *For each* $u,v\in V(H_{\mathbf{C}})$ *such that there is no* j *with* $u,v\in\{\boxminus_j,\boxplus_j\}$*, the number of* $s\in\mathcal{J}_0$ *such that* $u\in A_s^{\mathbf{C}}$ *and* $v\notin U_s^{\mathbf{D}-}$ *is* $(1\pm1000\gamma_{\mathbf{A}})\sum_{s\in\mathcal{J}_0}\frac{|A_s^{\mathbf{C}}|(n-|U_s^{\mathbf{D}-}|)}{n^2}$.

Proof. Suppose that $s\in\mathcal{J}_0$ is fixed. Recall that since after Stage C we have a $(100\gamma_{\mathbf{A}},L_{\mathbf{C}},d_{\mathbf{C}},d_{\mathbf{C}})$-quasirandom setup, and that for each $s\in\mathcal{J}_0$ we have $|I_s^{\mathbf{C}}|=(1\pm100\gamma_{\mathbf{A}})\frac{n}{2}\big(1-\frac{|U_s|}{n}\big)^2$. By (1) this means $|I_s^{\mathbf{C}}|>\lambda^*n$, and so we can select a set $J_s\subseteq I_s^{\mathbf{C}}$ uniformly at random of size λ^*n. For each s, we choose an arbitrary pairing of the paths of F_s to the set J_s, pairing path P to the index r_P.

We now aim to verify that (D-*iii*) holds with high probability. We go through the properties of Definition 32 one by one.

For (*Quasi* 1), the critical observation is the following. By Fact 23, for any given set $Y\subseteq I_s^{\mathbf{C}}$ of indices with $|Y|\geq cn$, when we reveal the choice of J_s, the probability that we do not have

$$|Y\cap J_s|=|Y|\frac{|J_s|}{|I_s^{\mathbf{C}}|}\pm n^{0.9} \tag{68}$$

is at most $\exp(-n^{0.5})$. If S_1,S_2,T_1,T_2,T_3 are any sets as in Definition 28, each of size at most $2L_{\mathbf{D}}$, and T' is any subset of $\mathcal{J}_0\setminus(T_1\cup T_2\cup T_3)$ of size at most $4L_{\mathbf{D}}$, by $(L_{\mathbf{C}},100\gamma_{\mathbf{A}},d_{\mathbf{C}},d_{\mathbf{C}})$-index-quasirandomness after Stage C we have

$$\big|\mathbb{U}_{H_{\mathbf{C}}}(S_1,S_2,T_1,T_2,T_3\cup T')\big|=(1\pm100\gamma_{\mathbf{A}})d_{\mathbf{C}}^{|S_1|+|S_2|}\tfrac{n}{2}\prod_{\ell\in T_1\cup T_2}\tfrac{n-|\operatorname{im}^{\mathbf{C}}(\ell)|}{n}\prod_{\ell\in T_3\cup T'}\tfrac{|I_\ell^{\mathbf{C}}|}{n/2}\,.$$

Applying (68) iteratively for each $\ell\in T'$ successively and taking the union bound over choices of all these sets, we see that with probability at least $1-\exp(-n^{0.4})$, for each choice of sets we have

$$\begin{aligned}\Big|\mathbb{U}_{H_{\mathbf{C}}}(S_1,S_2,&T_1,T_2,T_3\cup T')\cap\bigcap_{\ell\in T'}J_\ell\Big|\\&=(1\pm200\gamma_{\mathbf{A}})d_{\mathbf{C}}^{|S_1|+|S_2|}\tfrac{n}{2}\prod_{\ell\in T_1\cup T_2}\tfrac{n-|\operatorname{im}^{\mathbf{C}}(\ell)|}{n}\prod_{\ell\in T_3}\tfrac{|I_\ell^{\mathbf{C}}|}{n/2}\prod_{\ell\in T'}\tfrac{|I_\ell^{\mathbf{D}-}|}{n/2}\,.\end{aligned} \tag{69}$$

Suppose this likely event occurs. Note that for any $s\in\mathcal{J}_0$ we have $n-|\operatorname{im}^{\mathbf{D}-}(s)|=\big(n-|\operatorname{im}^{\mathbf{C}}(s)|\big)-2|I_s^{\mathbf{D}-}|$. Furthermore, given any sets S_1,S_2,T_1,T_2,T_3, consider the set

$$\Big(\mathbb{U}_{H_{\mathbf{C}}}(S_1,S_2,T_1,T_2,T_3)\cap\bigcap_{\ell\in T_3\cap\mathcal{J}_0}J_\ell\Big)\setminus\big(\cup_{\ell\in(T_1\cup T_2)\cap\mathcal{J}_0}J_\ell\big)$$

which is as in the definition of index-quasirandomness after stage $\mathbf{D}^-$. We can write the inclusion-exclusion formula for the size of this set difference, all of whose terms are of the form of the left hand side of (69). By (69), each of these terms is within a $(1\pm200\gamma_{\mathbf{A}})$ factor of the corresponding term obtained by multiplying out the final product in

$$d_{\mathbf{C}}^{|S_1|+|S_2|}\tfrac{n}{2}\prod_{\ell\in(T_1\cup T_2)\setminus\mathcal{J}_0}\tfrac{n-|\operatorname{im}^{\mathbf{C}}(\ell)|}{n}\prod_{\ell\in T_3}\tfrac{|I_\ell^{\mathbf{D}-}|}{n/2}\prod_{\ell\in(T_1\cup T_2)\cap\mathcal{J}_0}\big(\tfrac{n-|\operatorname{im}^{\mathbf{C}}(\ell)|}{n}-\tfrac{|J_\ell|}{n/2}\big)\,.$$

Since we have $n - |\operatorname{im}^{\mathbf{C}}(s)| - 2|J_s| = n - |\operatorname{im}^{\mathbf{D}-}(s)|$ for each $s \in \mathcal{J}_0$, and since for each $s \notin \mathcal{J} - 0$ we have $\operatorname{im}^{\mathbf{D}-}(s) = \operatorname{im}^{\mathbf{C}}(s)$, this formula is the same as appears in the definition of index-quasirandomness after stage $\mathbf{D}^-$. We conclude that after stage $\mathbf{D}^-$ we have $(2L_{\mathbf{D}}, 400 \cdot 2^{4L_{\mathbf{D}}}\gamma_{\mathbf{A}}, d_{\mathbf{C}}, d_{\mathbf{C}})$-index-quasirandomness, verifying $(Quasi\,1)$.

For $(Quasi\,2)$, observe that if $s \in \mathcal{J}_1 \cup \mathcal{J}_2$ then A_s^b has not changed from after Stage C, and hence the desired statement holds. For $s \in \mathcal{J}_0$, by construction exactly half of $A_s^{\mathbf{D}-}$ is in each of $V_{\boxminus}$ and $V_{\boxplus}$. Similarly, for $(Quasi\,3)$, nothing has changed for these statements from after Stage C.

We now prove $(Quasi\,4)$ and $(Quasi\,5)$. Observe that since for $s \in \mathcal{J}_0$ we have $A_s^{\boxplus} \cap V_{\boxminus} = \{\boxminus_i : i \in I_s^{\mathbf{D}-}\}$ and similarly for $A_s^{\boxminus} \cap V_{\boxplus}$, and since $A_s^{\boxminus} \cap V_{\boxminus} = A_s^{\boxplus} \cap V_{\boxplus} = \emptyset$, it suffices to prove the latter. For $s \in \mathcal{J}_1 \cup \mathcal{J}_2$, nothing has changed since after Stage C, so we only need to consider $s \in \mathcal{J}_0$. After Stage C, by $(Quasi\,4)$, for any given $u \in V(H_{\mathbf{C}})$ and $s \in \mathcal{J}_0$ we have

$$\big|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{C}}}(u) : i \in I_s^{\mathbf{C}}\}\big|, \big|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{C}}}(u) : i \in I_s^{\mathbf{C}}\}\big| = (1 \pm 100\gamma_{\mathbf{A}})d_{\mathbf{C}}|I_s^{\mathbf{C}}|\,.$$

When we select J_s uniformly at random from $I_s^{\mathbf{C}}$, by Fact 23, with probability at least $1 - \exp(-n^{0.5})$ we therefore have

$$\big|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{C}}}(u) : i \in J_s\}\big|, \big|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{C}}}(u) : i \in J_s\}\big| = (1 \pm 200\gamma_{\mathbf{A}})d_{\mathbf{C}}|J_s|\,.$$

Since for $s \in \mathcal{J}_0$, after stage $\mathbf{D}^-$ we have $A_s^{\mathbf{D}-} = \{\boxminus_i, \boxplus_i : i \in J_s\}$, taking the union bound over s and u, we see that with high probability $(Quasi\,5)$ holds after stage $\mathbf{D}^-$.

We next move to $(Quasi\,6)$. Since nothing has changed from after Stage C for $\mathcal{J}_1$ and $\mathcal{J}_2$, we only need to prove the statement for $\mathcal{J}_0$. Since $|J_s|, |I_s^{\mathbf{C}} \setminus J_s| = \Omega(n)$, we see that for any given disjoint sets Q, Q' of indices of size at most $6L_{\mathbf{D}}$, we have $\Pr(Q \subseteq J_s | Q' \cap J_s)$ equal to either $\big(\frac{2\lambda^*}{(\delta+10\lambda^*)^2}\big)^{|Q|} \pm n^{-0.5}$, if $Q \subseteq I_s^{\mathbf{C}}$, and 0 otherwise. We use this approximate independence to prove $(Quasi\,6)$ inductively. Our induction statement is the following $(*)_k$, where $0 \le k \le 6L_{\mathbf{D}}$ is an integer.

Let S be a set of vertices, and let T, T' be pairwise disjoint sets of indices. Suppose in addition that there is no i such that $\{\boxminus_i, \boxplus_i\} \subseteq S$, and that there is no $i \in T \cup T'$ such that $\{\boxminus_i, \boxplus_i\} \cap S \neq \emptyset$. Finally suppose that $|T'| \le k$, and that $|S| + |T| + |T'| \le 6L_{\mathbf{D}}$. Let

$$X(S, T, T') := \big\{s \in \mathcal{J}_0 : S \subseteq \operatorname{im}^{\mathbf{C}}(s), T \subseteq I_s^{\mathbf{C}}, T' \subseteq J_s\big\}\,.$$

Then we have

$$\big|X(S,T,T')\big| = |\mathcal{J}_0|(\delta + 10\lambda^*)^{2|T|}(1 - \delta - 10\lambda^*)^{|S|}(2\lambda^*)^{|T'|} \pm 100(k+1)\gamma_{\mathbf{A}} n\,,$$

and we reveal only the intersection of T' with the sets J_s to show this.

Observe that the statement $(*)_0$ is given by $(Quasi\,6)$ after Stage C. For a given $k < 6L_{\mathbf{D}}$, suppose that we have established $(*)_k$; we want to establish a given case S, T, T' of $(*)_{k+1}$. Let $i \in T'$. The set $X(S, T, T')$ consists of those $s \in X(S, T \cup \{i\}, T' \setminus \{i\})$ such that $i \in J_s$. As above, $(*)_k$ gives us the size of the former set, and then by Chernoff's inequality, with probability at least $1-\exp(-n^{0.5})$ we have

$$\begin{aligned}&X(S,T,T')\\&=\tfrac{2\lambda}{(\delta+10\lambda)^2}|\mathcal{J}_0|(\delta+10\lambda^*)^{2|T|+2}(1-\delta-10\lambda^*)^{|S|}(2\lambda^*)^{|T'|-1}\pm 100(k+1)\gamma_{\mathbf{A}}n\pm n^{0.5}\end{aligned}$$

which, rearranging, gives us the desired case of $(*)_{k+1}$.

We now suppose that all the above polynomially many good events occur, i.e. $(*)_{6L_{\mathbf{D}}}$ is true. Suppose that $|S|\le 4L_{\mathbf{D}}$ and $|T|\le 2L_{\mathbf{D}}$. Observe that as Q ranges over all subsets of S, the sets $X(S\setminus Q,\emptyset,T\cup Q)$ are pairwise disjoint and their union is precisely

$$Y(S,T):=\left\{s\in\mathcal{J}_0: S\subseteq \mathrm{im}^{\mathbf{D}-}(s), T\subseteq J_s\right\}.$$

We conclude by $(*)_{6L_{\mathbf{D}}}$ that

$$\big|Y(S,T)\big|=|\mathcal{J}_0|(1-\delta-8\lambda^*)^{|S|}(2\lambda^*)^{|T|}\pm 100\cdot 6L_{\mathbf{D}}2^{6L_{\mathbf{D}}}\gamma_{\mathbf{A}}n\,.$$

Finally, given sets S_1,S_2,T as in $(Quasi6)$, each of size at most $2L_{\mathbf{D}}$, we can write $\{s\in\mathcal{J}_0: S_1\cap\mathrm{im}^{\mathbf{D}-}(s)=\emptyset, S_2\subseteq\mathrm{im}^{\mathbf{D}-}(s), T\subseteq J_s\}$ by inclusion-exclusion in terms of the $|Y(S_2\cup Q,T)|$ as Q ranges over subsets of S_1. We obtain that the size of this last set is

$$|\mathcal{J}_0|(\delta+8\lambda^*)^{|S_1|}(1-\delta-8\lambda^*)^{|S_2|}(2\lambda^*)^{|T|}\pm 100\cdot 6L_{\mathbf{D}}4^{6L_{\mathbf{D}}}\gamma_{\mathbf{A}}n$$

which is as required for $(Quasi6)$ after stage $\mathbf{D}^-$ with error term $100\cdot 6L_{\mathbf{D}}4^{6L_{\mathbf{D}}}\delta^{4L_{\mathbf{D}}}(\lambda^*)^{2L_{\mathbf{D}}}\gamma_{\mathbf{A}}$.

For $(Quasi7)$, we only need to consider $\mathcal{J}_0$ since nothing has changed for $\mathcal{J}_1$ or $\mathcal{J}_2$. Given $v\in V(H_{\mathbf{C}})$, let i be such that $v\in\{\boxminus_i,\boxplus_i\}$. After Stage C, by $(Quasi6)$ with $S_1=S_2=\emptyset$ and $T=\{i\}$, the number of $s\in\mathcal{J}_0$ such that $i\in I_s^{\mathbf{C}}$ is $(1\pm 100\gamma_{\mathbf{A}})(\delta+10\lambda^*)^2|\mathcal{J}_0|$. Now for a given s from this collection, the probability that we pick $i\in J_s$ is $|J_s|/|I_s^{\mathbf{C}}|=(1\pm 100\gamma_{\mathbf{A}})\frac{2\lambda^*}{(\delta+10\lambda^*)^2}$. By Chernoff's inequality we conclude that the number of $s\in\mathcal{J}_0$ such that $i\in J_s$ is with probability at least $1-\exp(-n^{0.5})$ equal to $(1\pm 1000\gamma_{\mathbf{A}})(2\lambda^*)|\mathcal{J}_0|$, as required for $(Quasi7)$

For $(Quasi8)$, we again need only consider $\mathcal{J}_0$ since nothing has changed since Stage C for $\mathcal{J}_1$ and $\mathcal{J}_2$. Given $u,v\in V(H_{\mathbf{C}})$ with $u\neq v$ and $\{u,v\}$ not a terminal pair, let i be such that $u\in\{\boxminus_i,\boxplus_i\}$ and let j be such that $v\in\{\boxminus_j,\boxplus_j\}$. We want to estimate the number of $s\in\mathcal{J}_0$ such that $i\in J_s$ and both $v\notin\mathrm{im}^{\mathbf{C}}(s)$ and $j\notin J_s$ occur. Note that we can split these s up according to whether $j\in I_s^{\mathbf{C}}$. If $j\in I_s^{\mathbf{C}}$, then automatically $v\notin\mathrm{im}^{\mathbf{C}}(s)$. The number of $s\in\mathcal{J}_0$ such that $i,j\in I^{\mathbf{C}}s$ is by $(Quasi6)$ after Stage C equal to $(1\pm 100\gamma_{\mathbf{A}})|\mathcal{J}_0|(2|I_s^{\mathbf{C}}|/n)^2$, and the probability, when we reveal J_s, that any given one of these has $i\in J_s$ and $j\notin J_s$ is $|J_s|(|I_s^{\mathbf{C}}|-|J_s|)/|I_s^{\mathbf{C}}|^2\pm n^{-0.5}$. So by Chernoff's inequality, with probability at least $1-\exp(-n^{0.5})$, there are

$$(1\pm 200\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot 4|J_s|(|I_s^{\mathbf{C}}|-|J_s|)n^{-2}$$

$s\in\mathcal{J}_0$ such that $i\in J_s$ and $j\in I_s^{\mathbf{C}}\setminus J_s$. Next, we consider those $s\in\mathcal{J}_0$ such that $i\in I_s^{\mathbf{C}}$, $v\notin\mathrm{im}^{\mathbf{C}}(s)$ and $j\notin I_s^{\mathbf{C}}$. Using $(Quasi6)$ after Stage C twice, there are

$$|\mathcal{J}_0|\tfrac{2|I_s^{\mathbf{C}}|}{n}\cdot\big(\tfrac{n-|U_s^{\mathbf{C}}|}{n}-\tfrac{2|I_s^{\mathbf{C}}|}{n}\big)\pm 200\gamma_{\mathbf{A}}n$$

such s. Of these, when we reveal J_s, the probability that any one such s has $i \in J_s$ is $\frac{|J_s|}{|I_s^{\mathbf{C}}|}$, and so by Chernoff's inequality, with probability at least $1-\exp(-n^{0.5})$, there are

$$|\mathcal{J}_0|\tfrac{2|J_s|}{n}\cdot\big(\tfrac{n-|U_s^{\mathbf{C}}|}{n}-\tfrac{2|I_s^{\mathbf{C}}|}{n}\big)\pm 300\gamma_{\mathbf{A}}n$$

such s. Summing up, we have

$$\begin{aligned}(1\pm 200\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot 4|J_s|(|I_s^{\mathbf{C}}|-|J_s|)n^{-2}+|\mathcal{J}_0|\tfrac{2|J_s|}{n}\cdot\big(\tfrac{n-|U_s^{\mathbf{C}}|}{n}-\tfrac{2|I_s^{\mathbf{C}}|}{n}\big)\pm 300\gamma_{\mathbf{A}}n\\ =|\mathcal{J}_0|\tfrac{2|J_s|}{n}\tfrac{n-|U_s^{\mathbf{C}}|-2|J_s|}{n}\pm 500\gamma_{\mathbf{A}}n\,.\end{aligned}$$

Note that $|U_s^{\mathbf{D}-}|$ is precisely $|U_s^{\mathbf{C}}|+2|J_s|$, so that this formula is exactly what we need for (*Quasi* 8) after stage $\mathbf{D}^-$.

Finally, for (D⁻iv), we use a similar argument as for (*Quasi* 8). Given u,v, if $u\in A_s^{\mathbf{C}}$ and $v\notin U_s^{\mathbf{D}-}$, then we need $v\notin \mathrm{im}^{\mathbf{C}}(s)$ and we remove those v such that $v\in\{\boxminus_j,\boxplus_j\}$ for some $j\in J_s$. The latter event is contained in the former. From (*Quasi* 8) after Stage C, we have

$$\big|\{s\in\mathcal{J}_0: u\in A_s^{\mathbf{C}}, v\notin \mathrm{im}^{\mathbf{C}}(s)\}\big|=(1\pm 100\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot 2\lambda^*(\delta+10\lambda^*)\,.$$

By Lemma 33, letting j be the index such that $v\in\{\boxminus_j,\boxplus_j\}$, we have

$$\big|\{s\in\mathcal{J}_0: u\in A_s^{\mathbf{C}}, j\in I_s^{\mathbf{C}}\}\big|=(1\pm 100\gamma_{\mathbf{A}})\sum_{s\in\mathcal{J}_0}\tfrac{2|A_s^{\mathbf{C}}||I_s^{\mathbf{C}}|}{n^2}=(1\pm 100\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot 2\lambda^*(\delta+10\lambda^*)^2\,.$$

Note that the probability that a given s in this last set has $j\in J_s$ is $\frac{|J_s|}{|I_s^{\mathbf{C}}|}$, so by Chernoff's inequality, with probability at least $1-\exp(-n^{0.5})$ we have

$$\big|\{s\in\mathcal{J}_0: u\in A_s^{\mathbf{C}}, j\in J_s\}\big|=(1\pm 200\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot(2\lambda^*)^2\,.$$

Putting these together, we have

$$\big|\{s\in\mathcal{J}_0: u\in A_s^{\mathbf{C}}, v\notin U_s^{\mathbf{D}-}\}\big|=(1\pm 1000\gamma_{\mathbf{A}})|\mathcal{J}_0|\cdot 2\lambda^*(\delta+10\lambda^*-2\lambda^*)$$

as required. □

We are now in a position to complete the proof of Lemma 34. The main work now is to obtain the setup for Lemma 42.

Proof of Lemma 34. We set $\nu=\frac{1}{1000}2^{-4L_{\mathbf{D}}}\delta^{4L_{\mathbf{D}}}(\lambda^*)^{4L_{\mathbf{D}}}$. Let C' be such that $2^{-10L_{\mathbf{D}}}C'$ is returned by Lemma 42, and set $\gamma'=1000L_{\mathbf{C}}\cdot 4^{3L_{\mathbf{C}}}\delta^{2L_{\mathbf{C}}}(\lambda^*)^{L_{\mathbf{C}}}\gamma_{\mathbf{A}}$, so that Lemma 55 guarantees that the setup after stage $\mathbf{D}^-$ is a $(\gamma',2L_{\mathbf{D}},d_{\mathbf{C}},d_{\mathbf{C}})$-quasirandom setup.

Let $H=H_{\mathbf{C}}$, which has $2\lfloor\frac{n}{2}\rfloor$ vertices.

We let $s^*=|\mathcal{J}_0|\geq\nu n$. Temporarily abusing notation, we relabel the graphs G_s such that $\mathcal{J}_0=[s^*]$. For each $s\in\mathcal{J}_0$, we do the following. Let F_s denote the path-forest whose components are $\mathsf{leftpath}_4(P)$ and $\mathsf{rightpath}_4(P)$ for each $P\in\mathsf{SpecPaths}_s^*$, and let A_s be the leaves of F_s. Note that for each s, the path-forest F_s is *not* $F_s^{\mathbf{D}-}$, and the set A_s is *not* the set $A_s^{\mathbf{D}-}$ from the quasirandom setup after stage $\mathbf{D}-$; rather $A_s=A_s^{\mathbf{D}-}\cup A_s^{\mathbf{C}}$, where $A_s^{\mathbf{C}}$ is the set of anchors after Stage C, and we recall $\{\boxminus_i,\boxplus_i: i\in J_s\}=A_s^{\mathbf{D}-}$. By construction $A_s^{\mathbf{C}}$ and $A_s^{\mathbf{D}-}$ are disjoint.

We let $U_s = U_s^{\mathbf{D}-} = \mathrm{im}^{\mathbf{D}-}(s)$. We let ϕ_s be the restriction of $\phi_s^{\mathbf{D}-}$ to the leaves of F_s.

For each vertex $x \in A_s$, we let $\xi_s(x)$ be whichever of $V_\boxminus$ and $V_\boxplus$ contains $\phi_s(x)$. For each vertex $y \in V(F_s) \setminus A_s$, we choose $\xi_s(y) \in \{V_\boxminus, V_\boxplus\}$ uniformly at random.

By Lemma 29 and ($Quasi$ 1), H is $((2L_{\mathbf{D}}+2)\gamma', 2L_{\mathbf{D}})$-block-quasirandom, and for each $s \in \mathcal{J}_0$, (H, U_s) is $((2L_{\mathbf{D}}+2)\gamma', 2L_{\mathbf{D}})$-block-diet. In particular, letting $d_{\boxminus\boxminus}, d_{\boxplus\boxplus}$ denote the density of $H[V_\boxminus]$ and $H[V_\boxplus]$ respectively, and $d_{\boxminus\boxplus}$ denote the bipartite density of $H[V_\boxminus, V_\boxplus]$, we have that each of these three parameters is $(1 \pm (2L_{\mathbf{D}}+2)\gamma')d_{\mathbf{C}}$.

