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# The Švarc-Milnor lemma for braids and area-preserving diffeomorphisms

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# The Švarc-Milnor lemma for braids and area-preserving diffeomorphisms

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**Abstract**

We prove a number of new results on the large-scale geometry of the $L^p$-metrics on the group of area-preserving diffeomorphisms of each orientable surface. Our proofs use in a key way the Fulton-MacPherson type compactification of the configuration space of $n$ points on the surface due to Axelrod-Singer and Kontsevich. This allows us to apply the Švarc-Milnor lemma to configuration spaces, a natural approach which we carry out successfully for the first time. As sample results, we prove that all right-angled Artin groups admit quasi-isometric embeddings into the group of area-preserving diffeomorphisms endowed with the $L^p$-metric, and that all Gambaudo-Ghys quasi-morphisms on this metric group coming from the braid group on $n$ strands are Lipschitz. This was conjectured to hold, yet proven only for low values of $n$ and the genus $g$ of the surface.

## Contents



# 1 Introduction and main results

## 1.1 Introduction

The $L^2$-length of a path of volume-preserving diffeomorphisms, which describes a time-dependent flow of an ideal incompressible fluid, corresponds to the hydrodynamic action of the flow in the same way as the length of a path in a Riemannian manifold corresponds to its energy (cf. [50]). Indeed, it is the length of this path with respect to the formal right-invariant Riemannian metric on the group $\mathcal{G}$ of volume preserving diffeomorphisms introduced by Arnol'd in [1]. The $L^1$-length of the same path has a dynamical interpretation as the average length of a trajectory of a point under the flow.

Therefore, following the principle of least action, it makes sense to consider the infimum of the lengths of paths connecting two fixed volume-preserving diffeomorphisms. This gives rise to a right-invariant distance function (metric) on $\mathcal{G}$. Taking the identity transformation as the initial point, Arnol'd observes that a path whose $L^2$-length is minimal (and equal to the distance) necessarily solves the Euler equation of an ideal incompressible fluid.

It follows from works of Ebin and Marsden [21] that for diffeomorphisms in $\mathcal{G}$ that are $C^2$-close to the identity, the infimum is indeed achieved. Further, more global results on the corresponding Riemannian exponential map were obtained in [19],[51] (see also [20]). In [49, 50] Shnirel'man showed, among a number of surprising facts about this subject, that in the case of the ball of dimension 3, the diameter of the $L^2$-metric is bounded. This result is conjectured to hold for all compact simply connected manifolds of dimension 3 or larger (see [23, 38, 2]), while its analogue in the non-simply-connected case is false [23, 7]. Furthermore, Shnirel'man has conjectured that for compact manifolds of dimension 2, the $L^2$-diameter is infinite.

Shnirel'man's conjecture, and its analogues for $L^p$-metrics, with $p \geq 1$ are by now proven. It follows from results of Eliashberg and Ratiu [23] that on compact surfaces (possibly with boundary) other than $T^2$ and $S^2$, Shnirel'man's conjecture holds for all $p \geq 1$. Their arguments rely on the Calabi homomorphism Cal [16] from the compactly supported Hamiltonian group $\mathrm{Ham}_c(M, \sigma)$ to the real numbers in the case of a surface $M$ with non-empty boundary ($\sigma$ is the area form), and on non-trivial first cohomology combined with trivial center of the fundamental group in the closed case. For the two-torus $T^2$ Shnirel'man's conjecture holds by [14, Appendix A]. Finally, the case of $S^2$ was settled in [14] by means of differential forms on the configuration space related to the cross-ratio map. In [43] the second author gave a new uniform proof of Shnirel'man's conjecture for all compact surfaces.

The methods that were used to prove Shnirel'man's conjecture are two-dimensional in nature, and have to do with braiding of trajectories of time-dependent two-dimensional Hamiltonian flows (in extended phase space). Indeed, Shnirel'man has proposed to use relative rotation numbers to bound from below the $L^2$-lengths of two-dimensional Hamiltonian paths in [50]. This direction is related to the method of [23] by a theorem of Fathi [24] and Gambaudo and Ghys [27] (see also [48, 31, 33]). This theorem shows that the Calabi homomorphism is proportional to the relative rotation number of the trajectories of two distinct points in the two-disc $\mathbb{D}$ under a Hamiltonian flow, averaged over the configuration space of ordered pairs of distinct points $(x_1, x_2)$ in the two-disc.