We claim that with high probability, for each $s \in \mathcal{J}_0$ the random choice of ξ_s means that F_s has the $10\gamma'$-anchor distribution property with respect to H. Indeed, given $a, b, c \in \{\boxminus, \boxplus\}$, let $v \in V_b \setminus U_s$. Then by ($Quasi$ 5), applied after Stage C to $A_s^{\mathbf{C}}$ and after stage $\mathbf{D}^-$ to $A_s^{\mathbf{D}-}$, we have

$$\big|A_s \cap \mathsf{N}_H(v) \cap V_a\big| = (1 \pm \gamma')d_{\mathbf{C}} \cdot 2\lambda^* n\,.$$

Given a path $P \in F_s$ and a leaf of that path x anchored in $\mathsf{N}_H(v) \cap V_a$, the probability that the neighbour y and second neighbour z of x are assigned sets $\xi_s(y) = V_b$ and $\xi_s(z) = V_c$ is $\frac{1}{4}$. Since the choice of ξ_s on P affects at most the two leaves of P, we conclude by McDiarmid's inequality that with probability at least $1 - \exp(-n^{0.1})$ we have $(1 \pm 2\gamma')d_{\mathbf{C}} \cdot \frac{1}{2}\lambda^* n$ leaves x of F_s embedded to $\mathsf{N}_H(v) \cap V_a$, such that their neighbour and second neighbour are assigned to V_b and V_c respectively by ξ_s. Similarly, by ($Quasi$ 2) applied both after Stage C to $A_s^{\mathbf{C}}$ and after stage $\mathbf{D}^-$ to $A_s^{\mathbf{D}-}$, the number of leaves of F_s anchored in V_a in total is $(1 \pm \gamma') \cdot 2\lambda^* n$ and by a similar application of McDiarmid's inequality, with probability at least $1 - \exp(-n^{0.1})$, the number of these leaves whose neighbour is assigned by ξ_s to V_b and second neighbour to V_c is $(1 \pm 2\gamma') \cdot \frac{1}{2}\lambda^* n$. Taking the union bound over choices of a, b, c, s and v, with high probability we have the $10\gamma'$-anchor distribution property for each $s \in \mathcal{J}_0$.

Next, observe that for each $a \in \{\boxminus, \boxplus\}$ by ($Quasi$ 1) with $S_1 = S_2 = T_3 = \emptyset$ and either $T_1 = \{s\}, T_2 = \emptyset$ or vice versa, we have $|V_a \setminus U_s| = (1 \pm \gamma') \cdot \frac{1}{2}(\delta - 8\lambda^*)n$, while $\big|V(F_s) \setminus A_s\big| = 8\lambda^* n$. Even if ξ_s assigned all these vertices to V_a, by choice of ν we have that $\big|V_a \setminus U_s\big| - \big|\{x \in V(F_s) \setminus A_s : \xi_s(x) = V_a\}\big| \geq \nu n$. Furthermore, each path of F_s has five vertices, of which the middle three are assigned uniformly at random to $V_\boxminus$ or $V_\boxplus$ by ξ_s. By the Chernoff bound and the union bound over the at most n choices of s, in each F_s we have $\big|\xi_s^{-1}(\{V_a\}) \setminus A_s\big| \geq \lambda^* n \geq \nu n$ with probability at least $1 - \exp(-n^{0.1})$. Similarly, in each path of F_s, the two internal edges each have probability at least $\frac{1}{4}$ of being assigned by ξ_s to any of within $V_\boxminus$, within $V_\boxplus$, or between $V_\boxminus$ and $V_\boxplus$, and by McDiarmid's inequality, the total number of internal edges of F_s assigned by ξ_s to each of within $V_\boxminus$, within $V_\boxplus$, and between $V_\boxminus$ and $V_\boxplus$, is at least $\frac{1}{4}\lambda^* n \geq \nu n$ with probability at least $1 - \exp(-n^{0.1})$.

We claim that the random choice of the ξ_s gives the $3\gamma'$-pair distribution property. To that end, fix distinct vertices $u \in V_a$ and $v \in V_b$, where $a, b \in \{\boxminus, \boxplus\}$ Suppose that we have $uv \in E(H_{\mathbf{C}})$, and in particular by Lemma 33 we do not have $\{u, v\} = \{\boxminus_i, \boxplus_i\}$ for any i. We want to know the various $w_{uv;s}$ as s ranges over $\mathcal{J}_0$, as in Definition 41. Recall that there are three different possibilities for $w_{uv;s} > 0$. If $u, v \notin U_s$, then we claim it is likely that we have

$$w_{uv;s} = \frac{(1 \pm \gamma')2\lambda^* n}{(1 \pm \gamma')^2 \frac{1}{4}(\delta + 8\lambda^*)^2 n^2} = (1 \pm 4\gamma') \frac{8\lambda^*}{(\delta + 8\lambda^*)^2 n} .$$

The denominator uses estimates on $|V_a \setminus U_s|$ from $(Quasi\,1)$ as above. For the numerator we observe that F_s has $4\lambda^* n$ internal edges (and so $8\lambda^* n$ pairs span an internal edge), each of which has probability $\frac{1}{4}$ of being assigned by ξ_s to respectively V_a and V_b. The random choice of ξ_s on each component of F_s affects the sum by at most 4. Hence by McDiarmid's inequality, with probability at least $1 - \exp(-n^{0.1})$ we obtain the numerator. It remains to estimate the number of $s \in \mathcal{J}_0$ such that $u, v \notin U_s$. By $(Quasi\,6)$, with $S_1 = \{u, v\}$ and $S_2 = T = \emptyset$, we see that this number is $(1 \pm \gamma')(\delta + 8\lambda^*)^2 |\mathcal{J}_0|$. Thus the contribution to $\sum_{s \in \mathcal{J}_0} w_{uv;s}$ from s falling into this category is

$$(1 \pm 4\gamma') \frac{8\lambda^*}{(\delta + 8\lambda^*)^2 n} \cdot (1 \pm \gamma')(\delta + 8\lambda^*)^2 |\mathcal{J}_0| = (1 \pm 6\gamma') \frac{8\lambda^* |\mathcal{J}_0|}{n} .$$

Next, if $u \in A_s$ and $v \notin U_s$, then we can have $w_{uv;s} > 0$, and if we do, we have

$$w_{uv;s} = \tfrac{1}{(1 \pm \gamma')(\delta + 8\lambda^*) n/2}$$

by our estimate on $|V_a \setminus U_s|$ from $(Quasi\,1)$. This occurs when the vertex x of F_s anchored to u has neighbour y such that $\xi_s(y) = V_b$. There are two ways this can happen: either $u \in A_s^{\mathbf{C}}$, $v \notin U_s$, and $\xi_s(v) = V_b$, or $u \in A_s^{\mathbf{D}-}$, $v \notin U_s$, and $\xi_s(v) = V_b$. For the first of these, by $(D^{-}iv)$ we see that there are $(1 \pm 1000\gamma_{\mathbf{A}})|\mathcal{J}_0| 2\lambda^*(\delta + 8\lambda^*)$ choices of s for which $u \in A_s^{\mathbf{C}}$ and $v \notin U_s$, and of these by Chernoff's inequality with probability at least $1 - \exp(-n^{0.5})$ we see

$$(1 \pm 2000\gamma_{\mathbf{A}})|\mathcal{J}_0| \lambda^*(\delta + 8\lambda^*)$$

choices of s for which all three conditions hold. For the second, we use $(Quasi\,8)$ with the setup after $\mathbf{D}^-$ and Chernoff's inequality to obtain the same bound. Putting these together, the contribution to $\sum_{s \in \mathcal{J}_0} w_{uv;s}$ from s falling into this category is

$$\tfrac{1}{(1 \pm \gamma')(\delta + 8\lambda^*) n/2} \cdot 2(1 \pm 2000\gamma_{\mathbf{A}})|\mathcal{J}_0| \lambda^*(\delta + 8\lambda^*) = (1 \pm 2\gamma') \tfrac{4\lambda^* |\mathcal{J}_0|}{n} .$$

Finally, we can have $w_{uv;s} > 0$ if $v \in A_s$ and $u \notin U_s$. This is symmetric to the previous case, and we obtain the same bounds. We conclude that

$$\sum_{s \in \mathcal{J}_0} w_{uv;s} = (1 \pm 2\gamma') \frac{16\lambda^* |\mathcal{J}_0|}{n} ,$$

which is as required by Definition 41, since each F_s has $8\lambda n$ edges, hence $16\lambda n$ pairs (x, y) of vertices such that $xy \in E(F_s)$, of which by the random choice of the ξ_s and random anchor distribution $(1 \pm \gamma_{\mathbf{A}})4\lambda n$ pairs (x, y) have $\xi_s(x) = V_a$ and $\xi_s(y) = V_b$.

We define d'_{ab} for $a, b \in \{\boxminus, \boxplus\}$ as in Lemma 42 to be the densities of the graph obtained by successfully packing all the path-forests F_s into $H_{\mathbf{C}}$ according to ξ_s. We now aim to show that

each of these quantities is
$$(1 \pm \gamma')\big(d_{\mathbf{C}} - \tfrac{16\lambda^*|\mathcal{J}_0|}{n^2}\big)\,,$$
and in particular all are at least ν. To that end, we claim that with high probability, for each $s \in \mathcal{J}_0$, the number of edges $uv \in F_s$ such that $\xi_s(u), \xi_s(v) = V_\boxminus$ is $(1 \pm 1000\gamma_{\mathbf{A}})|\mathcal{J}_0| \cdot 2\lambda^* n$. The choice of ξ_s on different components of F_s is independent, and the effect of the choice on any given component is to change this count by at most 4, so by McDiarmid's inequality the probability that the actual number of edges $uv \in F_s$ such that $\xi_s(u), \xi_s(v) = V_\boxminus$ differs from its expectation by $n^{0.7}$ or more is at most $\exp(-n^{0.1})$. Now the probability that an edge of a path P which is not adjacent to a leaf of P has both vertices assigned to $V_\boxminus$ is exactly $\frac{1}{4}$. If uv is an edge of P such that u is a leaf vertex (so $\xi_s(u)$ is fixed) then we have $\xi_s(u), \xi_s(v) = V_\boxminus$ with probability either $\frac{1}{2}$ (if $\xi_s(u) = V_\boxminus$) or 0. By $(Quasi 8)$, applied after Stage C to $A_s^{\mathbf{C}}$, we see
$$|A_s \cap V_\boxminus| = (1 \pm 100\gamma_{\mathbf{A}})\lambda^* n + \lambda^* n\,,$$
and putting these together, the expected number of edges of F_s both of whose ends are assigned to $V_\boxminus$ is $(1 \pm 100\gamma_{\mathbf{A}}) \cdot 2\lambda^* n$, from which the claim follows. By the same argument, the same bound holds replacing $V_\boxminus$ with $V_\boxplus$, and taking the union bound, with high probability both bounds hold for each $s \in \mathcal{J}_0$. Hence
$$d'_{ab} = (1 \pm \gamma')d_{\mathbf{D}}\,, \quad \text{where} \quad d_{\mathbf{D}} = d_{\mathbf{C}} - \tfrac{8\lambda^*|\mathcal{J}_0|}{\binom{n}{2}}$$
for each $a, b \in \{\boxminus, \boxplus\}$ as required.

Suppose that the random choice of the ξ_s is such that all the above high probability events hold. Then by Lemma 42, with input $\nu, 2L_{\mathbf{D}}, \gamma = 100\gamma'$, when Algorithm 2 is executed, with probability at least $1 - \exp\big(-n^{0.2}\big)$ we obtain for each $s \in \mathcal{J}_0$ an embedding ϕ'_s of F_s into $H_{\mathbf{C}}\big[(V(H) \setminus U_s) \cup \phi_s(A_s)\big]$ which extends ϕ_s, such that $\phi'_s(x) \in \xi_s(x)$ for each $x \in V(F_s)$, and such that for each $uv \in E(H)$ there is at most one s such that ϕ'_s uses the edge uv. Suppose that this likely event occurs; we will in addition list polynomially many further likely events, which we presume all occur, and fix such an outcome.

We now define the setup after Stage D. We let $H_{\mathbf{D}}$ be the graph H' returned by Lemma 42. For $s\mathcal{J}_1 \cup \mathcal{J}_2$, we set $\phi_s^{\mathbf{D}} = \phi_s^{\mathbf{C}}$, with the same anchor and used sets and the same path-forests to embed. For $s \in \mathcal{J}_0$, we let $\phi_s^{\mathbf{D}}$ be the union of $\phi_s^{\mathbf{D}-}$ and the above obtained ϕ'_s (these overlap on the set A_s, but they agree on these vertices). We let $F_s^{\mathbf{D}}$ be the path-forest with paths $\mathsf{SpecShortPaths}_s^*$, with anchor set $A_s^{\mathbf{D}}$ the image under $\phi_s^{\mathbf{D}}$ of its leaves and $U_s^{\mathbf{D}} = \mathrm{im}^{\mathbf{D}}(s)$. We let $I_s^{\mathbf{D}} = \{i \in [\lfloor n/2 \rfloor] : \boxminus_i, \boxplus_i \in A_s^{\mathbf{D}}\}$. We now argue that this is a $(\gamma_{\mathbf{D}}, L_{\mathbf{D}}, d_{\mathbf{D}}, d_{\mathbf{D}})$-quasirandom setup, as required for Lemma 34.

For $(Quasi 1)$, fix S_1, S_2, T_1, T_2, T_3 as in Definition 28, each of size at most $L_{\mathbf{D}}$, with families of sets $(U_s^{\mathbf{D}-})_{s \in \mathcal{J}}$ and $(J_s)_{s \in \mathcal{J}_0}$, and let $X = \mathbb{U}_{H_{\mathbf{C}}}(S_1, S_2, T_1, T_2, T_3)$. Then we have $|X| \geq \nu n$, and so by $(PP5)$ it is likely that we have

$$\big|X\cap\mathbb{U}_{H_{\mathbf{D}}}(S_1,S_2,T_1\cap\mathcal{J}_0,T_2\cap\mathcal{J}_0)\big| = (1\pm C'\gamma')\big(\tfrac{d_{\mathbf{D}}}{d_{\mathbf{C}}}\big)^{|S_1|+|S_2|}\prod_{t\in T_1}\frac{|V_{\boxminus}\setminus U_t^{\mathbf{D}}|}{|V_{\boxminus}\setminus U_t^{\mathbf{D}-}|}\prod_{t\in T_2}\frac{|V_{\boxplus}\setminus U_t^{\mathbf{D}}|}{|V_{\boxplus}\setminus U_t^{\mathbf{D}-}|}|X|\,.$$

Plugging in the size $|X|$ given by $(Quasi\,1)$ after stage $\mathbf{D}^-$, and observing that the above set is precisely $\mathbb{U}_{H_{\mathbf{D}}}(S_1,S_2,T_1,T_2,T_3)$ with families of sets $(U_s^{\mathbf{D}})_{s\in\mathcal{J}}$ and $(J_s)_{s\in\mathcal{J}_0}$, this is what is required for $(L_{\mathbf{D}},2C'\gamma',d_{\mathbf{D}},d_{\mathbf{D}})$-index-quasirandomness.

For $(Quasi\,2)$, observe that for $\mathcal{J}_1$ and $\mathcal{J}_2$ nothing has changed from after stage $\mathbf{D}^-$, while for $\mathcal{J}_0$ the required equality holds even with zero error by construction. Similarly, for $(Quasi\,3)$ nothing has changed from after stage $\mathbf{D}^-$. For $(Quasi\,4)$, it suffices by construction to establish $(Quasi\,5)$ as in stage $\mathbf{D}^-$.

For $(Quasi\,5)$, fix $s\in\mathcal{J}$, $a\in\{\boxminus,\boxplus\}$ and $u\in V(H_{\mathbf{D}})\setminus U_s^{\mathbf{D}}$. By $(Quasi\,5)$ after stage $\mathbf{D}^-$, writing A_s^a for the set of anchors $v\in A_s^{\mathbf{D}}$ such that the path end not anchored to v is anchored in V_a, we have

$$\big|\{\boxminus_i\in\mathsf{N}_{H_{\mathbf{C}}}(u)\cap A_s^a\}\big| = (1\pm\gamma')d_{\mathbf{C}}|A_s^a\cap V_{\boxminus}|\quad\text{and}$$
$$\big|\{\boxplus_i\in\mathsf{N}_{H_{\mathbf{C}}}(u)\cap A_s^a\}\big| = (1\pm\gamma')d_{\mathbf{C}}|A_s^a\cap V_{\boxplus}|\,.$$

We define a weight function w on $V(H_{\mathbf{C}})$ by setting $w(v)=1$ if $v\in A_s^a$, and otherwise $w(v)=0$. Note that $\sum_{v\in V_b}w(v)$ is either equal to zero or at least νn by $(Quasi\,2)$ and by construction. Thus by $(PP4)$, it is likely that we have

$$\big|\{\boxminus_i\in\mathsf{N}_{H_{\mathbf{D}}}(u)\cap A_s^a\}\big| = (1\pm\gamma')d_{\mathbf{C}}|A_s^a\cap V_{\boxminus}|(1\pm C'\gamma')\tfrac{d_{\mathbf{D}}}{d_{\mathbf{C}}}\quad\text{and}$$
$$\big|\{\boxplus_i\in\mathsf{N}_{H_{\mathbf{D}}}(u)\cap A_s^a\}\big| = (1\pm\gamma')d_{\mathbf{C}}|A_s^a\cap V_{\boxplus}|(1\pm C'\gamma')\tfrac{d_{\mathbf{D}}}{d_{\mathbf{C}}}\,.$$

This is as required for $(Quasi\,5)$.

For $(Quasi\,6)$, fix S_1,S_2,T as in the statement, each of size at most $L_{\mathbf{D}}$. By $(Quasi\,6)$ after stage $\mathbf{D}^-$, for each $Q\subseteq S_2$ we have

$$\big|\{s\in\mathcal{J}_0:(S_1\cup Q)\cap U_s^{\mathbf{D}-}=\emptyset,T\subseteq J_s\}\big| = (1\pm\gamma')|\mathcal{J}_0|(\delta+8\lambda^*)^{|S_1|+|Q|}(2\lambda^*)^{|T|}\,,$$

and by $(PP6)$, and bounds on $|U_s^{\mathbf{D}}\cap V_{\boxminus}|$ and $|U_s^{\mathbf{D}-}\cap V_{\boxminus}|$ from $(Quasi\,1)$, applied to each of these sets (which all have size at least νn) it is likely that we have

$$\begin{aligned}\big|\{s\in\mathcal{J}_0:(S_1\cup Q)\cap U_s^{\mathbf{D}}=\emptyset,T\subseteq J_s\}\big|&\\ &= (1\pm\gamma')|\mathcal{J}_0|(\delta+8\lambda^*)^{|S_1|+|Q|}(2\lambda^*)^{|T|}(1+2C'\gamma')\big(\tfrac{\delta+2\lambda^*}{\delta+8\lambda^*}\big)^{|S_1|+|Q|}\big)\\ &= |\mathcal{J}_0|(\delta+2\lambda^*)^{|S_1|+|Q|}(2\lambda^*)^{|T|}\pm 3C'\gamma' n\,.\end{aligned}$$

Now we can write the desired $\big|\{s\in\mathcal{J}_0:S_1\cap U_s^{\mathbf{D}}=\emptyset,S_2\subseteq U_s^{\mathbf{D}},T\subseteq J_s\}\big|$ by inclusion-exclusion in terms of these sets, and we obtain

$$\big|\{s\in\mathcal{J}_0:S_1\cap U_s^{\mathbf{D}}=\emptyset,S_2\subseteq U_s^{\mathbf{D}},T\subseteq J_s\}\big| = |\mathcal{J}_0|(\delta+2\lambda^*)^{|S_1|}(1-\delta-2\lambda^*)^{|S_2|}(2\lambda^*)^{|T|}\pm 2^{L_{\mathbf{D}}+10}C'\gamma' n$$

which is as required for $(Quasi\,6)$.

For $(Quasi7)$, observe that nothing has changed since after stage $\mathbf{D}^{-}$. The same holds for $(Quasi8)$ for $i=1,2$. For $\mathcal{J}_0$, fix distinct $u,v\in V(H_{\mathbf{D}})$ such that for no j do we have $\{u,v\}=\{\boxminus_j,\boxplus_j\}$. Let $X=\{s\in\mathcal{J}_0 : u\in A_s^{\mathbf{D}}, v\notin U_s^{\mathbf{D}-}\}$. Note that $A_s^{\mathbf{D}}=A_s^{\mathbf{D}-}$. By $(Quasi8)$ after Stage $\mathbf{D}^{-}$, we have $|X|=(1\pm\gamma')|\mathcal{J}_0|\cdot 2\lambda^*(\delta+8\lambda)$. By $(PP6)$, and bounds on $|U_s^{\mathbf{D}}\cap V_{\boxminus}|$ and $|U_s^{\mathbf{D}-}\cap V_{\boxminus}|$ from $(Quasi1)$ after stages D and $\mathbf{D}^{-}$, we have

$$\left|\{s\in X : v\notin U_s^{\mathbf{D}}\}\right| = (1\pm 2C'\gamma')\tfrac{n-|U_s^{\mathbf{D}}|}{n-|U_s^{\mathbf{D}-}|}\cdot(1\pm\gamma')|\mathcal{J}_0|\cdot 2\lambda^*(\delta+8\lambda)$$

as required for $(Quasi8)$. □

12. Stage E (Proof of Lemma 36)

In this stage, we need to pack the paths $\mathsf{SpecPaths}_s^*$ for $s\in\mathcal{J}_2$. Recall that each of these paths has seven edges. For each $s\in\mathcal{J}_2$, the set $\mathsf{SpecPaths}_s$ contains exactly λn paths; in Stage C we packed at most $n^{0.6}$ of these paths to leave $\mathsf{SpecPaths}_s^*$. The goal of this section is to pack a specified number of edges of these paths into each of $V_{\boxminus}$ and $V_{\boxplus}$, and between the sides $V_{\boxminus}$ and $V_{\boxplus}$. Due to our choice $\lambda\gg\sigma_0,\sigma_1$, the density of the graph $H_{\mathbf{E}}$ after this stage will be much smaller than the density of $H_{\mathbf{D}}$. Thus we need to embed roughly half the edges of the paths $\bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}_s^*$ crossing between $V_{\boxminus}$ and $V_{\boxplus}$, with the other roughly half about equally split between sides. To begin with, we assign to each path a number of edges it should use within $V_{\boxminus}$, crossing between $V_{\boxminus}$ and $V_{\boxplus}$, and within $V_{\boxplus}$. Observe that a path with one anchor on each side necessarily has an *odd* number of crossing edges, while a path with both anchors on the same side has an *even* number of crossing edges.

We will use the following definition with $X=\bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}_s^*$, where our goal is to use N_1 edges within $V_{\boxminus}$, to use N_2 edges crossing between sides, and N_3 within $V_{\boxplus}$. The map m identifies which paths are odd and which even, while the occupancy assignment $\aleph(x)$ says how many edges the path x needs to use in $V_{\boxminus}$, crossing between sides, and in $V_{\boxplus}$ in that order.

Definition 56. *Suppose that X is a finite set, and $m : X\to\{odd, even\}$ is a map. Then we say that a triple (N_1,N_2,N_3) of non-negative integers is a* moderately balanced occupancy requirement *if*

(a) $N_1+N_2+N_3=7|X|$,

(b) $3|X|\le N_2\le 4|X|$ and $N_1,N_3\ge|X|$, and

(c) the parity of N_2 is the same as the parity of $|m^{-1}(odd)|$.

Given a moderately balanced occupancy requirement (N_1,N_2,N_3), we say that a map $\aleph : X\to\mathbb{Z}^3$ is an occupancy assignment *for (N_1,N_2,N_3) if we have the following. For each $x\in X$ it holds that*

(d) for each $x\in X$, we have that $\aleph(x)_1\ge 1$, $\aleph(x)_3\ge 1$ and $\aleph(x)_2\ge 2$,

(e) for each $x\in X$, we have that $\sum_{i=1}^{3}\aleph(x)_i=7$,

(f) for each $x\in X$, we have that $\aleph(x)_2$ is odd if and only if $m(x)=odd$,

(*g*) *for each* $i \in \{1,2,3\}$*, we have that* $N_i = \sum_{x\in X} \aleph(x)_i$.

We call vectors in the image of $\aleph$ occupancy vectors.

To understand the definition of moderately balanced occupancy requirement, (*a*) says that in total we assign all $7|X|$ edges of the paths and (*b*) that the desired assignment is not too far from the quarter-half-quarter distribution mentioned above.

The first condition (*d*) of an occupancy assignment is something we will need to help embed paths later in this stage; the remaining conditions simply say that we assign all seven edges of each path, that precisely the odd paths have an odd number of crossing edges assigned, and that in total we assign the desired number of edges to each side and crossing. What an occupancy assignment does *not* specify is which vertices of each path will be assigned to which side; we will do that later in the stage.

The next easy lemma says that given a moderately balanced occupancy requirement, we can find an occupancy assignment.

Lemma 57. *Suppose that* X *is a finite set, and* $m : X \to \{odd, even\}$ *is a map, and* (N_1, N_2, N_3) *is a moderately balanced occupancy requirement for* X *and* m*. Then there exists an occupancy assignment for* (N_1, N_2, N_3).

Proof. Let a set $X = \{x_1, x_2, \ldots, x_n\}$, a map $m : X \to \{odd, even\}$ and a moderately balanced occupancy requirement be given. An occupancy assignment $\aleph$ can be found by Algorithm 5.