This line of research was notably pursued in [29], and further in [4], [18], [7], [9], [39], [14] obtaining quasi-isometric and bi-Lipschitz embeddings of various groups (right-angled Artin groups and additive groups of finite-dimensional real vector spaces) into $\mathrm{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, into $\ker(\mathrm{Cal}) \subset \mathrm{Ham}_c(\mathbb{D}^2, dx \wedge dy)$, and into $\mathrm{Ham}(S^2, \sigma)$ endowed with their respective $L^p$-metrics (see [10] for similar embedding results on manifolds with a sufficiently complicated fundamental group). In all cases, the key technical estimate is an upper bound, via the $L^p$-length of an isotopy of volume-preserving diffeomorphisms, of the average, over all points in a configuration space of the manifold, of the word length in the fundamental group of the configuration space of the trace of the point under the induced isotopy (closed up to a loop by a system of short paths on the configuration space).

Such estimates were initially produced by means of analyzing relative rotation numbers of pairs of braids, or quadruples as in [14]. However, in the case of a single braid, as observed by Polterovich, a simpler estimate is possible via the Švarc-Milnor lemma [22, 54, 44]. A similar estimate in the case of braids on more than one strand is not as readily available, because the configuration space $X_n(M)$ of $n$ points on $M$ is not a compact metric space. In [43], the second author has introduced a new complete metric on $X_n(M)$ which has allowed for a new proof of Shnirel'man's conjecture. However, with this metric $X_n(M)$ still can not be considered to be a compact metric space from the point of view of the Švarc-Milnor lemma.

In this paper we show how to successfully carry out the strategy of the Švarc-Milnor lemma for configuration spaces. While hints of a similar approach can be discerned in [26] in the special case of the two-disk and of double collisions, it was not known earlier to be applicable in the general context discussed herein. Specifically, we consider a metric on $X_n(M)$ coming from a natural compactification

$\overline{X}_n(M)$ thereof, which is a compact geodesic metric space. This compactification is equivariant under the action of the diffeomorphism group $\mathrm{Diff}(M)$ on $X_n(M)$ in the sense that the action extends naturally to the compactification[1]. Furthermore, the map $\pi_1(X_n(M)) \to \pi_1(\overline{X}_n(M))$ induced by the inclusion is an isomorphism. Hence we may apply the Švarc-Milnor lemma to $\overline{X}_n(M)$. This, and further comparison to the metric from [43] allows us to prove our main estimate.

The compactification we use was introduced by Axelrod-Singer in [3] and by Kontsevich in [40, 41], inspired by the Fulton-MacPherson compactification in algebraic geometry [25]. The compactification $\overline{X}_n(M)$ is roughly speaking a certain positive oriented blow-up of $M^n$ along its multi-diagonals. For instance, one part of its codimension 1 boundary stratum is identified with the disjoint union of spaces of the form $N_1(D_{ij}(M))|_{D^0_{ij}(M)}$ where $N_1$ is the unit normal bundle, $D_{ij}(M) \subset M^n$ is the submanifold of points $(x_1, \ldots, x_n)$ where $x_i = x_j$ and $D^0_{ij}(M) \subset D_{ij}(M)$ is the open dense subset where $x_k \neq x_i = x_j$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$. Note that this is an $S^1$-bundle. Intuitively, from a physical perspective, this means that one resolves a double collision of points by recording the collision point and the direction in which they have collided. Other parts of the codimension 1 stratum correspond to simple $k$-tuple collisions, and correspond to $S^{2k-3}$-bundles normal to $k$-diagonals. Higher strata correspond to more complicated collisions modeled by suitable graphs. It will, however, be technically most convenient for us to use a model of this compactification recently constructed directly as a subspace of a Euclidean space by Sinha [52]. We describe this construction in Section 2 below.

Finally, we observe that lower bounds on the average word length can often be provided by quasimorphisms - functions that are additive with respect to the group multiplication - up to an error which is uniformly bounded (as a function of two variables). The quasimorphisms we consider here were introduced and studied by Gambaudo and Ghys in the beautiful paper [28] (see also [8, 45, 46, 47, 6, 14, 13]). These quasimorphisms essentially appear from invariants of braids traced out by the action of a Hamiltonian path on an ordered $n$-tuple of distinct points in the surface (suitably closed up), averaged over the configuration space $X_n(M)$ of $n$-tuples of distinct points on the surface $M$. As one of our results, we prove that all homogeneous Gambaudo-Ghys quasi-morphisms are Lipschitz in the $L^p$-metric for all $p \geq 1$. This subsumes all previous results in this direction and provides a maximally general result. It also contrasts a recent result of Khanevsky according to which none of these quasi-morphisms are continuous in Hofer's metric [37].