Algorithm 5: Generating an occupancy assignment

```
Input  : set X, map m : X → {odd, even} and m. b. o. requirement (N_1, N_2, N_3)
Output: occupancy assignment ℵ : X → ℤ^3
for k = 1, 2, ... |X| do
    ℵ(x_k)_1 = ℵ(x_k)_3 = 1 and ℵ(x_k)_2 = 2;
    if m(x_k) = odd
        then ℵ(x_k)_2 = 3;
end
while Σ_k ℵ(x_k)_2 < N_2 do
    t = min{ℓ ∈ [n] : ℵ(x_ℓ)_2 ≤ ℵ(x_k)_2 for all k};
    ℵ(x_t)_2 ← ℵ(x_t)_2 + 2;
end
while Σ_{i,k} ℵ(x_k)_i < Σ_i N_i do
    t = min{ℓ ∈ [n] : Σ_i ℵ(x_ℓ)_i < 7};
    j = min{i ∈ [3] : Σ_ℓ ℵ(x_ℓ)_i < N_i};
    ℵ(x_t)_j ← ℵ(x_t)_j + 1;
end
```

The for loop sets initial values for $\aleph(x_k)_i$ for every $k \in [n]$ and $i \in [3]$. Right at the end of that loop the following holds: We have $\sum_k \aleph(x_k)_1 = |X| \le N_1$, $\sum_k \aleph(x_k)_3 = |X| \le N_3$ and $\sum_k \aleph(x_k)_2 \le 3|X|$. By the If line, the parity of $\sum_k \aleph(x_k)_2$ is the same as the parity of $|m^{-1}(odd)|$

and hence, by (b) and (c), $N_2 - \sum_k \aleph(x_k)_2 \geq 0$ is even. Further, $|\aleph(x_{k_1})_2 - \aleph(x_{k_2})_2| \leq 2$ for every $k_1, k_2 \in [n]$.

During the subsequent while loop, the value of $\sum_k \aleph(x_k)_2$ gets increased by 2 in each iteration, hence forcing the loop to stop eventually with $N_2 = \sum_k \aleph(x_k)_2$. By the choice of x_t in each such iteration, we also ensure that $|\aleph(x_{k_1})_2 - \aleph(x_{k_2})_2| \leq 2$ is maintained for every $k_1, k_2 \in [n]$. In particular, using $N_2 \leq 4|X|$ from (b), we know that $\aleph(x_k)_2 \leq 5$ needs to hold for every $k \in [n]$ by the end of that while loop. Hence, at the end of this while loop, we have $\sum_i \aleph(x_\ell)_i \leq 7$ for every $\ell \in [n]$, and $\sum_\ell \aleph(x_\ell)_i \leq N_i$ for every $i \in [3]$.

Afterwards, during the last loop, $\aleph(x_k)_2$ is not updated for any $k \in [n]$. However, as long as $\sum_{i,k} \aleph(x_k)_i < \sum_i N_i$ holds, the following is done: We fix $t = \min\{\ell \in [n] : \sum_i \aleph(x_\ell)_i < 7\}$ which must exist, since otherwise we would have $\sum_i \aleph(x_\ell)_i \geq 7$ for every $\ell \in [n]$, which leads to $\sum_{i,k} \aleph(x_k)_i \geq 7|X| = \sum_i N_i$, in contradiction to the condition for processing an iteration of the while loop. We also fix $j = \min\{i \in [3] : \sum_\ell \aleph(x_\ell)_i < N_i\}$ whose existence is ensured analogously. Then we update $\aleph$ by increasing $\aleph(x_t)_j$ by 1. Therefore, since $\sum_{i,k} \aleph(x_k)_i$ is increased with every iteration, the while loop and hence the whole algorithm must end eventually. Inductively, using the choice of t and j, one verifies easily that throughout the while loop we always maintain $\sum_i \aleph(x_\ell)_i \leq 7$ for every $\ell \in [n]$, and $\sum_\ell \aleph(x_\ell)_i \leq N_i$ for every $i \in [3]$. Using all this, we will show now that the output $\aleph$ satisfies the Properties (d)-(g).

Property (d) already holds at the end of the for loop. Afterwards it is kept to be true, since the values $\aleph(x_k)_i$ never decrease throughout the algorithm.

Assume Property (e) was wrong. Then, by the observations for the second while loop, there would need to exists some $\ell \in [n]$ with $\sum_i \aleph(x_\ell)_i < 7$ while $\sum_i \aleph(x_k)_i \leq 7$ for every $k \neq \ell$. However, this would lead to $\sum_{i,k} \aleph(x_k)_i < 7n = \sum_i N_i$ by (a), in contradiction to the while loop having stopped. Property (g) is proven analogously.

Finally, Property (f) holds, since for every $k \in [n]$, the initial value for $\aleph(x_k)_2$ in the for loop is chosen such that it satisfies (f), and since in the subsequent while loops we never change the parity of $\aleph(x_k)_2$. □

The following lemma is the main work of this section. We check that the number of edges we need to use in $V_\boxminus$, crossing, and in $V_\boxplus$, in order to obtain (18), gives us a moderately balanced occupancy requirement, and then use Lemma 57 to find a corresponding occupancy assignment $\aleph$. We then need to create a map ξ which assigns sides to the unembedded vertices of $\bigcup_{s \in \mathcal{J}_2} \mathsf{SpecPaths}_s^*$ in order that each path, when we embed all vertices to their assigned by ξ sides, uses a number of edges within and between sides specified by $\aleph$.

What makes this more complicated is that we need also to ensure that the conditions of Lemma 42, which we use to actually perform the embedding of $\bigcup_{s \in \mathcal{J}_2} \mathsf{SpecPaths}_s^*$, are met; this means we need to use some randomness in constructing ξ. The following lemma states that ξ as desired exists.

For compactness, in this lemma and its proof, we write for each $a \in \{\boxminus, \boxplus\}$ and $s \in \mathcal{J}_2$

$$E^* := \bigcup_{s\in\mathcal{J}_2} E(\mathsf{SpecPaths}^*_s) \quad \text{and}$$
$$X^a_s := \left\{x \in \mathsf{SpecPaths}^*_s : \xi(x) = V_a, x \text{ is not an endvertex of } \mathsf{SpecPaths}^*_s\right\}.$$

Lemma 58. *There exists a map* $\xi : \bigcup_{s\in\mathcal{J}_2} V\left(\mathsf{SpecPaths}^*_s\right) \to \{V_\boxminus, V_\boxplus\}$ *with the following properties.*

(*i*) *For each* $P \in \bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s$ *and each* $x \in \{\mathsf{leftpath}_0(P), \mathsf{rightpath}_0(P)\}$ *we have* $\phi^{\mathbf{A}}(x) \in \xi(x)$.

(*ii*) *For each* $a \in \{\boxminus, \boxplus\}$ *and* $s \in \mathcal{J}_2$*, we have* $\left|X^a_s\right| \geq \frac{1}{4}\lambda n$.

(*iii*) *For each* $a, b \in \{\boxminus, \boxplus\}$ *and* $s \in \mathcal{J}_2$*, we have*
$$\left|\left\{(x, y) \,:\, xy \in E(\mathsf{SpecPaths}_s), x \in X^a_s, y \in X^b_s\right\}\right| \geq \tfrac{1}{16}\lambda n\,.$$

(*iv*) *We have*

$$\sum_{xy\in E^*} \mathbb{1}_{\xi(x),\xi(y)=V_\boxminus} = e_{H_\mathbf{D}}(V_\boxminus) - \left(\sigma_0\lambda^* n^2 + 2j_{\boxminus\boxminus} + 3(j_{\boxminus\boxplus} + j_{\boxplus\boxplus})\right)\ , \tag{70}$$

$$\sum_{xy\in E^*} \mathbb{1}_{\xi(x),\xi(y)=V_\boxplus} = e_{H_\mathbf{D}}(V_\boxplus) - \left(\sigma_0\lambda^* n^2 + 2j_{\boxplus\boxplus} + 3(j_{\boxminus\boxplus} + j_{\boxminus\boxminus})\right)\ , \tag{71}$$

$$\sum_{xy\in E^*} \mathbb{1}_{\{\xi(x),\xi(y)\}=\{V_\boxminus,V_\boxplus\}} = e_{H_\mathbf{D}}(V_\boxminus, V_\boxplus) - \left(\sigma_0\lambda^* n^2 + 6(j_{\boxminus\boxminus} + j_{\boxplus\boxplus}) + 5j_{\boxminus\boxplus}\right)\ . \tag{72}$$

(*v*) *For every* $s \in \mathcal{J}_2$*, the pair* $\left(\phi^{\mathbf{D}}_{\restriction \textit{endvertices of } \mathsf{SpecPaths}^*_s}, \xi_{\restriction \textit{inner vertices of } \mathsf{SpecPaths}^*_s}\right)$ *has the* $6\gamma_\mathbf{D}$*-anchor distribution property with respect to the path-forest* $\mathsf{SpecPaths}^*_s$*, partition* $(V_\boxminus, V_\boxplus)$ *and the graph* $H_\mathbf{D}$.

(*vi*) *The collection of path-forests* $(\mathsf{SpecPaths}^*_s)_{s\in\mathcal{J}_2}$ *together with used sets* $(\mathrm{im}^{\mathbf{D}}(s))_{s\in\mathcal{J}_2}$*, anchorings* $(\phi^{\mathbf{D}}_{\restriction \textit{endvertices of } \mathsf{SpecPaths}^*_s})_{s\in\mathcal{J}_2}$ *and assignments* $(\xi_{\restriction \mathsf{SpecPaths}^*_s})_{s\in\mathcal{J}_2}$ *has the* $20\gamma_\mathbf{D}$*-pair distribution property.*

Proof. Let us write $D_{\boxminus\boxminus}$, $D_{\boxplus\boxplus}$, and $D_{\boxminus\boxplus}$ for the quantities on the right-hand sides of (70), (71), and (72), respectively. Define a map $m : \bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s \to \{\textit{odd}, \textit{even}\}$ as follows. If $P \in \bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s$ is such that $\phi^{\mathbf{A}}(\mathsf{leftpath}_0(P))$ and $\phi^{\mathbf{A}}(\mathsf{rightpath}_0(P))$ both belong to $V_\boxminus$, or both to $V_\boxplus$ then we set $m(P) := \textit{even}$. Otherwise, set $m(P) := \textit{odd}$.

Claim 58.1. *The triple* $(D_{\boxminus\boxminus}, D_{\boxminus\boxplus}, D_{\boxplus\boxplus})$ *is a moderately balanced occupancy assignment for the set* $\bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s$ *and the map* m.

Proof of Claim 58.1. Let us first verify Definition 56(a). By summing up the right-hand sides of (70)-(72), we get

$$\begin{aligned} D_{\boxminus\boxminus} + D_{\boxplus\boxplus} + D_{\boxminus\boxplus} &= e(H_{\mathbf{D}}) - 3\sigma_0\lambda^* n^2 - 11(j_{\boxminus\boxminus} + j_{\boxplus\boxplus} + j_{\boxminus\boxplus}) \\ &= e(H_{\mathbf{D}}) - 3\sum_{s\in\mathcal{J}_0} |\mathsf{SpecShortPaths}_s| - 11\sum_{s\in\mathcal{J}_1} |\mathsf{SpecPaths}^*_s| \, . \end{aligned} \tag{73}$$

Recall that we plan to eventually use every edge of H for the packing (c.f. Section 6.2 and Definition 9(a)). Some edges of H were used prior to Stage E, $7\sum_{s\in\mathcal{J}_2} |\mathsf{SpecPaths}^*_s|$ edges will be used in Stage E, $11\sum_{s\in\mathcal{J}_1} |\mathsf{SpecPaths}^*_s|$ edges will be used in Stage F and $3\sum_{s\in\mathcal{J}_0} |\mathsf{SpecShortPaths}_s|$ edges will be used in Stage G. This gives

$$\begin{aligned} e(H) = &\,(e(H) - e(H_{\mathbf{D}})) \\ &+ 7\sum_{s\in\mathcal{J}_2} |\mathsf{SpecPaths}^*_s| + 3\sum_{s\in\mathcal{J}_0} |\mathsf{SpecShortPaths}_s| + 11\sum_{s\in\mathcal{J}_1} |\mathsf{SpecPaths}^*_s| \, . \end{aligned}$$

Plugging this back to (73), we get

$$D_{\boxminus\boxminus} + D_{\boxplus\boxplus} + D_{\boxminus\boxplus} = 7\sum_{s\in\mathcal{J}_2} |\mathsf{SpecPaths}^*_s| \, ,$$

as was needed.

Next, we verify Definition 56(c). Here we use the parity correction established in Stage B. We first consider the edges and paths anchored in $V_\boxminus$ after Stage B. By ($B.iii$), we have $\deg_{H_{\mathbf{B}}}(v) \equiv \mathrm{PathTerm}(v) \mod 2$ for every $v \in V(H_{\mathbf{B}})$. Summing over $v \in V_\boxminus$, we have

$$\sum_{v\in V_\boxminus} \deg_{H_{\mathbf{B}}}(v) \equiv \sum_{v\in V_\boxminus} \mathrm{PathTerm}(v) \mod 2 \, .$$

Observe that on the left-hand side, every edge in $V_\boxminus$ is counted twice, and on the right hand side every path with both terminals in $V_\boxminus$ is counted twice. Thus we see that the parity of the number of edges in $H_{\mathbf{B}}$ leaving $V_\boxminus$ is equal to the parity of the number of paths in $\bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}_s$ with exactly one anchor in $V_\boxminus$; we call these *crossing paths*. We claim that this property is maintained in Stages C and D: that is, the parity of the number of edges in $H_{\mathbf{D}}$ leaving $V_\boxminus$ is equal to the parity of the number of unembedded crossing paths. More formally, we claim

$$e_{H_{\mathbf{D}}}\big(V_\boxminus, V_\boxplus\big) \equiv \Big| \bigcup_{s\in\mathcal{J}_0} \mathsf{SpecShortPaths}_s \Big| + j_{\boxminus\boxplus} + \big|m^{-1}(odd)\big| \mod 2 \, . \tag{74}$$

To see that this is true, consider the embeddings in Stages C and D. In Stage C, we embed some complete paths. When we embed a path with both anchors in $V_\boxminus$, or both anchors not in $V_\boxminus$, we do not change the number of crossing paths, and we necessarily use an even number of edges leaving $V_\boxminus$, so that the parities of the two quantities remain the same. When we embed a crossing path, the number of crossing paths drops by one and so the parity changes, but we use an odd number of edges leaving $V_\boxminus$ and so that parity also changes to match. Thus after Stage C, the parity of the number of edges going from $V_\boxminus$ to $V(H_{\mathbf{B}}) \setminus V_\boxminus$ is equal to the parity of the number of unembedded crossing paths. Note that after Stage C, there are no edges of $H_{\mathbf{C}}$ at $\boxdot$ (if it exists) and nor are any unembedded paths anchored there. Thus we can ignore $\boxdot$, and we get that the parity of $e_{H_{\mathbf{C}}}\big(V_\boxminus, V_\boxplus\big)$ is equal to the parity of the number of unembedded crossing paths, i.e. the number of paths in $\bigcup_{s\in\mathcal{J}} \mathsf{SpecPaths}^*_s$ with exactly one anchor in $V_\boxminus$. This last number is equal to the number of paths in $\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecPaths}^*_s$ with exactly one anchor in $V_\boxminus$, plus $j_{\boxminus\boxplus} + |m^{-1}(odd)|$ (which account for the paths belonging to $\mathcal{J}_1 \cup \mathcal{J}_2$).

In Stage D, we embed parts of the paths $\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecPaths}^*_s$. Consider some $P \in \mathsf{SpecPaths}^*_s$. Each of the two pieces of P that we embed starts at an anchor of P and terminates at an anchor of the corresponding path P' in $\mathsf{SpecShortPaths}_s$. Note that P' is a crossing path by construction. If both anchors of P are in $V_\boxminus$, or both are in $V_\boxplus$, then when we embed the two pieces of P we use in total an odd number of edges leaving $V_\boxminus$ (we use an even number in one piece and an odd number in the other) and increase the number of crossing paths by one; if P is a crossing path, then we embed in total an even number of edges leaving $V_\boxminus$ (either both pieces use an even number of edges, or both pieces use an odd number of edges, leaving $V_\boxminus$) and the number of crossing paths does not change. This, with the observation that all paths in $\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecShortPaths}_s$ are crossing, gives (74).

Recalling that $D_{\boxminus\boxplus}$ is the right-hand side of (72) and hence

$$D_{\boxminus\boxplus} \equiv e_{H_{\mathbf{D}}}(V_\boxminus, V_\boxplus) - \sigma_0\lambda^* n^2 - j_{\boxminus\boxplus} \mod 2\,.$$

Since $\big|\bigcup_{s\in\mathcal{J}_0} \mathsf{SpecShortPaths}_s\big| = \sigma_0\lambda^* n^2$, we see from (74) that $D_{\boxminus\boxplus} \equiv |m^{-1}(odd)| \mod 2$ as desired.

Last, we verify Definition 56(b). By Lemma 29, we have $d\big(H_{\mathbf{D}}[V_\boxminus]\big) = (1 \pm 2\gamma_{\mathbf{D}})d_{\mathbf{D}}$. Thus we have

$$\begin{aligned} D_{\boxminus\boxminus} &= e_{H_{\mathbf{D}}}(V_\boxminus) - \big(\sigma_0\lambda^* n^2 + 2j_{\boxminus\boxminus} + 3(j_{\boxminus\boxplus} + j_{\boxplus\boxplus})\big) \\ \boxed{\text{by choice of } \sigma_1} \quad &= e_{H_{\mathbf{D}}}(V_\boxminus) \pm 2\sigma_0\lambda^* n^2 \\ &= \big(1 \pm 2\gamma_{\mathbf{D}}\big) d_{\mathbf{D}} \binom{|V_\boxminus|}{2} \pm 2\sigma_0\lambda^* n^2 \\ &= d_{\mathbf{D}} \cdot \frac{n^2}{8} \pm 3\sigma_0\lambda^* n^2 \overset{(17)}{=} 14\lambda^2 \cdot \frac{n^2}{8} \pm 4\sigma_0\lambda n^2\,. \end{aligned}$$

Similarly, we obtain $D_{\boxplus\boxplus} = 14\lambda^2 \cdot \frac{n^2}{8} \pm 4\sigma_0\lambda n^2$, and $D_{\boxminus\boxplus} = 28\lambda^2 \cdot \frac{n^2}{8} \pm 4\sigma_0\lambda n^2$. Definition 56($b$) follows then from (1). □

Having verified that we have a moderately balanced occupancy requirement, Lemma 57 can give an occupancy assignment. However, we cannot directly use the output of Lemma 57 to decide the number of edges that each path of $\bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s$ will embed to each side and crossing: this output could be imbalanced between the different $s \in \mathcal{J}_2$ and lead to failure of (v) or (vi). We put in an extra step and randomisation to deal with this as follows.

We split $\bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}^*_s$ into three sets $\mathcal{Q}_{\boxminus\boxminus}$, $\mathcal{Q}_{\boxminus,\boxplus}$, and $\mathcal{Q}_{\boxplus\boxplus}$,

$$\mathcal{Q}_{\boxminus\boxminus} := \left\{ P \in \bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}^*_s : \phi^{\mathbf{A}}(\mathsf{leftpath}_0(P)), \phi^{\mathbf{A}}(\mathsf{rightpath}_0(P)) \in V_{\boxminus} \right\}$$
$$\mathcal{Q}_{\boxplus\boxplus} := \left\{ P \in \bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}^*_s : \phi^{\mathbf{A}}(\mathsf{leftpath}_0(P)), \phi^{\mathbf{A}}(\mathsf{rightpath}_0(P)) \in V_{\boxplus} \right\},$$
$$\mathcal{Q}_{\boxminus,\boxplus} := \bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}^*_s \setminus (\mathcal{Q}_{\boxminus\boxminus}\cup\mathcal{Q}_{\boxplus\boxplus}) .$$

We define auxiliary sets $X = X_{\boxminus\boxminus}\dot\cup X_{\boxminus\boxplus}\dot\cup X_{\boxplus\boxplus}$, where the sizes of $X_{\boxminus\boxminus}$, $X_{\boxminus\boxplus}$, and $X_{\boxplus\boxplus}$ are $|\mathcal{Q}_{\boxminus\boxminus}|$, $|\mathcal{Q}_{\boxminus\boxplus}|$, and $|\mathcal{Q}_{\boxplus\boxplus}|$, respectively. These sets exist simply to have an occupancy assignment defined on them, which we can transfer to the $\mathcal{Q}_{\boxminus\boxminus}$ and so on by choosing random bijections.

We apply Lemma 57 with the set X, the map m which maps $X_{\boxminus\boxminus}\cup X_{\boxplus\boxplus}$ to *even* and $X_{\boxminus,\boxplus}$ to *odd*, and the tuple $(D_{\boxminus\boxminus}, D_{\boxminus\boxplus}, D_{\boxplus\boxplus})$. Let $\aleph$ be the occupancy assignment we obtain.

We pick a uniform random (and independent of $\aleph$) partition of X into sets $X^{\triangledown}$ and $X^{\blacktriangle}$. Let $\pi_{\boxminus\boxminus} : \mathcal{Q}_{\boxminus\boxminus}\to X_{\boxminus\boxminus}$, $\pi_{\boxminus\boxplus} : \mathcal{Q}_{\boxminus\boxplus}\to X_{\boxminus\boxplus}$, and $\pi_{\boxplus\boxplus} : \mathcal{Q}_{\boxplus\boxplus}\to X_{\boxplus\boxplus}$ be independent uniformly random bijections. We write $\pi : \bigcup_{s\in\mathcal{J}_2}\mathsf{SpecPaths}^*_s \to X$ for the union of these three bijections.

We can now define ξ on each set $V(P)$, $P\in\mathsf{SpecPaths}^*_s$, $s\in\mathcal{J}_2$ as follows. Let $P = x_1x_2\cdots x_8$, where in case $P\in\mathcal{Q}_{\boxminus\boxminus}\cup\mathcal{Q}_{\boxplus\boxplus}$ we take $x_1 = \mathsf{leftpath}_0(P)$ and in case $P\in\mathcal{Q}_{\boxminus\boxplus}$ we take the orientation of P such that $\phi^{\mathbf{A}}(x_1)\in V_{\boxminus}$. If $\pi(P)\in X_{\boxminus\boxminus}\cap X^{\triangledown}$ then we define $\xi_{\restriction V(P)}$ according to Table 2, whereas if $\pi(P)\in X_{\boxminus\boxminus}\cap X^{\blacktriangle}$, we define $\xi_{\restriction V(P)}$ according to Table 3.

TABLE 2. Defining $\xi_{\restriction V(P)}$ when $\pi(P)\in X_{\boxminus\boxminus}\cap X^{\triangledown}$.

value of $\aleph(\pi_{\boxminus\boxminus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(1,2,4)$	$x_1,x_2,x_8\mapsto V_{\boxminus}$, $x_3,x_4,\dots,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxplus)$	$(\boxminus,\boxplus,\boxplus)$
$(1,4,2)$	$x_1,x_2,x_4,x_8\mapsto V_{\boxminus}$, $x_3,x_5,x_6,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxplus)$	$(\boxminus,\boxplus,\boxplus)$
$(2,2,3)$	$x_1,x_2,x_3,x_8\mapsto V_{\boxminus}$, $x_4,x_5,x_6,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxminus)$	$(\boxminus,\boxplus,\boxplus)$
$(2,4,1)$	$x_1,x_2,x_3,x_5,x_8\mapsto V_{\boxminus}$, $x_4,x_6,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxminus)$	$(\boxminus,\boxplus,\boxplus)$
$(3,2,2)$	$x_1,x_2,x_3,x_4,x_8\mapsto V_{\boxminus}$, $x_5,x_6,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxminus)$	$(\boxminus,\boxplus,\boxplus)$
$(4,2,1)$	$x_1,x_2,\dots,x_5,x_8\mapsto V_{\boxminus}$, $x_6,x_7\mapsto V_{\boxplus}$	$(\boxminus,\boxminus,\boxminus)$	$(\boxminus,\boxplus,\boxplus)$

TABLE 3. Defining $\xi_{\restriction V(P)}$ when $\pi(P)\in X_{\boxminus\boxminus}\cap X^{\blacktriangle}$. Note that the table is the same as Table 2 if the paths are followed in the reversed direction.

value of $\aleph(\pi_{\boxminus\boxminus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(1,2,4)$	$x_1,x_7,x_8\mapsto V_{\boxminus}$, $x_2,x_3,\dots,x_6\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxplus)$
$(1,4,2)$	$x_1,x_5,x_7,x_8\mapsto V_{\boxminus}$, $x_2,x_3,x_4,x_6,\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxplus)$
$(2,2,3)$	$x_1,x_6,x_7,x_8\mapsto V_{\boxminus}$, $x_2,x_3,x_4,x_5,\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxminus)$
$(2,4,1)$	$x_1,x_4,x_6,x_7,x_8\mapsto V_{\boxminus}$, $x_2,x_3,x_5,\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxminus)$
$(3,2,2)$	$x_1,x_5,x_6,x_7,x_8\mapsto V_{\boxminus}$, $x_2,x_3,x_4\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxminus)$
$(4,2,1)$	$x_1,x_4,\dots,x_8\mapsto V_{\boxminus}$, $x_2,x_3\mapsto V_{\boxplus}$	$(\boxminus,\boxplus,\boxplus)$	$(\boxminus,\boxminus,\boxminus)$

The point of Tables 2 and 3 is that they satisfy requirement (i) of the Lemma, and the number of edges of P that are mapped into $V_{\boxminus}$, between $V_{\boxminus}$ and $V_{\boxplus}$, and into $V_{\boxplus}$, are $\aleph(\pi_{\boxminus\boxminus}(P))_1$, $\aleph(\pi_{\boxminus\boxminus}(P))_2$, and $\aleph(\pi_{\boxminus\boxminus}(P))_3$, respectively. Let us now explain the 'left pattern' and the 'right pattern' columns in Tables 2 and 3. The left pattern is the sequence which says where the three left-most vertices of P are mapped. The right pattern is the sequence which says where the three right-most vertices of P are mapped (starting with x_8).