We prove further stronger results on the large-scale geometry on the $L^p$-metric on $\mathcal{G}$. In particular, we provide bi-Lipschitz group monomorphisms of $\mathbb{R}^m$ endowed with the standard (say Euclidean) metric into $(\mathcal{G}, d_{L^p})$ for each positive integer $m$ and each $p \geq 1$. Finally, our methods combined with an argument of Kim-Koberda [39] (cf. Crisp-Wiest [18], Benaim-Gambaudo [4]) show the existence of quasi-isometric group monomorphisms from each right-angled Artin group to $(\mathcal{G}, d_{L^p})$ for each $p \geq 1$. We note that this was previously known only for $\mathbb{D}^2$ and $S^2$ [39, 14].

Let $M_1 \hookrightarrow M_2$ be a measure preserving embedding of surfaces. It is an open question if the induced monomorphism $\mathrm{Diff}_0(M_1, \sigma_1) \hookrightarrow \mathrm{Diff}_0(M_2, \sigma_2)$ is a quasi-isometric embedding. Note that our results provide a partial positive answer to this problem, see Remark 1.3.5.


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[1] It is curious to note that the same is *not* true for the action of $\mathrm{Homeo}(M)$ as was recently proven [42].

## 1.2 Preliminaries

### 1.2.1 The $L^p$-metric

Let $M$ denote a smooth oriented manifold without boundary that is either closed, or $M = X \setminus \partial X$ for a compact manifold $X$. Let $M$ be endowed with a Riemannian metric $g$ and smooth measure $\mu$ (given by an orientation on $M$ and volume form, which in our case of a surface $M$ is an area form $\sigma$). We require that $g$ and $\mu$ extend continuously to $X$ in the second case. Finally denote by

$$\mathcal{G} = \mathrm{Diff}_{c,0}(M, \mu)$$

the identity component of the group of compactly supported diffeomorphisms of $M$ preserving the smooth measure $\mu$. In other words, if $M = X \setminus \partial X$, it is the identity component of the group of measure preserving diffeomorphisms of $X$ fixing point-wise a neighbourhood of $\partial X$.

Fix $p \geq 1$. For a smooth isotopy $\{\phi_t\}_{t\in[0,1]}$, from $\phi_0 = 1$ to $\phi_1 = \phi$, we define the $L^p$-length by

$$l_p(\{\phi_t\}) = \int_0^1 \left(\frac{1}{\mathrm{vol}(M,\mu)} \cdot \int_M |X_t|^p d\mu\right)^{\frac{1}{p}} dt\,,$$

where $X_t = \frac{d}{dt'}|_{t'=t}\phi_{t'} \circ \phi_t^{-1}$ is the time-dependent vector field generating the isotopy $\{\phi_t\}$, and $|X_t|$ is its length with respect to the Riemannian structure on $M$. As is easily seen by a displacement argument, the $L^p$-length functional determines a non-degenerate norm on $\mathcal{G}$ by the formula

$$d_p(\mathbf{1}, \phi) = \inf\ l_p(\{\phi_t\}).$$

This in turn defines a right-invariant metric on $\mathcal{G}$ by the formula

$$d_p(\phi_0, \phi_1) = d_p(\mathbf{1}, \phi_1{\phi_0}^{-1}).$$

*Remark* 1.2.1. Consider the case $p = 1$. It is easy to see that the $L^1$-length of an isotopy is equal to the average Riemannian length of the trajectory $\{\phi_t(x)\}_{t\in[0,1]}$ (over $x \in M$, with respect to $\mu$). Moreover for each $p \geq 1$, by Jensen's (or Hölder's) inequality, we have

$$l_p(\{\phi_t\}) \geq l_1(\{\phi_t\}).$$

Denote by $\widetilde{\mathbf{1}}$ the identity element of the universal cover $\widetilde{\mathcal{G}}$ of $\mathcal{G}$. Similarly one has the $L^p$-pseudo-norm (that induces the right-invariant $L^p$-pseudo-metric) on $\widetilde{\mathcal{G}}$, defined for $\widetilde{\phi} \in \widetilde{\mathcal{G}}$ as

$$d_p(\widetilde{\mathbf{1}}, \widetilde{\phi}) = \inf\ l_p(\{\phi_t\}),$$

where the infimum is taken over all paths $\{\phi_t\}$ in the class of $\widetilde{\phi}$. Clearly $d_p(\mathbf{1}, \phi) = \inf d_p(\widetilde{\mathbf{1}}, \widetilde{\phi})$, where the infimum runs over all $\widetilde{\phi} \in \widetilde{\mathcal{G}}$ that map to $\phi$ under the natural epimorphism $\widetilde{\mathcal{G}} \to \mathcal{G}$.

Up to bi-Lipschitz equivalence of metrics ($d$ and $d'$ are equivalent if $\frac{1}{C}d \leq d' \leq Cd$ for a certain constant $C > 0$) the $L^p$-metric on $\mathcal{G}$ (and its pseudo-metric analogue on $\widetilde{\mathcal{G}}$) is independent of the choice Riemannian structure and of the volume form $\mu$ on $M$. In particular, the question of boundedness or unboundedness of the $L^p$-metric enjoys the same invariance property.

**Terminology:** For a positive integer $n$, we use $A, B, C > 0$ as generic notation for positive constants that depend only on $M, \mu, g$ and $n$.

### 1.2.2 Quasimorphisms

Some of our results have to do with the notion of a quasimorphism. Quasimorphisms are a helpful tool for the study of non-abelian groups, especially those that admit few homomorphisms to $\mathbb{R}$. A quasimorphism $r : G \to \mathbb{R}$ on a group $G$ is a real-valued function that satisfies

$$r(xy) = r(x) + r(y) + b_r(x, y),$$

for a function $b_r : G \times G \to \mathbb{R}$ that is uniformly bounded:

$$\delta(r) := \sup_{G\times G} |b_r| < \infty.$$

A quasimorphism $\overline{r} : G \to \mathbb{R}$ is called *homogeneous* if $\overline{r}(x^k) = k\overline{r}(x)$ for all $x \in G$ and $k \in \mathbb{Z}$. In this case, it is additive on each pair $x, y \in G$ of commuting elements: $r(xy) = r(x) + r(y)$ if $xy = yx$.

For each quasimorphism $r : G \to \mathbb{R}$ there exists a unique homogeneous quasimorphism $\overline{r}$ that differs from $r$ by a bounded function:

$$\sup_{G} |\overline{r} - r| < \infty.$$

It is called the *homogenization* of $r$ and satisfies

$$\overline{r}(x) = \lim_{n\to\infty} \frac{r(x^n)}{n}.$$

Denote by $Q(G)$ the real vector space of homogeneous quasimorphisms on $G$.

For a finitely-generated group $G$, with finite symmetric generating set $S$, define the word norm $|\cdot|_S : G \to \mathbb{Z}_{\geq 0}$ by

$$|g|_S = \min\{k \,|\, g = s_1 \cdot \ldots \cdot s_k,\ \forall\, 1 \leq j \leq k,\ s_j \in S\}$$

for $g \in G$. This is a norm on $G$, and as such it induces a right-invariant metric $d_S : G \times G \to \mathbb{Z}_{\geq 0}$ by $d_S(f, g) = |gf^{-1}|_S$. This metric is called the word metric. In this setting, any quasimorphism $r : G \to \mathbb{R}$ is controlled by the word norm. Indeed, for all $g \in G$,

$$|r(g)| \leq \left(\delta(r) + \max_{s\in S} |r(s)|\right) \cdot |g|_S.$$

When a specific symmetric generating set $S$ for $G$ can be fixed, we will usually denote $|\cdot|_S$ by $|\cdot|_G$.

We refer to [17] for more information about quasimorphisms.

### 1.2.3 Configuration spaces and braid groups

For a manifold $M$, which in this paper is usually of dimension 2, the configuration space $X_n(M) \subset M^n$ of $n$-tuples of points on $M$ is defined as

$$X_n(M) = \{(x_1, \ldots, x_n)|\ x_i \neq x_j,\ 1 \leq i < j \leq n\}.$$

That is

$$X_n(M) = M^n \setminus \bigcup_{1\leq i<j\leq n} D_{ij}$$

where for $1 \leq i < j \leq n$, the partial diagonal $D_{ij} \subset M^n$ is defined as $D_{ij} = \{(x_1, \ldots, x_n)|\ x_i = x_j\}$. Note that $D_{ij}$ is a submanifold of $M^n$ of codimension $\dim M$.