By symmetry, there is a similar assignment $\xi_{\restriction V(P)}$ if $\pi(P) \in X_{\boxplus\boxplus} \cap X^{\triangledown}$ and $\pi(P) \in X_{\boxplus\boxplus} \cap X^{\blacktriangle}$, which for completeness we give in Table 4 and Table 5, respectively.

TABLE 4. Defining $\xi_{\restriction V(P)}$ when $\pi(P) \in X_{\boxplus\boxplus} \cap X^{\triangledown}$.

value of $\aleph(\pi_{\boxplus\boxplus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(4,2,1)$	$x_1, x_2, x_8 \mapsto V_{\boxplus}$, $x_3, x_4, \dots, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxminus)$	$(\boxplus, \boxminus, \boxminus)$
$(2,4,1)$	$x_1, x_2, x_4, x_8 \mapsto V_{\boxplus}$, $x_3, x_5, x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxminus)$	$(\boxplus, \boxminus, \boxminus)$
$(3,2,2)$	$x_1, x_2, x_3, x_8 \mapsto V_{\boxplus}$, $x_4, x_5, x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(1,4,2)$	$x_1, x_2, x_3, x_5, x_8 \mapsto V_{\boxplus}$, $x_4, x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(2,2,3)$	$x_1, x_2, x_3, x_4, x_8 \mapsto V_{\boxplus}$, $x_5, x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(1,2,4)$	$x_1, x_2, \dots, x_5, x_8 \mapsto V_{\boxplus}$, $x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$

TABLE 5. Defining $\xi_{\restriction V(P)}$ when $\pi(P) \in X_{\boxplus\boxplus} \cap X^{\blacktriangle}$. Similarly as with Tables 2 and 3, the paths in this table are reversed copies from Table 4.

value of $\aleph(\pi_{\boxplus\boxplus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(4,2,1)$	$x_1, x_7, x_8 \mapsto V_{\boxplus}$, $x_2, x_3, \dots, x_6 \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxminus)$
$(2,4,1)$	$x_1, x_5, x_7, x_8 \mapsto V_{\boxplus}$, $x_3, x_3, x_4, x_6, x_7 \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxminus)$
$(3,2,2)$	$x_1, x_6, x_7, x_8 \mapsto V_{\boxplus}$, $x_2, x_3, x_4, x_5 \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxplus)$
$(1,4,2)$	$x_1, x_4, x_6, x_7, x_8 \mapsto V_{\boxplus}$, $x_2, x_3, x_5, \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxplus)$
$(2,2,3)$	$x_1, x_5, x_6, x_7, x_8 \mapsto V_{\boxplus}$, $x_2, x_3, x_4 \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxplus)$
$(1,2,4)$	$x_1, x_4, \dots, x_8 \mapsto V_{\boxplus}$, $x_2, x_3 \mapsto V_{\boxminus}$	$(\boxplus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxplus)$

Next, we define $\xi_{\restriction V(P)}$ in case when $P \in \mathcal{Q}_{\boxminus\boxplus}$. To this end we use Tables 6 and 7.

TABLE 6. Defining $\xi_{\restriction V(P)}$ when $\pi(P) \in X_{\boxminus\boxplus} \cap X^{\triangledown}$.

value of $\aleph(\pi_{\boxminus\boxplus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(1,3,3)$	$x_1, x_2, x_4 \mapsto V_{\boxminus}$, $x_3, x_5, x_6, x_7, x_8 \mapsto V_{\boxplus}$	$(\boxminus, \boxminus, \boxplus)$	$(\boxplus, \boxplus, \boxplus)$
$(1,5,1)$	$x_1, x_2, x_4, x_6 \mapsto V_{\boxminus}$, $x_3, x_5, x_7, x_8 \mapsto V_{\boxplus}$	$(\boxminus, \boxminus, \boxplus)$	$(\boxplus, \boxplus, \boxminus)$
$(2,3,2)$	$x_1, x_2, x_3, x_5 \mapsto V_{\boxminus}$, $x_4, x_6, x_7, x_8 \mapsto V_{\boxplus}$	$(\boxminus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxplus)$
$(3,3,1)$	$x_1, \dots, x_4, x_6 \mapsto V_{\boxminus}$, $x_5, x_7, x_8 \mapsto V_{\boxplus}$	$(\boxminus, \boxminus, \boxminus)$	$(\boxplus, \boxplus, \boxminus)$

Again, it is obvious that Tables 6 and 7 obey requirement (i) of the Lemma, and the number of edges of P that are mapped into $V_{\boxminus}$, between $V_{\boxminus}$ and $V_{\boxplus}$, and into $V_{\boxplus}$, are $\aleph(\pi_{\boxminus\boxplus}(P))_1$, $\aleph(\pi_{\boxminus\boxplus}(P))_2$, and $\aleph(\pi_{\boxminus\boxplus}(P))_3$, respectively. Thus requirement (iv) is inherited from Definition 56(g).

TABLE 7. Defining $\xi_{\restriction V(P)}$ when $\pi(P) \in X_{\boxminus\boxplus} \cap X^{\blacktriangle}$.

value of $\aleph(\pi_{\boxminus\boxplus}(P))$	the map ξ on $V(P)$	left pattern	right pattern
$(1,3,3)$	$x_1, x_6, x_7 \mapsto V_\boxminus$, $x_2, \dots, x_5, x_8 \mapsto V_\boxplus$	$(\boxminus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(1,5,1)$	$x_1, x_4, x_6, x_7 \mapsto V_\boxminus$, $x_2, x_3, x_5, x_8 \mapsto V_\boxplus$	$(\boxminus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(2,3,2)$	$x_1, x_5, x_6, x_7 \mapsto V_\boxminus$, $x_2, x_3, x_4, x_8 \mapsto V_\boxplus$	$(\boxminus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$
$(3,3,1)$	$x_1, x_4, \dots, x_7 \mapsto V_\boxminus$, $x_2, x_3, x_8 \mapsto V_\boxplus$	$(\boxminus, \boxplus, \boxplus)$	$(\boxplus, \boxminus, \boxminus)$

By inspecting Tables 2 and 3, we observe that in all entries one of x_2 and x_7 is allocated to $V_\boxminus$, and the other is allocated to $V_\boxplus$. Thus each path of $\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s$ contributes one vertex to X^a_s for each $a \in \{\boxminus, \boxplus\}$. Since by $(Quasi\,2)$ we have

$$\left|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\right| = (1 \pm \gamma_{\mathbf{D}})\tfrac{1}{4} \cdot 2(\lambda n \pm n^{0.6}) \geq \tfrac{1}{4}\lambda n$$

we conclude (ii).

Finally, inspecting Table 7, we see that in all entries in this table the edge x_6x_7 is assigned to $V_\boxminus$, and x_2x_3 is assigned to $V_\boxplus$, and (since at least three edges are assigned between $V_\boxminus$ and $V_\boxplus$) there is an edge other than x_1x_2 and x_7x_8 which is assigned between $V_\boxminus$ and $V_\boxplus$. Thus each path P of $\mathsf{SpecPaths}^*_s$ which is in $\mathcal{Q}_{\boxminus\boxplus}$ and has $\pi(P) = X^{\blacktriangle}$ contributes one pair to each of the three sets considered in (iii) (observe that the case $a = \boxminus, b = \boxplus$ is identical to $a = \boxplus, b = \boxminus$ by symmetry). As above, by $(Quasi\,2)$ we have $\left|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\right| \geq \frac{1}{4}\lambda n$, and by Chernoff's inequality of these paths P, with probability at least $1 - \exp(-n^{0.5})$, at least $\frac{1}{16}\lambda n$ have $\pi(P) \in X^{\blacktriangle}$. We conclude (iii).

What remains is to check (v) and (vi). To this end, we use that $\pi_{\boxminus\boxminus}$, $\pi_{\boxminus\boxplus}$ and $\pi_{\boxplus\boxplus}$ are random, and that we split randomly $X = X^{\triangledown} \cup X^{\blacktriangle}$.

Let us first focus on proving (v), which is the following claim.

Claim 58.2. *With high probability for every $s \in \mathcal{J}_2$, the pair*

$$\left(\phi^{\mathbf{D}}_{\restriction \textit{endvertices of } \bigcup_{s\in\mathcal{J}_2} \mathsf{SpecPaths}^*_s}, \xi_{\restriction \textit{inner vertices of } \mathsf{SpecPaths}^*_s}\right)$$

has the $6\gamma_{\mathbf{D}}$-anchor distribution property with respect to the path-forest $\mathsf{SpecPaths}^*_s$, partition $(V_\boxminus, V_\boxplus)$ and the graph $H_{\mathbf{D}}$.

Proof. We fix $s \in \mathcal{J}_2$, and aim to prove the $6\gamma_{\mathbf{D}}$-anchor distribution property holds for $\mathsf{SpecPaths}^*_s$ with sufficiently high probability to take a union bound over the at most n choices of s.

As in Definition 39, given $a, b, c \in \{\boxminus, \boxplus\}$ let $A_{a,b,c}$ denote the collection of those endvertices x of paths $\mathsf{SpecPaths}^*_s$ such that $\phi^{\mathbf{D}}(x) \in V_a$, and such that the neighbour y of x satisfies $\xi(y) = V_b$, and the next vertex z satisfies $\xi(z) = V_c$. For each $v \in V_b \setminus \mathrm{im}^{\mathbf{D}}(s)$ we claim that with probability at least $1 - \exp(-n^{0.4})$ we have

$$\left|\left\{x \in A_{a,b,c} : \phi^{\mathbf{A}}(x) \in \mathsf{N}_{H_{\mathbf{D}}}(v)\right\}\right| = (1 \pm 3\gamma_{\mathbf{D}})d_{\mathbf{D}}|A_{a,b,c}| \pm \gamma_{\mathbf{D}} n^{0.99}\,. \tag{75}$$

Note that in this setting all the densities in Definition 39 are $(1 \pm 2\gamma_{\mathbf{D}})d_{\mathbf{D}}$ by the fact that from Lemma 34 we have a $(\gamma_{\mathbf{D}}, L_{\mathbf{D}}, d_{\mathbf{D}}, d_{\mathbf{D}})$-quasirandom setup, and using Lemma 29 with $(Quasi\,1)$, so that establishing (75) for each pattern a, b, c and $v \in V(H_{\mathbf{D}})$ gives the required $6\gamma_{\mathbf{D}}$-anchor distribution property. To simplify notation, we suppose $a = \boxminus$; the other case is symmetric.

Observe that $|A_{\boxminus,b,c}|$ is the total number of left and right patterns (as in Tables 2–7) whose consecutive values are $\boxminus$, b, and c which are assigned to paths of $\mathsf{SpecPaths}^*_s$ by ξ. This can be either a left or a right pattern of a path from $\mathcal{Q}_{\boxminus\boxminus}$, or a left pattern of a path from $\mathcal{Q}_{\boxminus\boxplus}$. We split up $A_{\boxminus,b,c}$ into sets $A^p_{\boxminus,b,c}$, where $p \in \{\boxminus, \boxplus\}$, and a vertex x of $A^{p,q}_{\boxminus,b,c}$ is in a path of $\mathsf{SpecPaths}^*_s$ whose endvertex not anchored to x is anchored in V_p.

We reveal the partition of $X = X^\triangledown \cup X^\blacktriangle$. Whatever this partition turns out to be, it together with $\aleph$ determine the total number $N_\boxminus$ of left and right patterns of paths in $\mathcal{Q}_{\boxminus\boxminus}$ which are $\boxminus, b, c$, and also the total number $N_\boxplus$ of left patterns of paths in $\mathcal{Q}_{\boxminus\boxplus}$ which are $\boxminus, b, c$. When we reveal $\pi_{\boxminus\boxminus}$, since each corresponding path has two ends in $V_\boxminus$, the size $\big|A^{\boxminus}_{\boxminus,b,c}\big|$ follows the hypergeometric distribution with parameters $\big(2\big|\mathcal{Q}_{\boxminus\boxminus}\big|, N_\boxminus, 2\big|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\big|\big)$. (Recall this is the random experiment of choosing, from a set of size $2\big|\mathcal{Q}_{\boxminus\boxminus}\big|$ of which $N_\boxminus$ are marked, a subset uniformly at random of size $2\big|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\big|\big)$ and seeing how many are marked.) Similarly, the size $\big|A^{\boxminus}_{\boxminus,b,c} \cap \mathsf{N}_{H_{\mathbf{D}}}(v)\big|$ follows the hypergeometric distribution with parameters $\big(2\big|\mathcal{Q}_{\boxminus\boxminus}\big|, N_\boxminus, A_{v,\boxminus}\big)$, where $A_{v,q}$ is the number of vertices in $V_q \cap \mathsf{N}_{H_{\mathbf{D}}}$ which are anchors of paths in $\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s$. By Fact 23, with probability at least $1 - \exp(-\sqrt{n})$ we get

$$\big|A^{\boxminus}_{\boxminus,b,c}\big| = \frac{N_\boxminus \cdot 2\big|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\big|}{2\big|\mathcal{Q}_{\boxminus\boxminus}\big|} \pm n^{0.9} \quad \text{and} \quad \big|A^{\boxminus}_{\boxminus,b,c} \cap \mathsf{N}_{H_{\mathbf{D}}}(v)\big| = \frac{N_\boxminus \cdot A_{v,\boxminus}}{2\big|\mathcal{Q}_{\boxminus\boxminus}\big|} \pm n^{0.9}\,.$$

By similar logic, recalling that paths of $\mathcal{Q}_{\boxminus\boxplus}$ have only one end anchored in $V_\boxminus$, we get

$$\big|A^{\boxplus}_{\boxminus,b,c}\big| = \frac{N_\boxplus \cdot \big|\mathcal{Q}_{\boxminus\boxplus} \cap \mathsf{SpecPaths}^*_s\big|}{\big|\mathcal{Q}_{\boxminus\boxplus}\big|} \pm n^{0.9} \quad \text{and} \quad \big|A^{\boxplus}_{\boxminus,b,c} \cap \mathsf{N}_{H_{\mathbf{D}}}(v)\big| = \frac{N_\boxplus \cdot A_{v,\boxplus}}{\big|\mathcal{Q}_{\boxminus\boxplus}\big|} \pm n^{0.9}\,.$$

Suppose that all four of these equalities hold. By ($Quasi\,5$) (specifically (12) or (13), depending on whether $v \in V_\boxminus$ or $V_\boxplus$), we have $A_{v,\boxminus} = (1 \pm \gamma_{\mathbf{D}})d_{\mathbf{D}} \cdot 2\big|\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s\big|$, and similarly $A_{v,\boxplus} = (1 \pm \gamma_{\mathbf{D}})d_{\mathbf{D}} \cdot \big|\mathcal{Q}_{\boxminus\boxplus} \cap \mathsf{SpecPaths}^*_s\big|$. Substituting these into the above equations, we get

$$\begin{aligned} \big|A^{\boxminus}_{\boxminus,b,c} \cap \mathsf{N}_{H_{\mathbf{D}}}(v)\big| &= (1 \pm \gamma_{\mathbf{D}})d_{\mathbf{D}}\big|A^{\boxminus}_{\boxminus,b,c}\big| \pm n^{0.95} \\ \text{and} \quad \big|A^{\boxplus}_{\boxminus,b,c} \cap \mathsf{N}_{H_{\mathbf{D}}}(v)\big| &= (1 \pm \gamma_{\mathbf{D}})d_{\mathbf{D}}\big|A^{\boxplus}_{\boxminus,b,c}\big| \pm n^{0.95}\,. \end{aligned}$$

Summing these equations, since $A_{\boxminus,b,c} = A^{\boxminus}_{\boxminus,b,c} \dot\cup A^{\boxplus}_{\boxminus,b,c}$, we obtain the desired (75). Taking the union bound over the four equalities and the choices of a, b, c, v we see that it holds for all these choices with probability at least $1 - \exp(-n^{0.4})$ as required. Finally the union bound over the choices of s completes the proof of the claim. □

Last, we shall prove that with high probability the map ξ has the $20\gamma_{\mathbf{D}}$-pair distribution property.

Recall that graphs $(G_s)_{s \in \mathcal{J}_2}$ were (partially) packed in Stage A and Stage C. For such an $s \in \mathcal{J}_2$, after Stage A, each G_s had $(\delta + 6\lambda)n$ unembedded vertices (c.f. (8)). In Stage C, between 0 and $6n^{0.6}$ additional vertices of G_s were embedded. Hence,

$$|V(G_s) \setminus \mathrm{im}^{\mathbf{D}}(s)| = (\delta + 6\lambda)n \pm 6n^{0.6} \; . \tag{76}$$

Similarly, the number of endvertices of paths in $\mathsf{SpecPaths}_s^*$ is

$$2|\mathsf{SpecPaths}_s^*| = 2\lambda n \pm 2n^{0.6} \; . \tag{77}$$

Recall that $E^* := \bigcup_{s \in \mathcal{J}_2} E(\mathsf{SpecPaths}_s^*)$, and let $E^{**} \subseteq E^*$ be the edges that do not contain endvertices (i.e. the 5 edges in the middle of each path). We refer to the edges of E^{**} as *internal* and those of $E^* \setminus E^{**}$ as *peripheral*. For $a, b \in \{\boxminus, \boxplus\}$ let

$$f_{ab} := \sum_{(x,y):xy \in E^{**}} \mathbb{1}_{\xi(x)=V_a, \xi(y)=V_b} \; , \tag{78}$$

where, as in Definition 41 of the pair distribution property, an edge xy with $\xi(x) = \xi(y)$ counts twice in this sum, once as (x, y) and once as (y, x). The same applies to future similar sums in this section. For $s \in \mathcal{J}_2$, let

$$f_{ab,s} := \sum_{(x,y):xy \in E^{**} \cap E(\mathsf{SpecPaths}_s^*)} \mathbb{1}_{\xi(x)=V_a, \xi(y)=V_b} \; . \tag{79}$$

We define $B_{\boxminus\boxminus} := 2D_{\boxminus\boxminus}$, and $B_{\boxplus\boxplus} := 2D_{\boxplus\boxplus}$, and $B_{\boxminus\boxplus}, B_{\boxplus\boxminus} := D_{\boxminus\boxplus}$. Thus B_{ab} is the number of edges in total, counted with multiplicity as in the above sums, that we want to embed in this stage with one end in V_a and the other in V_b. Recall that the D_{ab} form a moderately balanced occupancy requirement, and in particular we conclude

$$B_{ab} \geq \tfrac{7}{4}\lambda n|\mathcal{J}_2| = \tfrac{7}{4}\lambda(\lambda - \sigma_0 - \sigma_1)n^2 \, . \tag{80}$$

We now prove the following claim.

Claim 58.3. *For $s \in \mathcal{J}_2$, with probability at least $1 - 2\exp(-3\sqrt{n})$ we have for $a, b \in \{\boxminus, \boxplus\}$*

$$f_{ab,s} = (1 \pm 5\gamma_{\mathbf{D}}) \left(\frac{B_{ab}}{(\lambda - \sigma_0 - \sigma_1)n} - \lambda n \right) \; .$$

Proof of Claim 58.3. For $a, b \in \{\boxminus, \boxplus\}$, we define $B_{ab,s}$ as the total number of pairs (x, y), $xy \in E(\mathsf{SpecPaths}_s^*)$ with $\xi(x) = a$ and $\xi(y) = b$. We claim that with probability $1 - \exp(-\sqrt{n})$, for every $s \in \mathcal{J}_2$ we have

$$B_{ab,s} = (1 \pm 2\gamma_{\mathbf{D}}) \frac{B_{ab}}{(\lambda - \sigma_0 - \sigma_1)n} \geq \tfrac{3}{2}\lambda n \; . \tag{81}$$

The inequality in (81) is an immediate consequence of (80) and choice of $\gamma_{\mathbf{D}}$, so the difficulty is to prove the equality. To prove the equality in (81), fix $a, b \in \{\boxminus, \boxplus\}$ and $s \in \mathcal{J}_2$ and let $\mathfrak{v}$ be in the range of $\aleph$. By definition for each $c, d \in \{\boxminus, \boxplus\}$ there are $\aleph^{-1}(\mathfrak{v}) \cap X_{cd}$ elements of X_{cd} which are assigned $\mathfrak{v}$. The number of these elements which are in $\pi_{cd}\big(\mathsf{SpecPaths}_s^* \cap \mathcal{Q}_{cd}\big)$ follows the hypergeometric distribution with parameters

$$\big(|X_{cd}| \; , \; |\mathsf{SpecPaths}_s^* \cap \mathcal{Q}_{cd}| \; , \; |X_{cd} \cap \aleph^{-1}(\mathfrak{v})|\big)$$

and by Fact 23, with probability at least $1-\exp(-n^{0.6})$ we have

$$\big|\aleph^{-1}(\mathfrak{v})\cap X_{cd}\cap\pi_{cd}\big(\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}\big)\big|=\frac{|\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}||X_{cd}\cap\aleph^{-1}(\mathfrak{v})|}{|X_{cd}|}\pm n^{0.9}\,. \tag{82}$$

Suppose that (82) holds for each choice of $\mathfrak{v}$ and of c,d,s; there are at most $100n$ such choices in total, so that this occurs with probability at least $1-\exp(-\sqrt{n})$.

Suppose for a moment that $a=b=\boxminus$. Consider the following weighted sum of (82) (and note that the intersection with X_{cd} is redundant since $\pi_{cd}\big(\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}\big)\subseteq X_{cd}$). We have

$$\sum_{\mathfrak{v}}2\mathfrak{v}_1\cdot\big|\aleph^{-1}(\mathfrak{v})\cap\pi_{cd}\big(\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}\big)\big|=\sum_{\mathfrak{v}}2\mathfrak{v}_1\cdot\tfrac{|\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}||X_{cd}\cap\aleph^{-1}(\mathfrak{v})|}{|X_{cd}|}\pm n^{0.91} \tag{83}$$

The left hand side of this equality is precisely the contribution to $B_{ab,s}$ made by paths in $\mathcal{Q}_{cd}$, while on the right hand side, $\sum_{\mathfrak{v}}2\mathfrak{v}_1|X_{cd}\cap\aleph^{-1}(\mathfrak{v})|$ is the contribution to B_{ab} made by paths in $\mathcal{Q}_{cd}$. By ($Quasi\,2$), if $c=d$ then we have

$$|\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}|=(1\pm\gamma_{\mathbf{D}})\tfrac{1}{4}|\mathsf{SpecPaths}^*_s|\overset{(77)}{=}(1\pm\gamma_{\mathbf{D}})\tfrac{1}{4}\big(\lambda n\pm n^{0.6}\big)$$

and $|\mathcal{Q}_{cd}|=|\mathcal{J}_2|\frac{1}{4}\big(\lambda n\pm n^{0.6}\big)$. If $c\neq d$, we replace the fraction $\frac{1}{4}$ by $\frac{1}{2}$. In either case, we have

$$\frac{|\mathsf{SpecPaths}^*_s\cap\mathcal{Q}_{cd}|}{|\mathcal{Q}_{cd}|}=\frac{1\pm\frac{3}{2}\gamma_{\mathbf{D}}}{|\mathcal{J}_2|}\,.$$

Plugging these observations in, and summing (83) over choices of c and d, we get

$$B_{ab,s}=(1\pm\tfrac{3}{2}\gamma_{\mathbf{D}})\frac{B_{ab}}{|\mathcal{J}_2|}\pm n^{0.95}$$

from which (81) follows since $|\mathcal{J}_2|=(\lambda-\sigma_0-\sigma_1)n$. To deal with the case $a=b=\boxplus$, we use the same argument but replace $2\mathfrak{v}1$ with $2\mathfrak{v}_3$ throughout; to deal with $\{a,b\}=\{\boxminus,\boxplus\}$ we replace $2\mathfrak{v}_1$ with $\mathfrak{v}_2$.

The difference between $B_{ab,s}$ and the required $f_{ab,s}$ is precisely that peripheral edges count towards the former, but not the latter.

Let us look first at $f_{\boxminus\boxminus,s}$. Here, peripheral edges count twice towards $B_{\boxminus\boxminus,s}$ (once for each orientation) and they are necessarily peripheral edges of paths either in $\mathcal{Q}_{\boxminus\boxminus}\cap\mathsf{SpecPaths}^*_s$, or in in $\mathcal{Q}_{\boxminus\boxplus}\cap\mathsf{SpecPaths}^*_s$. The former is easy to deal with: by ($Quasi\,2$) and (77) we have $\big|\mathcal{Q}_{\boxminus\boxminus}\cap\mathsf{SpecPaths}^*_s\big|=\frac{1}{4}(1\pm\gamma_{\mathbf{D}})(\lambda n\pm n^{0.6})$, and consulting Tables 2 and 3 we see that in all cases, in each such path, exactly one of the two peripheral edges is assigned by ξ within $V_\boxminus$. Thus the peripheral edge contribution from these paths to $B_{\boxminus\boxminus,s}$ is $\frac{1}{2}(1\pm\gamma_{\mathbf{D}})(\lambda n\pm n^{0.6})$. For a path $P\in\mathcal{Q}_{\boxminus\boxplus}$, on the other hand, only one of the two peripheral edges can possibly lie in $V_\boxminus$. From Tables 6 and 7, we see that whether it does so depends on whether $\pi_{\boxminus\boxplus}(P)$ is in $X^\blacktriangle$ or $X^\triangledown$. These two cases are equally likely and chosen independently for all paths in $\mathcal{Q}_{\boxminus\boxplus}$. We have by ($Quasi\,2$) and (77) that $\big|\mathcal{Q}_{\boxminus\boxplus}\cap\mathsf{SpecPaths}^*_s\big|=\frac{1}{2}(1\pm\gamma_{\mathbf{D}})(\lambda n\pm n^{0.6})$, so by the Chernoff bound, with probability at least $1-\exp(-\sqrt{n})$ the contribution from these paths to $B_{\boxminus\boxminus,s}$ is

$\frac{1}{2}(1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6}) \pm n^{0.9}$. Putting this together, we have

$$f_{\boxminus\boxminus,s} = B_{\boxminus\boxminus,s} - (1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6}) \pm n^{0.9}$$
$$\overset{(81)}{=} (1 \pm 5\gamma_{\mathbf{D}})\Big(\frac{B_{\boxminus\boxminus}}{(\lambda - \sigma_0 - \sigma_1)n} - \lambda n\Big)\,.$$

This proves the Claim for $f_{\boxminus\boxminus,s}$. The same argument, replacing $\boxminus\boxminus$ with $\boxplus\boxplus$ and looking at Tables 4 and 5, proves the Claim for $f_{\boxplus\boxplus,s}$.

What remains is to prove the Claim for $f_{\boxminus\boxplus,s} = f_{\boxplus\boxminus,s}$. Note that peripheral edges count only once towards $B_{\boxminus\boxplus,s}$. By exactly the same argument as above, the number of peripheral edges from paths of $\mathcal{Q}_{\boxminus\boxminus} \cap \mathsf{SpecPaths}^*_s$ which contribute to $B_{\boxminus\boxplus,s}$ is $\frac{1}{4}(1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6})$, and we obtain the same bound for paths of $\mathcal{Q}_{\boxplus\boxplus} \cap \mathsf{SpecPaths}^*_s$. For paths P of $\mathcal{Q}_{\boxminus\boxplus} \cap \mathsf{SpecPaths}^*_s$, of which there are as above $\frac{1}{2}(1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6})$, from Tables 6 and 7 we see that either both or neither peripheral edges contribute to $B_{\boxminus\boxplus,s}$, according to whether $\pi_{\boxminus\boxplus}(P)$ is in $X^{\blacktriangle}$ or $X^{\triangledown}$. Much as above, we see that with probability at least $1 - \exp(-\sqrt{n})$ the contribution from these paths to $B_{\boxminus\boxplus,s}$ is $\frac{1}{2}(1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6}) \pm 2n^{0.9}$, and summing we get

$$f_{\boxminus\boxplus,s} = B_{\boxminus\boxplus,s} - (1 \pm \gamma_{\mathbf{D}})(\lambda n \pm n^{0.6}) \pm 2n^{0.9}$$
$$\overset{(81)}{=} (1 \pm 5\gamma_{\mathbf{D}})\Big(\frac{B_{\boxminus\boxplus}}{(\lambda - \sigma_0 - \sigma_1)n} - \lambda n\Big)\,,$$

as required. □

As in Definition 41 suppose that we are given $a, b \in \{\boxminus, \boxplus\}$ and an edge $uv \in E(H_{\mathbf{D}})$, with $u \in V_a$ and $v \in V_b$, we claim that with probability at least $1 - 3\exp(-\sqrt{n})$ we have that

$$\sum_{s \in \mathcal{J}_2} w_{uv;s} = (1 \pm 3\gamma_{\mathbf{D}}) \frac{\sum_{xy \in E^*} \mathbb{1}_{\xi(x)=V_a, \xi(y)=V_b}}{|V_a||V_b|} \pm 3\gamma_{\mathbf{D}} n^{-0.01}\,, \tag{84}$$

where the quantity $w_{uv;s}$ is as defined in Definition 41. Assuming we have this, then by the union bound over all possible edges $uv \in E(H_{\mathbf{D}})$ we obtain that with high probability, ξ has the $5\gamma_{\mathbf{D}}$-pair distribution property. So we need to prove (84) holds with the claimed probability. We split the sum into the contribution coming from (23b) and (23c) (i.e. respectively u or v is an anchor of some path in $\mathsf{SpecPaths}^*_s$, and the other is not in $\operatorname{im}\phi^{\mathbf{D}}_s$) and that from (23a) ($u, v \notin \operatorname{im}\phi^{\mathbf{D}}_s$). These cases are not exhaustive, but for $s \in \mathcal{J}_2$ where none of the three cases occurs, we have $w_{uv;s} = 0$.

We first deal with the case (23a). The numerator of $w_{uv;s}$ in this case (23a) counts (with multiplicity) the number of pairs which are internal edges of $\mathsf{SpecPaths}^*_s$ with one end assigned in V_a and the other in V_b, in other words it is $f_{ab,s}$. The denominator is by $(Quasi\,1)$ (with $S_1 = S_2 = T_3 = \emptyset$ and one of T_1 and T_2 being $\{s\}$ and the other $\emptyset$) and (76) equal to $(1 \pm$

$\gamma_{\mathbf{D}})^2\frac{1}{4}\big((\delta+6\lambda)n\pm n^{0.6}\big)^2$, and therefore

$$w_{uv;s} = \frac{f_{ab,s}}{(1\pm\gamma_{\mathbf{D}})^2\frac{1}{4}\big((\delta+6\lambda)n\pm n^{0.6}\big)^2} \overset{\text{Claim 58.3}}{=} \frac{(1\pm 5\gamma_{\mathbf{D}})\big(\frac{B_{ab}}{(\lambda-\sigma_0-\sigma_1)n}-\lambda n\big)}{(1\pm\gamma_{\mathbf{D}})^2\frac{1}{4}\big((\delta+6\lambda)n\pm n^{0.6}\big)^2}$$
$$= 4(1\pm 8\gamma_{\mathbf{D}})\frac{B_{ab}-\lambda(\lambda-\sigma_0-\sigma_1)n^2}{(\delta+6\lambda)^2(\lambda-\sigma_0-\sigma_1)n^3}\,.$$

Thus we simply need to know the number of $s\in\mathcal{J}_2$ such that $u,v\notin\operatorname{im}\phi_s^{\mathrm{D}}$. By ($Quasi\,6$) (with $S_1=\{u,v\}$ and $S_2=\emptyset$) this number is

$$(1\pm\gamma_{\mathbf{D}})|\mathcal{J}_2|\big(\delta+6\lambda\pm n^{-0.4}\big)^2 = (1\pm 2\gamma_{\mathbf{D}})(\lambda-\sigma_0-\sigma_1)n(\delta+6\lambda)^2\,,$$

and so we get

$$\sum_{\substack{s\in\mathcal{J}_2\\ u,v\notin\operatorname{im}\phi_s^{\mathbf{D}}}} w_{uv;s} = 4(1+11\gamma_{\mathbf{D}})\big(B_{ab}-\lambda(\lambda-\sigma_0-\sigma_1)n^2\big)n^{-2}\,. \tag{85}$$

We now deal with the case (23b). The denominator of (23b) is $(1\pm 2\gamma_{\mathbf{D}})\frac{1}{2}(\delta+6\lambda)n$ by ($Quasi\,1$) as above, so we have

$$w_{uv;s} = 2(1\pm 3\gamma_{\mathbf{D}})(\delta+6\lambda)^{-1}n^{-1}\,.$$

As before, what we need to do is estimate the number of $s\in\mathcal{J}_2$ such that $u=\phi_s^{\mathbf{D}}(x)$ is an anchor of some path in $\mathsf{SpecPaths}_s^*$ whose first two vertices are xy, and $v\notin\operatorname{im}\phi_s^{\mathbf{D}}$, and $\xi(y)=V_b$. To do this, first we define

$$S_{uv} := \{s\in\mathcal{J}_2 \;:\; \phi_s^{-1}(u)\in \text{endvertices of } \mathsf{SpecPaths}_s^*, v\notin\operatorname{im}\phi_s^{\mathbf{D}}\}\,.$$

What we want, then, is to estimate the number of $s\in S_{uv}$ such that $\xi(y)=V_b$, where y is the neighbour of $\big(\phi^{\mathbf{D}}\big)^{-1}(u)$ in $\mathsf{SpecPaths}_s^*$. To begin with, substituting (76), (77) and (3) into ($Quasi\,8$), we have

$$|S_{uv}| = (1\pm\gamma_{\mathbf{D}})|\mathcal{J}_2|\cdot\frac{(2\lambda n\pm 2n^{0.6})((\delta+6\lambda)n\pm 6n^{0.6})}{n^2}$$
$$= (1\pm 2\gamma_{\mathbf{D}})\cdot 2\lambda(\delta+6\lambda)(\lambda-\sigma_0-\sigma_1)n\,.$$

Now, let $s\in S_{uv}$ be arbitrary. Let $x\in V(G_s)$ be the anchor that is mapped on u, and let y be its neighbour in the path $P\in\mathsf{SpecPaths}_s^*$ containing x. Whatever $\aleph\big(\pi(P)\big)$ is, observe from Tables 2–7 that $\xi(y)=V_b$ is true for exactly one of $\pi(P)\in X^{\blacktriangle}$ and $\pi(P)\in X^{\triangledown}$. In other words, when the random map π is fixed, the event $\xi(y)=V_b$ holds with probability $\frac{1}{2}$, and these events are independent for the various $s\in S_{uv}$. Thus with probability at least $1-\exp(-\sqrt{n})$ the number of indices $s\in\mathcal{J}_2$ counted in (23b) is $\frac{1}{2}|S_{\mathrm{b}}|\pm n^{0.9}$. Under this assumption, we get

$$\sum_{s\in\mathcal{J}_2:\text{(23b) applies}} w_{uv;s} = (\tfrac{1}{2}|S_{uv}|\pm n^{0.9})\cdot 2(1\pm 3\gamma_{\mathbf{D}})(\delta+6\lambda)^{-1}n^{-1} \tag{86}$$
$$= (1\pm 6\gamma_{\mathbf{D}})\cdot 2\lambda(\lambda-\sigma_0-\sigma_1)\,, \tag{87}$$

and similarly we obtain that with probability at least $1-\exp(-\sqrt{n})$,

$$\sum_{s\in\mathcal{J}_2:\text{(23c) applies}} w_{uv;s} = (1\pm 6\gamma_{\mathbf{D}})\cdot 2\lambda(\lambda-\sigma_0-\sigma_1)\,. \tag{88}$$

Summing up (85), (86) and (88), we see that

$$\sum_{s\in\mathcal{J}_2} w_{uv;s} = 4(1\pm 20\gamma_{\mathbf{D}})\frac{B_{ab}}{n^2}\,.$$

Taking the union bound over the various choices, we see that this holds for all u and v, giving the desired $20\gamma_{\mathbf{D}}$-pair distribution property. □

By combining Lemma 58 with Lemma 42 we can prove Lemma 36. This proof is broadly similar to the proof for Stage D, Lemma 34.

Proof of Lemma 36. We set $\nu = \frac{1}{1000}2^{-4L_{\mathbf{E}}}\delta^{4L_{\mathbf{E}}}(\lambda)^{4L_{\mathbf{E}}}$. Let C' be such that $\frac{1}{20(L_{\mathbf{D}}+2)}\cdot 2^{-10L_{\mathbf{E}}}C'$ is returned by Lemma 42 for input ν and $L_{\mathbf{D}}$. Note that $C'\gamma_{\mathbf{D}}\le\gamma_{\mathbf{E}}$. Let $H=H_{\mathbf{D}}$, which has $2\lfloor\frac{n}{2}\rfloor$ vertices. From Lemma 58 we obtain a map ξ.

We now translate our setting into the setup for Lemma 42 with ν and $L=L_{\mathbf{D}}$, and with $\gamma=20(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}}$. We let $H=H_{\mathbf{D}}$ with the partition $V_{\boxminus}\dot\cup V_{\boxplus}$. Temporarily abusing notation, let $s^*:=|\mathcal{J}_2|\ge\nu n$ and suppose $\mathcal{J}_2=[s^*]$. Given $s\in\mathcal{J}_2$ we obtain a path-forest $F_s=\mathsf{SpecPaths}^*_s$, whose leaves A_s are anchored by the embedding $\phi_s:=\phi^{\mathbf{D}}_s\big|_{A_s}$, with used set $U_s:=\operatorname{im}\phi^{\mathbf{D}}_s$. Recall that each path in F_s has eight vertices, which is less than $L_{\mathbf{D}}$. The required assignment of sides is provided by ξ_s defined by the restriction of ξ (as defined in Lemma 58) to $\mathsf{SpecPaths}^*_s$. We let $U_s=\operatorname{im}^{\mathbf{D}}(s)$.

We next verify the conditions of Lemma 42. Recall that by $(Quasi1)$ the graph $H_{\mathbf{D}}$ with partition $(V_{\boxminus},V_{\boxplus})$ is $(\gamma_{\mathbf{D}},L_{\mathbf{D}})$-index-quasirandom with respect to $\big(\operatorname{im}^{\mathbf{D}}(s)\big)_{s\in\mathcal{J}}$ and $\big(I_s\big)_{s\in\mathcal{J}_0}$, with with $I_s:=\{i\in[\frac{n}{2}]:\boxminus_i,\boxplus_i\in A_s\}$. By Lemma 29, we conclude that $H_{\mathbf{D}}$ is $((L_{\mathbf{D}}+2)\gamma_{\mathbf{D}},L_{\mathbf{D}})$-block-quasirandom and $((L_{\mathbf{D}}+2)\gamma_{\mathbf{D}},L_{\mathbf{D}})$-block-diet with respect to each set U_s for $s\in\mathcal{J}_2$. By choice of γ we have $\gamma\ge(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}}$.

By Lemma 58(v), for each $s\in\mathcal{J}_2$ the forest F_s with anchors $\phi^{\mathbf{D}}_s|_{A_s}$ has the $6\gamma_{\mathbf{D}}$-anchor distribution property with respect to $H_{\mathbf{D}}$, and by Lemma 58(vi) the collection of forests with anchors and assignment ξ has the $20\gamma_{\mathbf{D}}$-pair distribution property. By choice of γ, in particular we have the γ-anchor distribution and γ-pair distribution properties. Furthermore, (ii) gives us $|\xi_s^{-1}(\{V_a\})\setminus A_s|\ge\frac14\lambda n\ge\nu n$ for each $a\in\{\boxminus,\boxplus\}$ and $s\in\mathcal{J}_2$, and (iii) gives us

$$\big|\{(x,y)\in E(F_s):x\in\xi_s^{-1}(\{V_a\})\setminus A_s, y\in\xi_s^{-1}(\{V_b\})\setminus A_s\}\big|\ge\tfrac{1}{16}\lambda n\ge\nu n$$

for each $a,b\in\{\boxminus,\boxplus\}$ and $s\in\mathcal{J}_2$.

For each $s\in\mathcal{J}_2$, by $(Quasi1)$ we have, for each $a\in\{\boxminus,\boxplus\}$, $|V_a\setminus U_s|=(1\pm\gamma_{\mathbf{D}})\cdot\frac12\big(n-|U_s|\big)\ge\frac14\delta n+6\lambda n$, where we use the fact that $\lambda\ll\delta$. Since $|\mathsf{SpecPaths}^*_s|\le\lambda n$ and each path has 6 interior vertices, even if all the vertices of the paths $\mathsf{SpecPaths}^*_s$ were assigned to V_a we obtain $|V_a\setminus U_s|-\big|\{x\in V(F_s)\setminus A_s:\xi_s(x)=V_a\}\big|\ge\frac14\delta n\ge\nu n$.

We define $d'_{\boxminus\boxminus}, d'_{\boxminus\boxplus}, d'_{\boxplus\boxplus}$ to be the densities obtained by a successful packing of the F_s according to ξ_s. By Lemma 58, we have $d'_{\boxminus\boxminus}, d'_{\boxminus\boxplus}, d'_{\boxplus\boxplus} \geq \nu$.

This is the setup for Lemma 42, and the conclusion of that lemma is that *PathPacking* w.h.p. succeeds in packing each F_s as specified. That is, we obtain for each $s \in \mathcal{J}_2$ an embedding ϕ'_s of F_s into $H_{\mathbf{D}}\big[(V(H)\setminus U_s)\cup\phi_s(A_s)\big]$ which extends ϕ_s, such that $\phi'_s(x) \in \xi_s(x)$ for each $x \in V(F_s)$, and such that for each $uv \in E(H)$ there is at most one s such that ϕ'_s uses the edge uv. Suppose that this likely event occurs, and as before we will list polynomially many further events which we presume all occur, and fix such an outcome. Note that the existence of a packing according to ξ, by Lemma 58 (iv), in particular gives us the required edge counts (18).

We now define the setup after stage E and check it satisfies the conditions of Lemma 36. This is much the same as for Stage D. We let $H_{\mathbf{E}}$ be the graph H' returned by Lemma 42. For $s \in \mathcal{J}_0 \cup \mathcal{J}_1$, we set $\phi^{\mathbf{E}}_s = \phi^{\mathbf{D}}_s$, with the same anchor and used sets and the same path-forests to embed. Recall that the conclusion of Lemma 36 has paths indexed by $\mathcal{J}_0 \cup \mathcal{J}_1 \cup \emptyset$. We now argue that this is a $(\gamma_{\mathbf{E}}, L_{\mathbf{E}}, d^*_{\mathbf{E}}, \bar{d_{\mathbf{E}}})$-quasirandom setup, as required for Lemma 36.

For $(Quasi1)$, fix S_1, S_2, T_1, T_2, T_3 as in Definition 28, each of size at most $L_{\mathbf{E}}$, with families of sets $(U^{\mathbf{E}}_s)_{s\in\mathcal{J}_0\cup\mathcal{J}_1}$ and $(J_s)_{s\in\mathcal{J}_0}$, and let $X = \mathbb{U}_{H_{\mathbf{D}}}(S_1, S_2, T_1, T_2, T_3)$. Then we have $|X| \geq \nu n$, and so by $(PP5)$ it is likely that we have

$$\big|X \cap \mathbb{U}_{H_{\mathbf{E}}}(S_1, S_2, \emptyset, \emptyset)\big| = (1 \pm C'(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}})\big(\tfrac{d^*_{\mathbf{E}}}{d_{\mathbf{D}}}\big)^{|S_1\cap V_\boxminus|+|S_2\cap V_\boxplus|}\big(\tfrac{\bar{d_{\mathbf{E}}}}{d_{\mathbf{D}}}\big)^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}\,.$$

Plugging in the size $|X|$ given by $(Quasi1)$ after stage D, and observing that the above set is precisely $\mathbb{U}_{H_{\mathbf{E}}}(S_1, S_2, T_1, T_2, T_3)$ with families of sets $(U^{\mathbf{E}}_s)_{s\in\mathcal{J}_0\cup\mathcal{J}_1}$ and $(J_s)_{s\in\mathcal{J}_0}$, this is what is required for $(L_{\mathbf{E}}, \gamma_{\mathbf{E}}, d^*_{\mathbf{E}}, \bar{d_{\mathbf{E}}})$-index-quasirandomness.

For $(Quasi2)$, observe that for $\mathcal{J}_1$ nothing has changed from after stage D, and we no longer consider $\mathcal{J}_2$. Similarly, for $(Quasi3)$ nothing has changed from after stage D for $\mathcal{J}_1$. For $(Quasi4)$, it suffices by construction to establish $(Quasi5)$ as in stage D.

For $(Quasi5)$, fix $s \in \mathcal{J}_0 \cup \mathcal{J}_1$, $a \in \{\boxminus, \boxplus\}$ and $u \in V(H_{\mathbf{E}}) \setminus U^{\mathbf{E}}_s$. By $(Quasi5)$ after stage D, writing A^a_s for the set of anchors $v \in A^{\mathbf{E}}_s$ such that the path end not anchored to v is anchored in V_a, we have

$$\begin{aligned}\big|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{D}}}(u) \cap A^a_s\}\big| &= (1\pm\gamma_{\mathbf{D}})d_{\mathbf{D}}|A^a_s \cap V_\boxminus| \quad \text{and}\\ \big|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{D}}}(u) \cap A^a_s\}\big| &= (1\pm\gamma_{\mathbf{D}})d_{\mathbf{D}}|A^a_s \cap V_\boxplus|\,.\end{aligned}$$

We define a weight function w on $V(H_{\mathbf{D}})$ by setting $w(v) = 1$ if $v \in A^a_s$, and otherwise $w(v) = 0$. Note that $\sum_{v\in V_b} w(v)$ is either equal to zero or at least νn by $(Quasi2)$ and by construction. Thus by $(PP4)$, if $u \in V_\boxminus$ it is likely that we have

$$\begin{aligned}\big|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{E}}}(u) \cap A^a_s\}\big| &= (1\pm\gamma_{\mathbf{D}})d_{\mathbf{D}}|A^a_s \cap V_\boxminus|(1 \pm C'(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}})\tfrac{d^*_{\mathbf{E}}}{d_{\mathbf{D}}} \quad \text{and}\\ \big|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{E}}}(u) \cap A^a_s\}\big| &= (1\pm\gamma_{\mathbf{D}})d_{\mathbf{D}}|A^a_s \cap V_\boxplus|(1 \pm C'(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}})\tfrac{\bar{d_{\mathbf{E}}}}{d_{\mathbf{D}}}\,.\end{aligned}$$

Similarly if $u \in V_{\boxplus}$ it is likely that we have

$$\left|\{\boxminus_i \in \mathsf{N}_{H_{\mathbf{E}}}(u) \cap A_s^a\}\right| = (1 \pm \gamma_{\mathbf{D}}) d_{\mathbf{D}} |A_s^a \cap V_{\boxminus}| (1 \pm C'(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}}) \tfrac{\bar{d}_{\mathbf{E}}}{d_{\mathbf{D}}} \quad \text{and}$$
$$\left|\{\boxplus_i \in \mathsf{N}_{H_{\mathbf{E}}}(u) \cap A_s^a\}\right| = (1 \pm \gamma_{\mathbf{D}}) d_{\mathbf{D}} |A_s^a \cap V_{\boxplus}| (1 \pm C'(L_{\mathbf{D}}+2)\gamma_{\mathbf{D}}) \tfrac{d^*_{\mathbf{E}}}{d_{\mathbf{D}}} \,.$$

This is as required for $(Quasi\,5)$.

For $(Quasi\,6)$, observe that nothing has changed for $\mathcal{J}_0 \cup \mathcal{J}_1$ and that we no longer consider $\mathcal{J}_2$.

For $(Quasi\,7)$, observe that nothing has changed since after stage D for $\mathcal{J}_0$ and $\mathcal{J}_1$, while we no longer consider $\mathcal{J}_2$. The same holds for $(Quasi\,8)$. □

13. Stage F (Proof of Lemma 37)

In this stage we embed the paths $\{\mathsf{SpecPaths}^*_s, s \in \mathcal{J}_1\}$ to precisely adjust the degrees condition of the vertices. We need to obtain (20), that is

$$\deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxplus}) = \deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxminus}) + t(\boxminus_i) \;\text{ and }\; \deg_{H_{\mathbf{F}}}(\boxplus_i, V_{\boxminus}) = \deg_{H_{\mathbf{F}}}(\boxplus_i, V_{\boxplus}) + t(\boxplus_i)$$

for each $i \in [\lfloor n/2 \rfloor]$.