Finally, if $\dim M = 2$, we define the pure braid group of $M$ as

$$P_n(M) = \pi_1(X_n(M)).$$

Noting that the symmetric group $S_n$ on $n$ elements acts on $X_n(M)$, we form the quotient $C_n(M) = X_n(M)/S_n$ and define the full braid group of $M$ as

$$B_n(M) = \pi_1(C_n(M)).$$

We note that $P_n(M)$ and $B_n(M)$ enter the exact sequence $1 \to P_n(M) \to B_n(M) \to S_n \to 1$. In particular $P_n(M)$ is a normal subgroup of $B_n(M)$ of finite index. We refer to [36] for further information about braid groups.

### 1.2.4 Short paths and the Gambaudo-Ghys construction

Let $M$ be a compact oriented surface. Given a real valued quasimorphism $r$ on $P_n(M) = \pi_1(X_n(M), q)$ for a fixed basepoint $q \in X_n(M)$ there is a natural way to construct a real valued quasimorphism on the universal cover $\widetilde{\mathcal{G}}$ of the group $\mathcal{G} = \mathrm{Diff}_{c,0}(M, \sigma)$ of area preserving diffeomorphisms of the surface $M$. We shall see that in the case of $M \neq T^2$ this induces a quasimorphism on $\mathcal{G}$ itself, because the fundamental group of $\mathcal{G}$ is finite. The same is true for $M = T^2$ where we consider the group $\mathcal{G} = \mathrm{Ham}(M, \sigma)$ of Hamiltonian diffeomorphisms instead. This is not a restrictive condition from the viewpoint of large-scale geometry, since by a small modification of [14, Proposition A.1], the inclusion $(\mathrm{Ham}(T^2, \sigma), d_{L^p}) \hookrightarrow (\mathrm{Diff}_0(T^2, \sigma), d_{L^p})$ is a quasi-isometry for all $p \geq 1$.

The construction is carried out by the following steps (cf. [28, 45, 6]).

1. For all $x \in X_n(M) \setminus Z$, with $Z$ a closed negligible subset (e.g. a union of submanifolds of positive codimension) choose a smooth path $\gamma(x) : [0, 1] \to X_n(M)$ between the basepoint $q \in X_n(M)$ and $x$. Make this choice continuous in $X_n(M) \setminus Z$. We first choose a system of paths on $M$ itself. Then we consider the induced coordinate-wise paths in $M^n$, and pick $Z$ to ensure that these induced paths actually lie in $X_n(M)$. After choosing the system of paths $\{\gamma(x)\}_{x \in X_n(M) \setminus Z}$ we extend it measurably to all $x \in X_n(M)$ (obviously, no numerical values computed in the paper will depend on this extension). We call the resulting choice a "system of short paths".

2. Given a path $\{\phi_t\}_{t \in [0,1]}$ in $\mathcal{G}$ starting at $Id$, and a point $x \in X_n(M)$ consider the path $\{\phi_t \cdot x\}$, to which we then catenate the corresponding short paths. That is consider the loop

$$\lambda(x, \{\phi_t\}) := \gamma(x) \# \{\phi_t \cdot x\} \# \gamma(y)^{-1}$$

in $X_n(M)$ based at $q$, where $^{-1}$ denotes time reversal. Hence we obtain for each $x \in X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)$ an element $[\lambda(x, \{\phi_t\})] \in \pi_1(X_n(M), q)$ (or rather for each $x \in X_n(M)$ after the measurable extension in Step 1).

3. Consequently applying the quasimorphism $r : \pi_1(X_n(M), q) \to \mathbb{R}$ we obtain a measurable function $f : X_n(M) \to \mathbb{R}$. Namely $f(x) = r([\lambda(x, \{\phi_t\})])$. The quasimorphism $\Phi$ on $\widetilde{\mathcal{G}}$ is defined by

$$\Phi([\{\phi_t\}]) = \int_{X_n(M)} f \, d\mu^{\otimes n}.$$

It is immediate to see that this function is well-defined by topological reasons. The quasimorphism property follows by the quasimorphism property of $r$ combined with finiteness of volume. The fact that the function $f$ is absolutely integrable can be shown to hold a-priori by a reduction to the case of the disc. We note, however, that by Tonelli's theorem this fact follows as a by-product of the proof of our main theorem, and therefore requires no additional proof.