For each $s \notin \mathcal{J}_1$ we set $\phi_s^{\mathbf{F}} := \phi_s^{\mathbf{E}}$. We will construct embeddings $\phi_s^{\mathbf{F}}$ extending $\phi_s^{\mathbf{E}}$ for each $s \in \mathcal{J}_1$ that do not use any edge of H multiple times and give us (20) for each i. First, we argue that any such maps give us the required quasirandom setup.

We begin with $(Quasi\,1)$. For an arbitrary set $T \subseteq \mathcal{J}_0$ we have $\bigcup_{s\in T} \mathrm{im}^{\mathbf{F}}(s) = \bigcup_{s \in T} \mathrm{im}^{\mathbf{E}}(s)$. As we are extending the mappings only on paths from $\{\mathsf{SpecPaths}^*_s, s \in \mathcal{J}_1\}$, and that for each $s \in \mathcal{J}_1$ the mapping $\phi_s^{\mathbf{F}}$ is an embedding, observe that for each $v \in V(H)$, we have

$$|\mathsf{N}_{H_{\mathbf{E}}}(v) \setminus \mathsf{N}_{H_{\mathbf{F}}}(v)| \le 2|\mathcal{J}_1| = 2\sigma_1 n \,. \tag{89}$$

Also observe that the densities in the graph $H_{\mathbf{E}}$ and in the graph $H_{\mathbf{F}}$ are nearly identical. Indeed, a very rough estimate, gives us that $d^*_{\mathbf{E}} - d^*_{\mathbf{F}} \le 100\sigma_1^2$, and $\bar{d}_{\mathbf{E}} - \bar{d}_{\mathbf{F}} \le 100\sigma_1^2$.

Hence, for sets $S \subseteq V(H_{\mathbf{E}})$ and $T \subseteq \mathcal{J}_0$, we have

$$\left|\mathsf{N}_{H_{\mathbf{F}}}(S) \setminus \bigcup_{s\in T} \mathrm{im}^{\mathbf{F}}(s)\right| \overset{(89)}{=} \left|\mathsf{N}_{H_{\mathbf{E}}}(S) \setminus \bigcup_{s\in T} \mathrm{im}^{\mathbf{E}}(s)\right| \pm |S| \cdot 2\sigma_1 n \,. \tag{90}$$

Assume that we are given sets $S_1, S_2 \subseteq V_{\boxminus} \cup V_{\boxplus}$ and pairwise disjoint sets $T_1, T_2, T_3 \subseteq \mathcal{J}_0$ with $|S_i| \le L_{\mathbf{E}}$ and $|T_j| \le L_{\mathbf{E}}$ for $i \in [2]$ and $j \in [3]$. Combining (90) with the fact that $(H_{\mathbf{E}}, V_{\boxminus}, V_{\boxplus})$ is $(L_{\mathbf{E}}, \gamma_{\mathbf{E}}, d^*_{\mathbf{E}}, \bar{d}_{\mathbf{E}})$-index-quasirandom with respect to the collections $(\mathrm{im}^{\mathbf{E}}(s))_{s\in\mathcal{J}_0\cup\mathcal{J}_1}$ and $(I_s)_{s\in\mathcal{J}_0}$, with $I_s := \{i \in [\frac{n}{2}] : \boxminus_i, \boxplus_i \in A_s\}$ (see ($Quasi\,1$)), we obtain

$$\begin{aligned}|\mathbb{U}_{H_{\mathbf{F}}}(S_1,S_2,T_1,T_2,T_3)| &= |\mathbb{U}_{H_{\mathbf{E}}}(S_1,S_2,T_1,T_2,T_3)| \pm (|S_1|+|S_2|)\cdot 2\sigma_1 n \\ &\overset{\text{D28}}{=} (1\pm\gamma_{\mathbf{E}})(d^*_{\mathbf{E}})^{|S_1\cap V_\boxminus|+|S_2\cap V_\boxplus|}\cdot(\bar{d}_{\mathbf{E}})^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|} \\ &\qquad\cdot \prod_{s\in T_1\cup T_2}\left(1-\frac{\mathrm{im}^{\mathbf{E}}(s)}{n}\right)\cdot(2\lambda^*)^{|T_3|}\cdot\frac{n}{2}\pm(|S_1|+|S_2|)2\sigma_1 n \\ &= (1\pm\gamma_{\mathbf{F}})(d^*_{\mathbf{F}})^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}(\bar{d}_{\mathbf{F}})^{|S_1\cap V_\boxplus|+|S_2\cap V_\boxminus|}\cdot(\delta+2\lambda^*)^{|T_1|+|T_2|}\cdot(2\lambda^*)^{|T_3|}\cdot\frac{n}{2}\,,\end{aligned}$$

proving that $(H_{\mathbf{F}}, V_\boxminus, V_\boxplus)$ is $(L_{\mathbf{E}}, \gamma_{\mathbf{F}}, d_{\mathbf{F}^*}, \bar{d}_{\mathbf{F}})$-index-quasirandom with respect to the collections $(\mathrm{im}^{\mathbf{F}}(s))_{s\in\mathcal{J}_0}$, and $(I_s)_{s\in\mathcal{J}_0}$, with $I_s := \{i \in [\frac{n}{2}] : \boxminus_i, \boxplus_i \in A_s\}$, as required for ($Quasi\,1$).

For ($Quasi\,2$), ($Quasi\,3$), ($Quasi\,4$), ($Quasi\,6$), ($Quasi\,7$) and ($Quasi\,8$), either nothing has changed from Stage E, or the condition is vacuous since we ask for a quasirandom setup with index sets $\mathcal{J}_0, \emptyset, \emptyset$. It remains to verify ($Quasi\,5$), which follows from the above calculation and the observation that for a given u and v, the change in any of (12)–(15) in going from Stage E to Stage F is at most $\deg_{H_{\mathbf{E}}}(u) - \deg_{H_{\mathbf{F}}}(u)$ or $\deg_{H_{\mathbf{E}}}(v) - \deg_{H_{\mathbf{F}}}(v)$ respectively. By the above calculation, this change is absorbed in the larger error term $\gamma_{\mathbf{F}}$. This completes the proof that we obtain the required quasirandom setup for Lemma 37.

We now construct the $\phi_s^{\mathbf{F}}$ for $s \in \mathcal{J}_1$. Let $\mathcal{P}_0 := \bigcup_{s\in\mathcal{J}_1} \mathsf{SpecPaths}^*_s$. Let $v \in V_\boxminus \cup V_\boxplus$. Let $\{V_v, \bar{V}_v\} = \{V_\boxminus, V_\boxplus\}$ be such that $v \in V_v$ and $v \notin \bar{V}_v$. Embedding each path of $\mathcal{P}_0$ anchored at v will decrease the degree of v by 1; all other paths will decrease the degree by either 2 or 0, depending on whether we use v in the embedding or not. It turns out to be convenient to use the following rule: if a path of $\mathcal{P}_0$ is anchored at v and its other endpoint is also anchored in V_v, then we embed the edge from v in $E(V_\boxminus, V_\boxplus)$, while if the other endpoint is anchored in $\bar{V}_v$, then we embed the edge from v within V_v. To keep track of how many edges this uses, for a given set of paths $\mathcal{P}^* \subseteq \mathcal{P}_0$ we define $j(v, \mathcal{P}^*)$ being the number of paths $P \in \mathcal{P}^*$ anchored at v with its second anchor in V_v and $\bar{j}(v, \mathcal{P}^*)$ being the number of paths $P \in \mathcal{P}^*$ anchored at v with its second anchor in $\bar{V}_v$.

To satisfy (20), by packing $\mathcal{P}_0$ we need to achieve that $\deg_{H_{\mathbf{F}}}(v, V_v) - t(v) = \deg_{H_{\mathbf{F}}}(v, \bar{V}_v)$. We denote the *discrepancy degree* (defined with respect to a host graph $H^* \subseteq H$ and a set $\mathcal{P}^* \subseteq \mathcal{P}_0$) of a vertex $v \in V_\boxminus \cup V_\boxplus$ by

$$d_{H^*,\mathcal{P}^*}(v) = \deg_{H^*}(v, V_v) - t(v) - \deg_{H^*}(v, \bar{V}_v) + a(v, \mathcal{P}^*)\,,$$

where $a(v, \mathcal{P}^*) = j(v, \mathcal{P}^*) - \bar{j}(v, \mathcal{P}^*)$. Observe that by ($Quasi\,3$) we have

$$j(v, \mathcal{P}_0) = (1 \pm \gamma_{\mathbf{D}})\sigma_1^2 n \qquad \text{and} \qquad \bar{j}(v, \mathcal{P}_0) = (1 \pm \gamma_{\mathbf{D}})\sigma_1^2 n\,. \tag{91}$$

Hence, our ultimate goal is to obtain that $d_{H_{\mathbf{F}},\emptyset}(v) = 0$ for all $v \in V(H_{\mathbf{F}})$.

The two following claims state that the discrepancy degree is zero on average and does not deviate much from this value. We will then embed paths one by one, preserving the condition that the average discrepancy degree is zero.

Claim 58.4. *We have*
$$\sum_{v\in V_\boxminus} d_{H_\mathbf{E},\mathcal{P}_0}(v)=\sum_{v\in V_\boxplus} d_{H_\mathbf{E},\mathcal{P}_0}(v)=0\ .$$

Proof. We prove that the average of the discrepancy degree of vertices from $V_\boxminus$ is 0. The average of the discrepancy degree for $V_\boxplus$ is done analogously.
$$\begin{aligned}\sum_{v\in V_\boxminus} d_{H_\mathbf{E},\mathcal{P}_0}(v)&=\sum_{v\in V_\boxminus}\big(\deg_{H_\mathbf{E}}(v,V_\boxminus)-t(v)+\deg_{H_\mathbf{E}}(v,V_\boxplus)+a(v,\mathcal{P}_0)\big)\\
&=2e_{H_\mathbf{E}}(V_\boxminus)-e_{H_\mathbf{E}}(V_\boxminus,V_\boxplus)-\sum_{v\in V_\boxminus}t(v)+\sum_{v\in V_\boxminus}(j(v,\mathcal{P}_0)-\bar{j}(v,\mathcal{P}_0))\\
&\overset{\text{L36}}{=}2(\sigma_0\lambda^*n^2+2j_{\boxminus\boxminus}+3(j_{\boxminus\boxplus}+j_{\boxplus\boxplus}))-(\sigma_0\lambda^*n^2+6(j_{\boxminus\boxminus}+j_{\boxplus\boxplus})+5j_{\boxminus\boxplus})-\sigma_0\lambda^*n^2\\
&+(2j_{\boxminus\boxminus}-j_{\boxminus\boxplus})\\
&=0\ .\end{aligned}$$
□

Claim 58.5. *For each $v\in V_\boxminus\cup V_\boxplus$ we have $|d_{H_\mathbf{E},\mathcal{P}_0}(v)|\le\gamma_\mathbf{D}n$.*

Proof. Fix an arbitrary $v\in V_\boxminus\cup V_\boxplus$. By definition of $d_{H_\mathbf{E},\mathcal{P}_0}(\cdot)$, we have
$$|d_{H_\mathbf{E},\mathcal{P}_0}(v)|=\big|\deg_{H_\mathbf{E}}(v,V_v)-t(v)-\deg_{H_\mathbf{E}}(v,\bar{V}_v)+j(v,\mathcal{P}_0)-\bar{j}(v,\mathcal{P}_0)\big|\ .$$
Now, after stage E we have a $(\gamma_\mathbf{E},L_\mathbf{E},d^*_\mathbf{E},\bar{d_\mathbf{E}})$-quasirandom setup. By (*Quasi* 7), we have $j(v,\mathcal{P}_0)-\bar{j}(v,\mathcal{P}_0)=\pm3\gamma_\mathbf{E}\cdot2\sigma_1^2n$. Also by (*Quasi* 7) we have $t(v)=(1\pm\gamma_\mathbf{E})\cdot2\lambda^*\sigma_0n$. Finally by (*Quasi* 1), we have $\deg_{H_\mathbf{E}}(v,V_v)=(1\pm\gamma_\mathbf{E})d^*_\mathbf{E}n/2$ and $\deg_{H_\mathbf{E}}(v,\bar{V}_v)=(1\pm\gamma_\mathbf{E})\bar{d_\mathbf{E}}n/2$. Plugging this in, and using the definitions of $d^*_\mathbf{E}$ and $\bar{d_\mathbf{E}}$, we get
$$\begin{aligned}&|d_{H_\mathbf{E},\mathcal{P}_0}(v)|=\big|(1\pm\gamma_\mathbf{E})d^*_\mathbf{E}\tfrac{n}{2}-(1\pm\gamma_\mathbf{E})2\lambda^*\sigma_0n-(1\pm\gamma_\mathbf{E})\bar{d_\mathbf{E}}\tfrac{n}{2}\pm3\gamma_\mathbf{E}\cdot2\sigma_1^2n\big|\\
=&\big|(1\pm\gamma_\mathbf{E})\big(8(\lambda^*\sigma_0+\tfrac{11}{4}\sigma_1^2)\big)\tfrac{n}{2}-(1\pm\gamma_\mathbf{E})2\lambda^*\sigma_0n-(1\pm\gamma_\mathbf{E})\big(4(\lambda^*\sigma_0+\tfrac{11}{2}\sigma_1^2)\big)\tfrac{n}{2}\pm3\gamma_\mathbf{E}\cdot2\sigma_1^2n\big|\\
\le&\gamma_\mathbf{E}n\ .\end{aligned}$$
□

The mappings $(\phi_s^\mathbf{F})_{s\in\mathcal{J}_1}$ will be defined in roughly $\sigma_1^2n^2$ steps, where in each step i we define a mapping ϕ_i of one path $P_i\in\{\mathsf{SpecPaths}^*_s,s\in\mathcal{J}_1\}$. The reason for 'roughly' is that in Stage C we packed some (between 0 and $n^{0.6}$) paths from each $\mathsf{SpecPaths}_s$ with $s\in\mathcal{J}_1$. Let $m:=\sum_{s\in\mathcal{J}_1}|\mathsf{SpecPaths}^*_s|$ denote the precise number of paths we need to pack in this stage; then we have $\frac{1}{2}\sigma_1^2n^2\le m\le\sigma_1^2n^2$.

Denote by $\mathcal{P}_i$ the set of paths from $\{\mathsf{SpecPaths}^*_s,s\in\mathcal{J}_1\}$ that were not yet packed after step i, in particular $\mathcal{P}_m=\emptyset$. Set $\mathrm{im}_0(s)=\mathrm{im}^\mathbf{E}(s)$ be the image of the partial embedding of G_s at the beginning of Stage F. Let us assume that we are in step $i\in[m]$ and are about to embed a path $P_i\in\mathcal{P}_i$. For $s\in\mathcal{J}_1$, denote by $\mathrm{im}_i(s)$ be the image of G_s just after step i, i.e., $\mathrm{im}_i(s)=\mathrm{im}_{i-1}(s)$ if $P_i\notin\mathsf{SpecPaths}^*_s$ and $\mathrm{im}_i(s)=\mathrm{im}_{i-1}(s)\cup\phi_i(V(P_i))$, otherwise. Let H_i be the host graph just after step i (i.e., the union of edges not yet used by the packing after step i, the graph H_0 being $H_{\mathbf{E}}$). For the first (at most) $\gamma_{\mathbf{E}} n^2$ steps, we shall be correcting the degree discrepancy of the vertices. During these steps, we shall be careful not to use any one vertex more than $\gamma_{\mathbf{E}}\big(1 + 10(\sigma_0\lambda^*)^{-2}\delta\big)n$ times. After that, we shall just pack the rest of $\{\mathsf{SpecPaths}^*_s : s \in \mathcal{J}_1\}$ in such a way as not to destroy the degree discrepancy of any vertex.

First, we state an auxiliary claim that will ensure us the existence of the mappings $\phi_i, i \in [m]$ we shall define in the proofs of Lemmas 59 and 60.

Claim 58.6. *For any $i \le \sigma_1^2 n^2$, for any $s \in \mathcal{J}_1$, for any set $S \subseteq V(H)$ with $|S| \le 2$, and for $V^* \in \{V_\boxminus, V_\boxplus\}$, we have $|\mathsf{N}_{H_{i-1}}(S) \cap V^* \setminus \mathrm{im}_{i-1}(s)| > (\sigma_0\lambda^*)^2\delta n$.*

Proof. From (*Quasi* 1), by setting $S_1 := S$, $S_2 := \emptyset$, $T_1 := \{s\}$, and $T_2 = T_3 := \emptyset$, for $V^* = V_\boxminus$, and setting $S_1 := \emptyset$, $S_2 := S$, $T_1 := \emptyset$, $T_2 := \{s\}$, and $T_3 := \emptyset$, if $V^* = V_\boxplus$, we obtain that

$$|\mathsf{N}_{H_{\mathbf{E}}}(S) \cap V^* \setminus \mathrm{im}^{\mathbf{E}}(s)| \ge \frac{1}{2} \cdot \min\{d^*_{\mathbf{E}}, \bar{d}_{\mathbf{E}}\}^2 \cdot \frac{n}{2} \cdot 10\delta \ge \frac{10 \cdot 16}{4}(\sigma_0\lambda^*)^2\delta n,$$

where the last inequality comes from Fact 35.

Observe that for an arbitrary $v \in V_\boxminus \cup V_\boxplus$, the difference between $|\mathsf{N}_{H_{\mathbf{E}}}(v)|$ and $|\mathsf{N}_{H_{i-1}}(v)|$ is at most $2\sigma_1 n$, as $|\mathcal{J}_1| = \sigma_1 n$ and each path from $\{\mathsf{SpecPaths}^*_s, s \in \mathcal{J}_1\}$ has used at most 2 edges incident to v. Also observe that the difference between $|\,\mathrm{im}^{\mathbf{E}}(s)|$ and $|\,\mathrm{im}_{i-1}(s)|$ is at most $10\sigma_1 n$, being the maximal number of vertices in $\{\mathsf{SpecPaths}^*_s\}$ embedded in Stage E. Therefore, we have by (1)

$$\begin{aligned}|\mathsf{N}_{H_{i-1}}(S) \cap V^* \setminus \mathrm{im}_{i-1}(s)| &\ge |\mathsf{N}_{H_{\mathbf{E}}}(S) \cap V^* \setminus \mathrm{im}^{\mathbf{E}}(s)| - 4\sigma_1 n - 10\sigma_1 n \\ &\ge 40(\sigma_0\lambda^*)^2\delta n - 14\sigma_1 n > (\sigma_0\lambda^*)^2\delta n\,. \end{aligned}$$

□

In Lemma 59 below we claim that we can embed a path from $\mathcal{P}_i$ in such a way that we can improve by 2 the degree discrepancy of two vertices, without spoiling the other degree discrepancy (and thus keeping the average degree discrepancy equal to zero). Observe that in Stage B we ensured that the degree discrepancy of each vertices is even, so that repeating this procedure can indeed terminate with all vertices having degree discrepancy zero.

Lemma 59. *Let $i \le \gamma_{\mathbf{E}} n^2$. Suppose we have two vertices $u, w \in V^*$, $V^* \in \{V_\boxminus, V_\boxplus\}$ such that $d_{H_{i-1},\mathcal{P}_{i-1}}(u) > 0$ and $d_{H_{i-1},\mathcal{P}_{i-1}}(w) < 0$.*

Then there exists a path $P_i \in \mathcal{P}_{i-1}$ and there exists a mapping ϕ_i of P_i in H_{i-1} such that the discrepancy degree $d_{H_i,\mathcal{P}_i}(u) = d_{H_{i-1},\mathcal{P}_{i-1}}(u) - 2$, $d_{H_i,\mathcal{P}_i}(w) = d_{H_{i-1},\mathcal{P}_{i-1}}(w) + 2$, and $d_{H_i,\mathcal{P}_i}(v) = d_{H_{i-1},\mathcal{P}_{i-1}}(v)$ for all $v \in V(H) \setminus \{u, w\}$ and ϕ_i satisfies the following.

(*a*) *If one anchor of P_i is in $V_\boxminus$ and the other in $V_\boxplus$, then the image of ϕ_i uses 3 edges in $V_\boxminus$, 3 edges in $V_\boxplus$, and 5 edges between $V_\boxminus$ and $V_\boxplus$.*

(*b*) *If both anchors are in $V_\boxminus$, then ϕ_i uses 2 edges in $V_\boxminus$, 3 edges in $V_\boxplus$, and 6 edges between $V_\boxminus$ and $V_\boxplus$.*

(*c*) *If both anchors are in $V_\boxplus$, then ϕ_i uses 2 edges in $V_\boxplus$, 3 edges in $V_\boxminus$, and 6 edges between $V_\boxminus$ and $V_\boxplus$.*

Moreover, during the whole $\leq \gamma_{\mathbf{E}} n^2$ *times using this lemma, no vertex is used more than* $\gamma_{\mathbf{E}}(1 + 10(\sigma_0\lambda^*)^{-2}\delta)n$ *times for the embedding of the paths from* $\mathcal{P}_i$.

Proof. The next auxiliary claim helps us pick a suitable path P_i from the set $\mathcal{P}_{i-1}$.

Claim 59.1. *Suppose that* $i \leq \gamma_{\mathbf{E}} n^2$. *Then there exists a path* $P_i \in \mathcal{P}_{i-1} \cap \mathsf{SpecPaths}^*_s$ *with* s *such that* $\{u, w\} \cap \mathrm{im}_{i-1}(s) = \emptyset$.

Proof. From $(Quasi6)$, by setting $S_1 := \{u, w\}$, $S_2 = \emptyset$ and looking in $\mathcal{J}_1$ (where for each $s \in \mathcal{J}_1$ we have $n - |\mathrm{im}^{\mathbf{E}}(s)| \geq \delta n$), we obtain that

$$|\{s \in \mathcal{J}_1 \,:\, \mathrm{im}^{\mathbf{E}}(s) \cap \{u, w\} = \emptyset\}| \geq (1 - 2\gamma_{\mathbf{E}})\delta^2 \cdot \sigma_1 n \geq \tfrac{1}{2}\delta^2 \cdot \sigma_1 n \,.$$

At step $i \leq \gamma_{\mathbf{E}} n^2$, we have used either u or w at most $\gamma_{\mathbf{E}}\big(1 + 10(\sigma_0\lambda^*)^{-2}\delta\big)n$ times, and in particular in at most that many different graphs G_s, in Stage F. We have embedded at most $\gamma_{\mathbf{E}} n^2$ paths. Hence we have at least $\big(\frac{1}{2}\delta^2 \cdot \sigma_1 n - \gamma_{\mathbf{E}}\big(1 + 10(\sigma_0\lambda^*)^{-2}\delta\big)n\big)\sigma_1 n - \gamma_{\mathbf{E}} n^2 > 0$ paths to choose P_i from. □

Assume that $u, w \in V_{\boxminus}$. The other case is dealt with symmetrically and we omit the details. Let $P_i = \{x_1, \ldots, x_{12}\}$ be a path given by Claim 59.1, let s be such that $P_i \in \mathsf{SpecPaths}^*_s$ and let $v_1 := \phi^{\mathbf{E}}_s(x_1)$ and $v_{12} := \phi^{\mathbf{E}}_s(x_{12})$.

First consider the case when $v_1 \in V_{\boxminus}$ and $v_{12} \in V_{\boxplus}$ (or the other way around, which is done analogously). We set $v_5 = \phi_i(x_5) := w$ and $v_9 = \phi_i(x_9) := u$. Each vertex $x_j \in V(P_i) \setminus \{x_1, x_5, x_9, x_{12}\}$ (which we call the *connection vertices*) is successively mapped to v_j chosen in

$$\mathsf{N}_{H_{i-1}}(v_{j-1}) \cap V_* \setminus \big(\mathrm{im}(\phi^{\mathbf{E}}(G_s)) \cup \{v_2, \ldots, v_{i-1}\} \cup \{v_5, v_9\}\big) \tag{92}$$

or in

$$\mathsf{N}_{H_{i-1}}(v_{j-1}) \cap \mathsf{N}_{H_{i-1}}(v_{j+1}) \cap V_* \setminus \big(\mathrm{im}(\phi^{\mathbf{E}}(G_s)) \cup \{v_2, \ldots, v_{i-1}\} \cup \{v_5, v_9\}\big)\,, \tag{93}$$

if v_{j+1} is already defined, where $V_* = V_{\boxminus}$, for $j \in \{2, 8, 10\}$ and $V_* = V_{\boxplus}$ for $j \in \{3, 4, 6, 7, 11\}$. We insist additionally (and will in the following cases also) on choosing for each of these steps a vertex to which we embedded a connection vertex in Stage F less than $10\gamma_{\mathbf{E}}(\sigma_0\lambda^*)^{-2}\delta n$ times. By Claim 58.6 the set from which v_j is chosen is non-empty, since the total number of paths we embed while applying this lemma is at most $\gamma_{\mathbf{E}} n^2$ and each has 10 vertices to embed. Note that when embedding paths in this lemma, we use vertices either as connection vertices or because their degree discrepancy is non-zero; by Claim 58.5 we use any vertex at most $\gamma_{\mathbf{E}} n$ times for the latter, and this implies that we use any vertex in total at most $\gamma_{\mathbf{E}}\big(1 + 10(\sigma_0\lambda^*)^{-2}\delta\big)n$ times while applying this lemma. This satisfies the *Moreover* part of the lemma.

Observe that $\deg_{\phi_i(P_i)}(u, V_{\boxminus}) = 2$, $\deg_{\phi_i(P_i)}(u, V_{\boxplus}) = 0$, $\deg_{\phi_i(P_i)}(w, V_{\boxminus}) = 0$, $\deg_{\phi_i(P_i)}(w, V_{\boxplus}) = 2$, $\deg_{\phi_i(P_i)}(v_j, V_{\boxminus}) = \deg_{\phi_i(P_i)}(v_j, V_{\boxplus}) = 1$, for $j \in \{2, 3, 4, 6, 7, 8, 10, 11\}$, and $\deg_{\phi_i(P_i)}(v_1, V_{\boxminus}) = 1$, $\deg_{\phi_i(P_i)}(v_{12}, V_{\boxplus}) = 1$, while there is one less path anchored at $v_1 \in V_{\boxminus}$ and $v_{12} \in V_{\boxplus}$ in

$\mathcal{P}_i$ than in $\mathcal{P}_{i-1}$. This ensures the requirements for the degree discrepancy $d_{H_i,\mathcal{P}_i}(v)$ are fulfilled for all $v \in V_\boxminus \cup V_\boxplus$. Observe that ϕ_i satisfies (a), as it uses the following 5 edges between $V_\boxminus$ and $V_\boxplus$: $\{v_2, v_3\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_7, v_8\}$, and $\{v_{10}, v_{11}\}$, the following 3 edges inside $V_\boxminus$: $\{v_1, v_2\}, \{v_8, v_9\}$, and $\{v_9, v_{10}\}$, and the following 3 edges inside $V_\boxplus$: $\{v_3, v_4\}, \{v_6, v_7\}$, and $\{v_{11}, v_{12}\}$.

Second, consider the case when $v_1, v_{12} \in V_\boxminus$. We set $v_4 = \phi(x_4) := w$ and $v_8 = \phi(x_8) := u$. Each vertex $x_j \in V(P_i) \setminus \{x_1, x_4, x_8, x_{12}\}$ is successively mapped to v_j chosen as in (92) or (93) where $V_* = V_\boxminus$, for $j \in \{7, 9\}$ and $V_* = V_\boxplus$ for $j \in \{2, 3, 5, 6, 10, 11\}$. Similarly as above, Claim 58.6 ensures that we can define v_j without any vertex having too many connection vertices embedded to it. Similarly as above, the assignment of V_*, ensures the requirements for $d_{H_i,\mathcal{P}_i}(\cdot)$ are satisfied. Observe that $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_6, v_7\}, \{v_9, v_{10}\}, \{v_{11}, v_{12}\}\} \subseteq V_\boxminus \times V_\boxplus$, $\{\{v_7, v_8\}, \{v_8, v_9\}\} \subseteq \binom{V_\boxminus}{2}$, and $\{\{v_2, v_3\}, \{v_5, v_6\}, \{v_{10}, v_{11}\}\} \subseteq \binom{V_\boxplus}{2}$, satisfying (a).

Last, consider the case when $v_1, v_{12} \in V_\boxplus$. We set $v_3 = \phi(x_3) := u$ and $v_7 = \phi(x_7) := w$. Each vertex $x_j \in V(P_i) \setminus \{x_1, x_3, x_7, x_{12}\}$ is successively mapped to v_j chosen in (92) or in (93), where $V_* = V_\boxminus$ for $j \in \{2, 4, 10, 11\}$ and $V_* = V_\boxplus$ for $j \in \{5, 6, 8, 9\}$. Again we can define v_j due to Claim 58.6 and the requirements of $d_{H_i,\mathcal{P}_i}(\cdot)$ are satisfied. For the *Moreover part*, observe that ϕ_i uses 6 edges between $V_\boxminus$ and $V_\boxplus$, 3 edges inside $V_\boxminus$, and 2 edges inside $V_\boxplus$. □

Lemma 60. *Let $i \in [m]$. For any path $P_i \in \mathcal{P}_{i-1}$, there exists a mapping ϕ_i of P_i in H_{i-1} such that the discrepancy degree $d_{H_i,\mathcal{P}_i}(v) = d_{H_{i-1},\mathcal{P}_{i-1}}(v)$ for all $v \in V_\boxminus \cup V_\boxplus$. Moreover, ϕ_i satisfies the following.*

- *If one anchor of P_i is in $V_\boxminus$ and the other in $V_\boxplus$, then the image of ϕ_i uses 3 edges in $V_\boxminus$, 3 edges in $V_\boxplus$, and 5 edges between $V_\boxminus$ and $V_\boxplus$.*
- *If both anchors are in $V_\boxminus$, then ϕ_i uses 2 edges in $V_\boxminus$, 3 edges in $V_\boxplus$, and 6 edges between $V_\boxminus$ and $V_\boxplus$.*
- *If both anchors are in $V_\boxplus$, then ϕ_i uses 2 edges in $V_\boxplus$, 3 edges in $V_\boxminus$, and 6 edges between $V_\boxminus$ and $V_\boxplus$.*

Proof. Let $P_i = \{x_1, \ldots, x_{12}\}$ and let s be such that $P_i \in \mathsf{SpecPaths}^*_s$ and let $v_1 := \phi^{\mathbf{E}}_s(x_1)$ and $v_{12} := \phi^{\mathbf{E}}_s(x_{12})$.

First, consider the case when $v_1 \in V_\boxminus$ and $v_{12} \in V_\boxplus$ (or the other way around, which is done analogously). Each vertex $x_j \in V(P_i) \setminus \{x_1, x_{12}\}$ is successively mapped to v_j chosen in (92), or in (93) if $j = 11$, where $V_* = V_\boxminus$, for $j \in \{2, 5, 6, 9, 10\}$ and $V_* = V_\boxplus$ for $j \in \{3, 4, 7, 8, 11\}$. By Claim 58.6 the set from which v_j is chosen, is non-empty, as ϕ_i is eventually defined on 10 vertices. Similarly as in the proof of Lemma 59, it is easy to observe that the requirements on $d_{H_i,\mathcal{P}_i}(\cdot)$ are satisfied. For the *Moreover part*, observe that ϕ_i uses the edges $\{v_2, v_3\}, \{v_4, v_5\}, \{v_6, v_7\}, \{v_8, v_9\}$, and $\{v_{10}, v_{11}\}$ from $V_\boxminus \times V_\boxplus$. The edges $\{v_1, v_2\}, \{v_5, v_6\}$, and $\{v_9, v_{10}\}$ lie inside $V_\boxminus$ and the edges $\{v_3, v_4\}, \{v_7, v_8\}$, and $\{v_{11}, v_{12}\}$ lie inside $V_\boxplus$.

Second, consider the case when $v_1, v_{12} \in V_\boxminus$ (the case when $v_1, v_{12} \in V_\boxplus$ is done analogously). Each vertex $x_j \in V(P_i) \setminus \{x_1, x_{12}\}$ is successively mapped to v_j chosen in (92), or in (93) if

$j = 11$, where $V_* = V_{\boxminus}$, for $j \in \{4,5,8,9\}$ and $V_* = V_{\boxplus}$ for $j \in \{2,3,6,7,10,11\}$. By Claim 58.6 the set from which v_j is chosen, is non-empty, as ϕ_i is eventually defined on 10 vertices. Similarly as in the proof of Lemma 59, it is easy to observe that the requirements on $d_{H_i,\mathcal{P}_i}(\cdot)$ are satisfied. As just above, one can easily check that ϕ_i uses 2 edges in $V_{\boxminus}$, 3 edges in $V_{\boxplus}$, and 6 edges between $V_{\boxminus}$ and $V_{\boxplus}$. □

Once we have $d_{H_i,\mathcal{P}_i}(v) = 0$ for all $v \in V_{\boxminus} \cup V_{\boxplus}$, we use Lemma 60 to pack the left-over paths from $\mathcal{P}_i$. So, it is left to prove that we manage to correct all the degree discrepancy within the first $\gamma_{\mathbf{D}} n^2$ steps by using Lemma 59. Observe that by Claim 58.5 we have

$$\sum_{v \in V_{\boxminus} \cup V_{\boxplus}} |d_{H_0,\mathcal{P}_0}(v)| \le \gamma_{\mathbf{D}} n^2 \ .$$

Each use of Lemma 59 decreases $|d_{H_0,\mathcal{P}_0}(v)|$ by 2 for two vertices, and hence decreases

$$\sum_{v \in V_{\boxminus} \cup V_{\boxplus}} |d_{H_0,\mathcal{P}_0}(v)|$$

by 4. Hence, after at most $\frac{1}{4}\gamma_{\mathbf{D}} n^2$ steps, we have $\sum_{v \in V_{\boxminus} \cup V_{\boxplus}} |d_{H_0,\mathcal{P}_0}(v)| = 0$, implying $d_{H_0,\mathcal{P}_0}(v) = 0$, for every $v \in V_{\boxminus} \cup V_{\boxplus}$. This completes the proof of Lemma 37.

14. Stage G (Proof of Lemma 38)

In this section we shall prove Lemma 38 by obtaining the setting of Proposition 14 on an auxiliary multigraph $\mathcal{M}$.

Proposition 14 with input parameters $d_{P14} := \lambda\sigma_0$ and $\sigma_{P14} := \sigma_0$ outputs parameters L_{P14}, n_0, and γ_{P14}. By (1), we have $n \ge 2n_0$, $L_{\mathbf{F}} \ge L_{P14}$ and $\gamma_{\mathbf{F}} \le \min\{\gamma_{P14}, \sigma_0\lambda\delta/8\}$.

From our graph H and our paths in $\{\mathsf{SpecShortPaths}_s, s \in \mathcal{J}_0\}$, we define $\mathcal{M} = ([n/2] \sqcup \mathcal{J}_0, \vec{E}_1, E_2, E_3, E_4, E_5, E_6)$ with $\vec{E}_1 \subseteq [n/2]^2$, $E_2, E_3 \subseteq \binom{[n/2]}{2}$, and $E_4, E_5, E_6 \subseteq [n/2] \times \mathcal{J}_0$, as follows. Each pair $\boxminus_i, \boxplus_i$ corresponds to the vertex $i \in [n/2]$. For each edge $\{\boxminus_i, \boxplus_j\} \in E(H)$ insert an oriented edge $(i,j) \in \vec{E}_1$. For each edge $\{\boxminus_i, \boxminus_j\} \in E(H)$ insert an edge $\{i,j\} \in E_2$. For each edge $\{\boxplus_i, \boxplus_j\} \in E(H)$ insert an edge $\{i,j\} \in E_3$. Insert an edge $\{i,s\} \in E_6$, if there is a path $P \in \mathsf{SpecShortPaths}_s$ with $(\phi_s^{\mathbf{F}})^{-1}(\{\boxminus_i, \boxplus_i\}) = \{\mathsf{leftpath}_0(P), \mathsf{rightpath}_0(P)\}$. Insert an edge $\{i,s\} \in E_4$, if $\boxminus_i \notin \mathrm{im}(s)$. Insert an edge $\{i,s\} \in E_5$, if $\boxplus_i \notin \mathrm{im}(s)$.

As H is a simple graph, none of the edge sets E_i, $i \in 2, \ldots, E_6$, nor $\vec{E}_1$ has multiple edges that would be oriented in the same direction. As none of the edges $\{\boxminus_i, \boxplus_i\}, i \in [n/2]$ are present in $E(H)$, there is no loop in $\vec{E}_1$. Hence, the multigraph $\mathcal{M}$ is a chest.

Moreover, observe that $|\vec{E}_1| = |E_2| = |E_3| = |E_6| = |\bigcup_{s \in \mathcal{J}_0} \mathsf{SpecShortPaths}_s|$ and that $d_1 = \frac{|\vec{E}_1|}{n^2/4} = \bar{d_{\mathbf{F}}} = 4\lambda^*\sigma_0 > \lambda\sigma_0$, $d_4, d_5 = \delta + 2\lambda^* \ge d_6 = 2\lambda^* > \lambda\sigma_0$, $|V(\mathcal{M})| = n/2 + |\mathcal{J}_0| > n_0$, and $\min\{n/2, |\mathcal{J}_0|\} = \sigma_0 n > \sigma_0 |V(\mathcal{M})|$.

Lemma 61. *The chest $\mathcal{M}$ fulfils the degree conditions from Proposition 14.*

Proof. Fix any vertex $i \in \left[\lfloor \frac{n}{2} \rfloor\right]$. Condition (i) of Proposition 14 that $\deg_{E_2}(i) = \deg^{out}_{\vec{E}_1}(i) + \deg_{E_4}(i)$ translates to $\deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxminus}) = \deg_{H_{\mathbf{F}}}(\boxminus_i, V_{\boxplus}) + |\{s \in \mathcal{J}_0 : i \in I_s\}|$, with $I_s := \{i \in [\frac{n}{2}] :$

$\boxminus_i, \boxplus_i \in A_s\}$. This is given by (20) holding as stated in Lemma 37. Condition (ii) of Proposition 14 that $\deg_{E_3}(i) = \deg^{in}_{E_1}(i) + \deg_{E_4}(i)$ translates to $\deg_{H_{\mathbf{F}}}(\boxplus_i, V_\boxplus) = \deg_{H_{\mathbf{F}}}(\boxplus_i, V_\boxminus) + |\{s \in \mathcal{J}_0, i \in I_s\}|$ again given by (20). As for Condition itm:designs:iii of Proposition 14 we need to show that

$$|\{s \in \mathcal{J}_0, \boxminus_i \notin \mathrm{im}^{\mathbf{F}}(s)\}| \geq \deg_{H_{\mathbf{F}}}(\boxminus_i, V_\boxplus)\ .$$

From $(Quasi\,6)$ with $S_1 = \{\boxminus_i\}$ and $T = \emptyset$ we obtain that the left-hand side is at least $\frac{1}{2}(\delta + 2\lambda^*)\sigma_0 n$. By $(Quasi\,1)$, $(H_{\mathbf{F}}, V_\boxminus V_\boxplus)$ is $(L_{\mathbf{F}}, \gamma_{\mathbf{F}}, d^*_{\mathbf{F}}, \bar{d}_{\mathbf{F}})$-index-quasirandom with respect to the collections $(\mathrm{im}^{\mathbf{F}}(s))_{s\in\mathcal{J}_0}$, and $(I_s)_{s\in\mathcal{J}_0}$, so putting $S_1 = T_1 = T_2 = T_3 = \emptyset$ and $S_2 = \{\boxminus_i\}$ in Definition 28 the right-hand side is at most $(1+\gamma_{\mathbf{F}})\bar{d}_{\mathbf{F}}\frac{n}{2} < \frac{1}{4}\sigma_0^2 n$. By similar logic we obtain

$$|\{s \in \mathcal{J}_0; \boxplus_i \notin \mathrm{im}^{\mathbf{F}}(s)\}| \geq \deg_{H_{\mathbf{F}}}(\boxplus_i, V_\boxminus)\ ,$$

leading to Condition (iv) of Proposition 14.

Now fix any vertex $s \in \mathcal{J}_0$. Observe that

$$\begin{aligned}
\deg_{E_5}(s) &= |V_\boxminus \setminus \mathrm{im}^{\mathbf{F}}(s)| = |\mathbb{U}_{H_{\mathbf{F}}}(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{s\},\emptyset)|\,,\\
\deg_{E_6}(s) &= |V_\boxplus \setminus \mathrm{im}^{\mathbf{F}}(s)| = |\mathbb{U}_{H_{\mathbf{F}}}(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{s\})|\,,\ \text{and}\\
\deg_{E_4}(s) &= |I_s| = |\mathbb{U}_{H_{\mathbf{F}}}(\emptyset,\emptyset,\emptyset,\emptyset,\{s\},\emptyset,\emptyset)|\,.
\end{aligned}$$

Hence, by index-quasirandomness of $(H_{\mathbf{F}}, \mathrm{im}^{\mathbf{F}}(s))$,we have that $|V_\boxminus \setminus \mathrm{im}^{\mathbf{F}}(s)| = (1\pm\gamma_{\mathbf{F}})(\delta+2\lambda^*)\frac{n}{2}$, $|V_\boxplus \setminus \mathrm{im}^{\mathbf{F}}(s)| = (1\pm\gamma_{\mathbf{F}})(\delta+2\lambda^*)\frac{n}{2}$, and we have $|I_s| = (2\lambda^*)\frac{n}{2}$, implying (v) and (vi) of Proposition 14. □

To apply Proposition 14, it remains to prove the quasirandomness of the chest $\mathcal{M}$.

Lemma 62. *The chest $\mathcal{M}$ is $(\gamma_{\mathbf{F}}, L_{\mathbf{F}})$-quasirandom.*

Proof. The $(\gamma_{\mathbf{F}}, L_{\mathbf{F}})$-quasirandomness of the chest corresponds to the following conditions in our original graph $H_{\mathbf{F}}$. Suppose we are given mutually disjoint sets $X_1, X_1', X_2, X_3, X_4', X_5', X_6' \subseteq [n/2]$ and $X_4, X_5, X_6 \subseteq \mathcal{J}_0$ of total size at most $L_{\mathbf{F}}$. Let d_i be the edge-density of E_i. We need to show that

$$|\mathsf{N}^{out}_{\vec{E}_1}(X_1) \cap \mathsf{N}^{in}_{\vec{E}_1}(X_1') \cap \bigcap_{i=2}^{6} \mathsf{N}_{E_i}(X_i)| = (1\pm\gamma_{\mathbf{F}}) d_1^{|X_1|+|X_1'|} \cdot \prod_{i=2}^{6} d_i^{|X_i|}\frac{n}{2}\ ,\ \text{and} \tag{94}$$

$$|\mathcal{J}_0 \cap \bigcap_{i=4}^{6} \mathsf{N}_{E_i}(X_i')| = (1\pm\gamma_{\mathbf{F}}) \prod_{i=4}^{6} d_i^{|X_i'|} |\mathcal{J}_0|\ . \tag{95}$$

Define sets $Q_1, Q_2, Q_5' \subseteq V_\boxminus$, $Q_1', Q_3, Q_6' \subseteq V_\boxplus$, such that $Q_1 = \{\boxminus_i, i \in X_1\}$, $Q_1' = \{\boxplus_i, i \in X_1'\}$, $Q_2 = \{\boxminus_i, i \in X_2\}$, $Q_3 = \{\boxplus_i, i \in X_3\}$, $Q_5' = \{\boxminus_i, i \in X_5'\}$, and $Q_6'\{\boxplus_i, i \in X_6'\}$.

Then

- $\mathsf{N}^{out}_{\vec{E}_1}(X_1)$ corresponds to $\{i \in [n/2] : \boxplus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_1)\}$
- $\mathsf{N}^{in}_{\vec{E}_1}(X_1')$ corresponds to $\{i \in [n/2] : \boxminus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_1')\}$
- $\mathsf{N}_{E_2}(X_2)$ corresponds to $\{i \in [n/2] : \boxminus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_2)\}$

- $\mathsf{N}_{E_3}(X_3)$ corresponds to $\{i \in [n/2] : \boxplus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_3)\}$
- $\mathsf{N}_{E_4}(X_4)$ corresponds to $\bigcap_{s \in S_4} I_s$
- $\mathsf{N}_{E_5}(X_5)$ corresponds to $\{i \in [n/2] : \boxminus_i \notin \bigcup_{s \in X_5} \mathrm{im}^{\mathbf{F}}(s)\}$
- $\mathsf{N}_{E_6}(X_6)$ corresponds to $\{i \in [n/2] : \boxplus_i \notin \bigcup_{s \in X_6} \mathrm{im}^{\mathbf{F}}(s)\}$
- $\mathsf{N}_{E_4}(X_4')$ corresponds to $\{s \in \mathcal{J}_0, X_4' \subseteq I_s\}$
- $\mathsf{N}_{E_5}(X_5')$ corresponds to $\{s \in \mathcal{J}_0 : \mathrm{im}^{\mathbf{F}}(s) \cap Q_5' = \emptyset\}$
- $\mathsf{N}_{E_6}(X_6')$ corresponds to $\{s \in \mathcal{J}_0 : \mathrm{im}^{\mathbf{F}}(s) \cap Q_6' = \emptyset\}$

Hence, (94) and (95) translate as

$$\begin{aligned}\Bigg|\Bigg\{ i \in [\lfloor \tfrac{n}{2} \rfloor] \cap \bigcap_{s \in X_4} I_s : \Big(\boxminus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_1' \cup Q_2) \setminus \bigcup_{s \in X_5} \mathrm{im}^{\mathbf{F}}(s)\Big) \\ \& \Big(\boxplus_i \in \mathsf{N}_{H_{\mathbf{F}}}(Q_1 \cup Q_3) \setminus \bigcup_{s \in X_6} \mathrm{im}^{\mathbf{F}}(s)\Big)\Bigg\}\Bigg| \\ = (1 \pm \gamma_{\mathbf{F}}) d_1^{|Q_1|+|Q_1'|} \cdot \prod_{i=2}^{3} d_i^{|Q_i|} \cdot \prod_{i=4}^{6} d_i^{|X_i|} \cdot \frac{n}{2}\end{aligned} \tag{96}$$

$$|\{s \in \mathcal{J}_0 : X_4' \subseteq I_s \& \, \mathrm{im}^{\mathbf{F}}(s) \cap (Q_5' \cup Q_6') = \emptyset\}| = (1 \pm \gamma_{\mathbf{F}}) d_4^{|X_4'|} d_5^{|Q_5'|} d_6^{|Q_6'|} \lambda n \,, \tag{97}$$

where $d_1 = \bar{d}_{\mathbf{F}} = 4\lambda^* \sigma_0$, $d_2 = d_3 = d_{\mathbf{F}}^* = 8\lambda^*\sigma_0$, $d_6 = 2\lambda^*$, and $d_4, d_5 = \delta + 2\lambda^*$.

Condition (96) follows directly from ($Quasi1$) (see Definition 28) by setting $S_1 := Q_1' \cup Q_2$, $S_2 := Q_1 \cup Q_3$, $T_1 := X_5$, $T_2 := X_6$ and $T_3 := X_4$, while Condition (97) follows from ($Quasi6$) by setting $S_1 := Q_5' \cup Q_6'$, $S_2 = \emptyset$ and $T := X_4'$. □

By Proposition 14, we obtain that the chest $\mathcal{M}$ has a diamond core-decomposition. Next, we shall explain how we use the diamond-core decomposition given by Proposition 14 to pack the left-over path-forest.

For each coloured K_4 of the core-decomposition, the edge in E_4 gives us information on which path $P \in \bigcup_{s \in \mathcal{J}_0} \mathsf{SpecShortPaths}_s$ we shall pack using this K_4: if the E_4-edge lies between $s \in \mathcal{J}_0$ and $i \in [\frac{n}{2}]$, the path P is the only path in $\mathsf{SpecShortPaths}_s$ anchored at $\{\boxminus_i, \boxplus_i\}$. The edges in $E_2, \vec{E}_1, E_3$ between say i, j, and k give us information where the three edges of the path P will be mapped to. So if the E_2-edge between i and j ensures there is an edge in $H_{\mathbf{F}}$ between $\boxminus_i$ and $\boxminus_j$, the $\vec{E}_1$-edge between j and k ensures there is an edge in $H_{\mathbf{F}}$ between $\boxminus_j$ and $\boxplus_k$, and the E_3-edge between k and i ensures there is an edge in $H_{\mathbf{F}}$ between $\boxplus_k$ and $\boxplus_i$. The edges from E_4 and E_5 ensure that $\boxminus_j$ and $\boxplus_k$ are not already used by G_s.

As we have a diamond-core decomposition, we are ensured that every edge in $H_{\mathbf{F}}$ exactly once, and that every path in $\bigcup_{s \in \mathcal{J}_0} \mathsf{SpecShortPaths}_s$ is packed.

15. Concluding remarks

Theorem 10 allows to perfectly pack large degenerate graphs of maximum degree $O(n/\log n)$, most of which may be spanning, and some of which are non-spanning and give us quadratically many bare paths and linearly many odd degree vertices. As discussed, this implies the tree packing conjecture for large n and trees with maximum degree $O(n/\log n)$. Hence, to resolve this conjecture in full (for large n), it only remains to consider the case of trees with large degree vertices. As we explained in Section 1.2, our methods crucially need the maximum degree assumption, witnessed by the fact that they also work for pseudorandom graphs, in which the maximum degree assumption is sharp (up to constants). It would be very interesting to solve this last remaining case of the tree packing conjecture.

One could ask more generally which families of trees pack into K_n. Obviously, the number of edges in such a family of trees may not exceed $\binom{n}{2}$. We further argued in Section 1.2 that $\Omega(n/\log n)$ of the trees must be at least $\Omega(n/\log n)$-far from spanning. In Section 9.2 of [4] it was explained that if only an aggregate bound on the maximum degrees is imposed, then this bound cannot be less than $n/2$. This motivates the following conjecture.

Conjecture 63. *There exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ any family of trees $(T_s)_{s\in[N]}$ with $\Delta(T_s) \leq n/2$ and $v(T_s) \leq n$ for all $s \in [N]$, with $\delta n \leq v(T_s) \leq (1-\delta)n$ for all $s \in [\delta n]$, and with $\sum_{s\in[N]} e(T_s) \leq \binom{n}{2}$, packs into K_n.*

We further remark that we solely use the required odd degree vertices for parity corrections. In scenarios where this is not needed our methods would also work without odd degree vertices. Let us give an example. The Oberwolfach problem on packings of cycle-factors was solved by Glock, Joos, Kim, Kühn, and Osthus [9], and generalised by Keevash and Staden [17], who proved that any large and sufficiently quasirandom $2r$-regular graph can be decomposed into any family of r two-factors. Our methods would give a variant of this result where we do not require the host graph to be regular but need some non-spanning 2-regular graphs in the family. We could pack into any sufficiently quasirandom graph with all vertex degrees even any family of 2-regular graphs, of which δn have at least δn and at most $(1-\delta)n$ vertices and contain δn cycles of length at least 12. This last restriction on cycle lengths comes from the way we work with bare paths and could probably be removed with some extra work.

The motivation for the name 'Oberwolfach Problem' is that conference attendees sit around circular tables for their meals. The Oberwolfach Problem is to determine when it is possible that every mathematician sits next to every other one exactly once at meals. This is exactly the problem of packing K_n with n-vertex 2-regular graphs. Due to the Schwarzwald scenery, however, some mathematicians go on hikes, so that at some meals the number of vertices in the corresponding 2-factor to pack is smaller than n. We therefore propose the following 'Oberwolfach Problem with hikes' as a more practical version of the original. Given n and a collection of 2-regular graphs $G_1, \dots, G_t$ each on at most n vertices with $\sum_{i=1}^{t} e(G_i) = \binom{n}{2}$, when is it true that $G_1, \dots, G_t$ pack into K_n?

Concerning the packing of r-regular graphs with $r \geq 2$, Glock, Joos, Kim, Kühn, and Osthus [9] formulated the following conjecture, which already for $r = 3$ seems challenging.

Conjecture 64. *Given r, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ any family of n-vertex graphs $(G_s)_{s\in[N]}$ such that G_s is r_s-regular with $r_s \leq r$ for all $s \in [N]$ and such that $\sum_{s\in[N]} r_s = n - 1$, packs into K_n.*

Considering, more generally, D-degenerate graphs, the situation certainly is more complex. It is clear that an appropriate analogue of the tree packing conjecture with trees replaced by D-degenerate graphs cannot hold: If one of the graphs in the family is $K_{2,n-2}$, its embedding would isolate an edge in the host graph. However, an analogue of Ringel's conjecture could still hold.

Conjecture 65. *Given D, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ the following holds. Let G be any D-degenerate graph on n vertices and with $m \geq n - 1$ edges. Then $2m + 1$ copies of G pack into K_{2m+1}.*

16. Acknowledgments

This work was commenced during the Southwestern German Workshop on Graph Theory in August 2018. We thank the organisers, Maria Axenovich, Yury Person, and Dieter Rautenbach, and we thank (PA, JB, JH, DP) Pavel Hladký for babysitting our children during that time.

Appendix A. Deducing Theorem 21 from [16]

In this appendix we deduce Theorem 21 from Theorem 19 of [16]. This deduction is on the one hand fairly pedestrian, on the other hand it requires understanding some of the fairly complicated theory from [16]. Our paper and [16] use similar terms (such as "divisibility"), but our definitions are tailored to a more specialised setting. In this appendix, when referring to terminology or theorem, definition, and page numbers from [16], we use blue text. For the convenience of those readers who are using the arXiv version of [16] instead, we also occasionally add (in brackets and in *slanted text*) the corresponding theorem, definition, and page numbers from this arXiv version [13]. We remark that [16] contains numerous sample applications of Theorem 19 that are similar to our application here. Two further applications of Theorem 19 are used in [17, 18] to deduce the main results there.

We now explain how to deduce Theorem 21. Let $q, D \in \mathbb{N}$ and $\sigma > 0$ be given and let q' be as defined in Theorem 21. Let n_0 and ω_0 be the output of Theorem 19 (*Theorem 7.4*) with input $q := q', D$ and for uniformity $r := 2$. It is easy to check that the constants h and δ given on page 283 (*page 4*) are consistent with those in the statement of Theorem 21. Observe that $\sigma n \geq \frac{n}{h}$ by choice of q' and h.

Let $n > n_0$, set $n_1 := n$, and let $\omega \in (n^{-\delta}, \omega_0)$, as is required in our Theorem 21 but also in Theorem 19. Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be a partition of $[q]$ and $\mathcal{P}' = \{P'_1, \ldots, P'_t\}$ be a partition of

$[n]$ with $|P_i'| \geq \sigma n \geq \frac{n}{h}$ for each $i \in [t]$ and trivially $|P_i'| \leq n_1 = n$. We extend $\mathcal{P}$ by adding $q' - q$ isolated vertices to P_t. Abusing notation, we call this new partition also $\mathcal{P}$.

Let $\mathcal{H}$ be a simple $\mathcal{P}$-canonical family of $[D]$-edge-coloured digraph on $[q]$. We extend each $H \in \mathcal{H}$ by adding $q' - q$ isolated vertices and by abuse of notation call this new family $\mathcal{H}$. This step is only necessary to ensure that each digraph H is on vertex set $[q']$. The reason for moving from vertex set $[q]$ to vertex set $[q']$ in turn is that [16] requires that $h = 2^{50(q')^3}$ where $[q']$ is the vertex set of the digraphs we want to decompose into, but here we need that h is big in order to guarantee $\sigma n \geq \frac{n}{h}$. The added isolated vertices will not play any other role in the following.

Let G be a general $[D]$-edge-coloured digraph on $[n]$ such that $(G, \mathcal{P}')$ is $(\mathcal{H}, \mathcal{P})$-divisible, $(\mathcal{H}, \omega^{h^{20}}, \omega)$-regular and $(\mathcal{H}, \mathcal{P}, \sqrt[q'h]{\omega}, h)$-vertex-extendable. Note that the notion of divisibility, regularity and vertex-extendability was not changed by adding the $q' - q$ isolated vertices to digraphs of $\mathcal{H}$.

We shall check that G satisfies the conditions of Theorem 19. First of all, this theorem requires $\mathcal{H}$ to be canonical in the sense of Definition 35 (*Definition 7.1*). Observe that without loss of generality, we may assume that every edge in any graph from $\mathcal{H}$ between two parts P_i and P_j with $i < j$ is oriented towards P_j. (If this is not the case, that is, there is some colour d with colour location (j, i), then simply reverse all the edges of colour d in $\mathcal{H}$ and G, and accordingly change the colour location of d to (i, j).) This assumption is necessary so that we are consistent with Definition 35. This definition now requires us to specify an index vector $\mathbf{i}^d \in \mathbb{N}_0^t$ for each colour d such that $\mathbf{i}^d$ encodes the colour location of d as follows: Let d be a colour with colour location (i, j). If $i = j$, that is, edges of colour d lie within P_i, then we let $\mathbf{i}^d$ be the vector with a 2 at coordinate $i = j$ and 0 everywhere else. If $i \neq j$, on the other hand, that is, all edges of colour d run between P_i and P_j, then we let $\mathbf{i}^d$ be the vector with a 1 at position i and a 1 at position j, and 0 everywhere else. Definition 35 then defines a partition $R(\mathbf{i}^d) = (R(\mathbf{i}^d)_1, \ldots, R(\mathbf{i}^d)_t)$ of $[r] = [2]$ for each of these vectors $\mathbf{i}^d$. In our case, if the colour location (i, j) of d satisfies $i = j$, then we get $R(\mathbf{i}^d)_i = \{1, 2\}$ and $R(\mathbf{i}^d)_{i'} = \emptyset$ for all $i' \neq i$; if $i < j$, then we get $R(\mathbf{i}^d)_i = \{1\}$, $R(\mathbf{i}^d)_j = \{2\}$ and $R(\mathbf{i}^d)_{i'} = \emptyset$ for all $i' \neq i, j$. Further, Definition 35 requires us to specify for each colour d and each $j' \in [t]$ a permutation group $\Lambda^d_{j'}$ on $R(\mathbf{i}^d)_{j'}$. The purpose of these permutation groups is to allow for a mixture of (generalisations of partly) directed and undirected edges within parts; for us they will all be either trivial or the symmetric group $\mathbb{S}_2$ on elements $\{1, 2\}$. Indeed, we let $\Lambda^d_{j'}$ be the trivial group Id on $R(\mathbf{i}^d)_{j'}$ only containing the identity whenever $R(\mathbf{i}^d)_{j'}$ is empty, contains only one element, or if d is is an oriented colour. If d is an unoriented colour with colour index (j', j'), on the other hand, then we let $\Lambda^d_{j'}$ be $\mathbb{S}_2$. As in Definition 35, we set $\Lambda^d = \prod_{j'} \Lambda^d_{j'}$ and $\Lambda := (\Lambda^d : d \in [D])$.

We now claim that $\mathcal{H}$ is $(\mathcal{P}, \Lambda)$-canonical as defined in Definition 35. Indeed, Definition 35i just says that that for each colour d edges of colour d respect the colour location of d. Definition 35ii translates to the following: For each $H \in \mathcal{H}$ individually,

(a) ("for $\theta' \in Bij([r], Im(\theta))$ we have $\theta' \notin H \setminus H^d$") H does not have parallel or antiparallel edges of different colours, and (*b*) ("$\theta' \in H^d$ iff $\theta^{-1}\theta' \in \Lambda^d$") that edges of colour d with colour location (j, j) come in pairs forming directed 2-cycles if and only if the corresponding permutation group Λ_j^d equals $\mathbb{S}_2$.

Hence, Definition 35 is consistent with our Definition 16: loops are not allowed (see definition of r-digraph in Definition 28 (*Definition 6.1*)) and in Definition 16 we consider the special case of simple digraphs, i.e., we forbid parallel edges of the same colour.

Let us now turn to interpreting Definition 36. This definition uses the set $I_{q'}^r$ of injective maps from $[r]$ to $[q']$ (in [16] each edge in a directed r-graph on vertex set $[q']$ is such an injection; so $I_{q'}^r$ is the set of all possible directed r-edges on $[q']$). Here, we work with $r = 2$. Further, Definition 36 works with a vector formalism in which each $H \in \mathcal{H}$ is encoded as $H \in (\mathbb{N}_0^D)^{I_{q'}^2}$ in the following way. The element $H_{d,f}$ with $d \in [D]$ and $f \in I_{q'}^2$ equals 1 if there is an edge of colour d starting in vertex $f(1)$ and ending in vertex $f(2)$, and equals to 0 otherwise. The group Σ in Definition 36 is the permutation group on $[q']$ containing all permutations σ that preserve the partition $\mathcal{P}$, that is, $\sigma(P_i) = P_i$ for all $i \in [t]$. The Σ-adapted $[q']$-complex Φ on vertex set $[n]$ is used to encode where copies of $H \in \mathcal{H}$ are potentially allowed to be mapped in a graph on $[n]$. For us this just means that the partitions $\mathcal{P}$ and $\mathcal{P}'$ have to be respected: We let Φ be the the complete $\mathcal{P}$-partite $[q']$-complex on $[n]$ with partition $\mathcal{P}'$, that is, Φ firstly contains every (ordered) edge $(v_1, \ldots, v_{q'})$ with pairwise distinct vertices such that $v_i \in P_i'$ whenever $i \in P_i$; and secondly contains all (ordered) edges in the down-closure of any such edge $(v_1, \ldots, v_{q'})$. The (directed) q'-graph $\Phi_{q'}$ then consists of all the q'-edges of Φ. Similarly, Φ_2 contains all underlying directed 2-edges, that is, all directed 2-edges within each part P_i' and all directed 2-edges between any two parts P_i' and P_j' with $i < j$ that are directed towards P_j'. Further, $\mathcal{H}(\Phi)$ denotes the set of coloured labelled copies H^* on $[n]$ of digraphs $H \in \mathcal{H}$ corresponding to edges $\phi \in \Phi_{q'}$, that is, vertices of H from P_i are mapped to vertices from P_i' in H^* (and edges of H are mapped to edges in Φ_2). The host graph G then is defined as a coloured multi-digraph, in which every edge from Φ_2 may appear multiple times and in different colours, which is consistent with our Definition 17.

So now it remains to verify the three conditions of Theorem 19, namely that G is $\mathcal{H}$-divisible in Φ and $(\mathcal{H}, c, \omega)$-regular in Φ, and that all (Φ, G^H) are (ω, h)-extendable. These definitions are related to Definitions 18, 19, and 20, but we need to make the correspondence precise.

For $\mathcal{H}$-divisibility we first explain the notion of the degree vector $G(\psi)^* \in \mathbb{N}_0^{[D]\times I_2^i}$. In our setting, ψ is an injective map from $[i]$ to $[n]$, where $i = 0, 1, 2$. For $i = 0$, the only choice is the trivial one, $\psi = \emptyset_n$, and also I_2^0 only contains the trivial map $\emptyset_2$. Hence, $G(\emptyset_n)^*$ has only one coordinate $G(\emptyset_n)^*{}_{d,\emptyset_2}$ for each colour $d \in [D]$, which equals the total number of edges of colour d in G. For $i = 1$, a given ψ corresponds to a choice of a vertex $v \in [n]$ and I_2^1 contains the two elements $1 \mapsto 1$ and $1 \mapsto 2$. So in this case, $G(v)^*$ is a list of $2D$ numbers encoding the in- and out-degree of each colour at vertex v in the D colours. For $i = 2$, a given ψ corresponds to a choice of an ordered pair uv of vertices $[n]$ and I_2^2 contains the two possible permutations on 2 elements. In this case, $G(uv)^*$ is a list of $2D$ numbers encoding the number of edges going between u and v in each colour and each of the two directions. The degree vector $H(\theta)^*$ is defined analogously for $H \in \mathcal{H}$ and θ an injective map from $[i]$ to $[q']$ with $i = 0, 1, 2$.

Let us clarify one minor point in the definition of $\mathcal{H}$-divisibility in Definition 36:

- For $\mathbf{i}' \in \mathbb{N}_0^t$ we let $H\langle \mathbf{i}' \rangle = \langle H(\theta)^* : i_{\mathcal{P}}(\theta) = \mathbf{i}' \rangle$. We say G is $\mathcal{H}$-divisible (in Φ) if $G(\psi)^* \in H\langle \mathbf{i}' \rangle$ whenever $i_{\mathcal{P}'}(\psi) = \mathbf{i}'$.

should read

- For $\mathbf{i}' \in \mathbb{N}_0^t$ we let $\mathcal{H}\langle \mathbf{i}' \rangle = \langle H(\theta)^* : H \in \mathcal{H}, i_{\mathcal{P}}(\theta) = \mathbf{i}' \rangle$. We say G is $\mathcal{H}$-divisible (in Φ) if $G(\psi)^* \in \mathcal{H}\langle \mathbf{i}' \rangle$ whenever $i_{\mathcal{P}'}(\psi) = \mathbf{i}'$.

What this says is the following. For a fixed index vector $\mathbf{i}' \in \mathbb{N}_0^t$, specifying how many vertices we want to fix in each part of our partition, we let i be the sum $\sum \mathbf{i}'$ of the entries of $\mathbf{i}'$. We then consider all injective maps θ from $[i]$ to $[q']$ with this index vector, that is, θ sends exactly $\mathbf{i}'_j$ vertices to P_j. Each of these θ (specifying which vertices we are fixing) and each $H \in \mathcal{H}$ then gives us a degree vector $H(\theta)^*$. The lattice generated by all these vectors is denoted by $\mathcal{H}\langle \mathbf{i}' \rangle$. The graph G then is $\mathcal{H}$-divisible if for any mapping ψ from $[n]$ to some $[i]$ (fixing i vertices of G) the following holds. We let $\mathbf{i}' \in \mathbb{N}_0^t$ be the index vector of ψ. We then require that the degree vector $G(\psi)^*$ is in $\mathcal{H}\langle \mathbf{i}' \rangle$.

Accordingly, this definition has to be checked for each "$\mathbf{i}' \in \mathbb{N}_0^t$" and each ψ with "$i_{\mathcal{P}'}(\psi) = \mathbf{i}'$". For us, there are three cases to distinguish, depending on whether $\sum \mathbf{i}'$ is 0, 1, or 2; for higher values of $\sum \mathbf{i}'$ the condition in Definition 36 is void. We now show that these three cases correspond to 0-divisibility, 1-divisibility, and 2-divisibility in Definition 18, respectively.

- First suppose that $\sum \mathbf{i}' = 0$. Then the condition requires that $G(\emptyset)^*$ is a linear combination of the degree vectors $H(\emptyset)^*$ with $H \in \mathcal{H}$ with integer coefficients m_H. This means exactly that the number of edges of colour d in G is $\sum_{H \in \mathcal{H}} m_H c_{d,H}$, where $c_{d,H}$ is the number of edges of colour d in H, which is 0-divisibility.
- If $\sum \mathbf{i}' = 1$, the condition mandates the following for each choice of vertex $v \in [n]$. Let $j \in [t]$ be such that $v \in P'_j$. The degree vector $G(v)^*$ has to be a linear combination with integer coefficients of the degree vectors $H(\theta)^*$, with $H \in \mathcal{H}$ and $\theta : [1] \to [q']$ such that $\theta(1) \in P_j$. This is exactly 1-divisibility.
- If $\sum \mathbf{i}' = 2$, then the condition asks that for each choice of a pair uv from $[n]$ the following holds. Let $j, j' \in [t]$ be such that $u \in P'_j$, $v \in P'_{j'}$. The degree vector $G(uv)^*$ has to be a linear combination with integer coefficients of the degree vectors $H(\theta)^*$ with $H \in \mathcal{H}$ and $\theta : [2] \to [q']$ such that $\theta(1) \in P_j$ and $\theta(2) \in P_{j'}$. This means that an edge of colour d can appear in G between P'_j and $P'_{j'}$ only if there is an edge of colour d between P_j and $P_{j'}$ in some $H \in \mathcal{H}$. Further, since for unoriented colours d with colour index (j, j) the edges of colour d in each $H \in \mathcal{H}$ come in pairs that form unoriented 2-cycles, the same has to be true for edges of colour d in G. This is exactly 2-divisibility.

Let us now turn to $(\mathcal{H}, c, \omega)$-regularity, which requires that for each $H \in \mathcal{H}$ and each coloured copy ϕH of H in G with $\phi \in \Phi_{q'}$ (that is, ϕ maps vertices from P_j to P'_j) we can define a weight

$y^H_\phi \in [\omega n^{2-q}, \omega^{-1} n^{2-q}]$ so that for each edge of $e \in E(G)$, when we sum these weights over all copies ϕH which use e we get $(1 \pm c)$. Since in our setting, any coloured copy of any $H \in \mathcal{H}$ maps vertices from P_j to P'_j, this is identical to Definition 19.

The last notion is that of extendability for all $H \in \mathcal{H}$. By a remark before Definition 37, the required (ω, h)-extendability of (Φ, G^H) follows from $(\sqrt[q'h]{\omega}, h)$-vertex-extendability of (Φ, G). According to Definition 37 $(\sqrt[q'h]{\omega}, h)$-vertex-extendability of (Φ, G) mandates the following. For every vertex $x \in [q']$ and for any choice of vertex disjoint sets A_i with $i \in [q'] \setminus \{x\}$ of size at most h in G such that for every collection of $v_i \in A_i$ we have that $(i \mapsto v_i : i \in [q'] \setminus \{x\}) \in \Phi$ (which for us just means that $v_i \in P'_j$ where j is such that $i \in P_j$), there have to be at least $\sqrt[q'h]{\omega} n$ vertices $v \in \Phi^0_x$ (for us Φ^o_x is the part $P'_{j'}$ such that $x \in P_{j'}$) satisfying the following properties. Firstly, $(i \mapsto v_i : i \in [q']) \in \Phi$ whenever $v_x = v$ and $v_i \in A_i$ for $i \neq x$, which, since $v \in \Phi^0_x$ is always satisfied. Secondly, for every colour d and every edge θ of H which is of colour d and contains x, we have that all edges $(i \mapsto v_i : i \in [2])$ are edges of G in colour d whenever $v_j = v$ for $j = \theta^{-1}(x)$ and $v_i \in A_{\theta(i)}$ for all $i \neq j$. This means that for any edge xi in H of colour d we have that all edges vv_i with $v_i \in A_i$ of colour d are present in G; and that for any edge ix in H of colour d we have that all edges $v_i v$ with $v_i \in A_i$ of colour d are present in G. This corresponds exactly to Definition 20.

Hence the host graph G from Theorem 21 satisfies the conditions of Theorem 19 and thus G has an $\mathcal{H}$-decomposition as claimed.

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References

[1] P. Allen, J. Böttcher, D. Clemens, and A. Taraz, *Perfectly packing graphs with bounded degeneracy and many leaves*, arXiv:1906.11558.

[2] P. Allen, J. Böttcher, J. Hladký, and D. Piguet, *Packing degenerate graphs*, Adv. Math. **354** (2019), 106739.

[3] J. Balogh and C. Palmer, *On the Tree Packing Conjecture*, SIAM J. Discrete Math. **27** (2013), no. 4, 1995–2006.

[4] J. Böttcher, J. Hladký, D. Piguet, and A. Taraz, *An approximate version of the tree packing conjecture*, Israel J. Math. **211** (2016), no. 1, 391–446.

[5] A. Ferber, C. Lee, and F. Mousset, *Packing spanning graphs from separable families*, Israel J. Math. **219** (2017), no. 2, 959–982.

[6] A. Ferber and W. Samotij, *Packing trees of unbounded degrees in random graphs*, J. Lond. Math. Soc. (2) **99** (2019), no. 3, 653–677.

[7] D. A. Freedman, *On tail probabilities for martingales*, Ann. Probability **3** (1975), 100–118.

[8] S. Glock, D. Kühn, A. Lo, and D. Osthus, *The existence of designs via iterative absorption: hypergraph F-designs for arbitrary F*, arXiv:1611.06827, version 3.

[9] S. Glock, F. Joos, J. Kim, D. Kühn, and D. Osthus, *Resolution of the Oberwolfach problem*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 8, 2511–2547.

[10] A. Gyárfás and J. Lehel, *Packing trees of different order into K_n*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam, 1978, pp. 463–469.

[11] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience, 2000.

[12] F. Joos, J. Kim, D. Kühn, and D. Osthus, *Optimal packings of bounded degree trees*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 12, 3573–3647.

[13] P. Keevash, *Coloured and directed designs*, arXiv:1807.05770v2.

[14] ———, *The existence of designs*, arXiv:1401.3665.

[15] ———, *The existence of designs II*, arXiv:1802.05900.

[16] ———, *Coloured and directed designs*, Building Bridges II, Bolyai Society Mathematical Studies, vol. 28, Springer, Berlin, 2019, pp. 279–315.

[17] P. Keevash and K. Staden, *The generalised Oberwolfach problem*, arXiv:2004.09937.

[18] ———, *Ringel's tree packing conjecture in quasirandom graphs*, arXiv:2004.09947.

[19] J. Kim, D. Kühn, D. Osthus, and M. Tyomkyn, *A blow-up lemma for approximate decompositions*, Trans. Amer. Math. Soc. **371** (2019), no. 7, 4655–4742.

[20] J. Komlós, G. Sárközy, and E. Szemerédi, *Spanning trees in dense graphs*, Combin. Probab. Comput. **10** (2001), no. 5, 397–416.

[21] C. McDiarmid, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), London Math. Soc. Lecture Note Ser., vol. 141, Cambridge Univ. Press, Cambridge, 1989, pp. 148–188.

[22] S. Messuti, V. Rödl, and M. Schacht, *Packing minor-closed families of graphs into complete graphs*, J. Combin. Theory Ser. B **119** (2016), 245–265.

[23] R. Montgomery, A. Pokrovskiy, and B. Sudakov, *A proof of Ringel's conjecture*, arXiv:2001.02665.

[24] G. Ringel, *Problem 25*, Theory of Graphs and its Applications (Proc. Int. Symp. Smolenice 1963), Czech. Acad. Sci., Prague, 1963.

[25] V. Rödl, *On a packing and covering problem*, European J. Combin. **6** (1985), no. 1, 69–78.

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