4. Of course this quasimorphism can be homogenized, to obtain a homogeneous quasimorphism $\overline{\Phi}$.

*Remark* 1.2.2. If $M \neq S^2$ and $M \neq T^2$, then $\pi_1(\mathcal{G}) = 0$ and for $M = T^2$ the same is true for $\mathcal{G} = \mathrm{Ham}(M, \sigma)$. In the case $M = S^2$, by the result of Smale [53] $\pi_1(\mathcal{G}) = \mathbb{Z}/2\mathbb{Z}$. Hence the quasimorphisms descend to quasimorphisms on $\mathcal{G}$, e.g. by minimizing over the two-element fibers of the projection

$\widetilde{\mathcal{G}} \to \mathcal{G}$. For $\overline{\Phi}$, the situation is easier since by homogeneity it vanishes on $\pi_1(\mathcal{G}) \subset Z(\widetilde{\mathcal{G}})$, and therefore depends only on the image in $\mathcal{G}$ of an element in $\widetilde{\mathcal{G}}$. We keep the same notations for the induced quasimorphisms.

### 1.3 Main results

Recall that we work with the group $\mathcal{G} = \mathrm{Diff}_{c,0}(M, \sigma)$ for $(M, \sigma)$ a compact oriented surface with an area form. Our main technical result is the following.

**Theorem 1.** *For an isotopy $\overline{\phi} = \{\phi_t\}$ in $\mathcal{G}$, the average word norm of a trajectory $\lambda(x, \overline{\phi})$ is controlled by the $L^1$-length of $\overline{\phi}$ :*

$$W(\overline{\phi}) = \int_{X_n(M)\setminus Z \cup (\phi_1)^{-1}(Z)} |[\lambda(x, \overline{\phi})]|_{P_n(M)} \, d\mu^{\otimes n}(x) \leq A \cdot l_1(\overline{\phi}) + B,$$

*for certain constants $A, B > 0$.*

*Remark* 1.3.1. Note that $W(\overline{\phi})$ depends only on the class $\widetilde{\phi} = [\overline{\phi}] \in \widetilde{\mathcal{G}}$ of $\overline{\phi}$ in the universal cover $\widetilde{\mathcal{G}}$ of $\mathcal{G}$. Hence for all closed surfaces of positive genus $W(\overline{\phi})$ does not depend on the chosen isotopy $\overline{\phi}$, but only on the diffeomorphism $\phi_1$. This is because $\mathcal{G}$ is simply connected.

Theorem 1 has a number of consequences concerning the large-scale geometry of the $L^1$-metric on $\mathcal{G}$. Firstly, as any quasimorphism on a finitely generated group is controlled by the word norm, we immediately obtain the following statement.

**Corollary 1.** *The homogenization $\overline{\Phi}$ of each Gambaudo-Ghys quasimorphism $\Phi$ satisfies*

$$|\overline{\Phi}(\phi)| \leq C \cdot d_1(\phi, 1).$$

In particular, Corollary 1 implies that all Gambaudo-Ghys quasimorphisms are continuous in the $L^1$-metric (and hence in the $L^p$-metric, see Remark 1.3.4), a fact which was known so far only for the genus zero case (see [14, 7]) and for the higher genus case only when one considers Gambaudo-Ghys quasimorphisms coming from the fundamental group $P_1(M)$, see [7]. Note that none of these quasimorphisms are continuous in the Hofer metric by a recent result by Khanevsky [37].

By a theorem of Ishida [34] in the genus zero case and its generalisation to any compact oriented surface [12, Theorem 2.2] (see all well Brandenbursky-Kedra-Shelukhin [11] in the genus one case, and Brandenbursky [8] in the higher genus case), the image of the map $Q(P_n(M)) \xrightarrow{GG} Q(\mathcal{G})$ is infinite dimensional for $n \geq 4$. Thus $Q(\mathcal{G})$ is an infinite-dimensional vector space. Hence by Corollary 1 we obtain in particular the following.

**Corollary 2.** *The $L^1$-diameter of $\mathcal{G}$ is infinite.*

Let $Ent^k$ be the set of products of at most $k$ entropy zero diffeomorphisms. Theorem 1 in [12] and Corollary 1 imply the following.

**Corollary 3.** *For each positive integer $k$, the complement in $\mathcal{G}$ of the set $Ent^k$ contains a ball of any arbitrarily large radius in the $L^1$-metric. In particular, the set of non-autonomous diffeomorphisms contains a ball of any arbitrarily large radius in the $L^1$-metric.*

In what follows, we apply an argument of Kim-Koberda [39] (cf. Benaim-Gambaudo [4] and Crisp-Wiest [18]), and use Theorem 1 in order to obtain the following statement, which generalizes an answer to a question of Kapovich [35] in the case of $S^2$ [14] to arbitrary compact oriented surfaces.

**Corollary 4.** *The metric group $(\mathcal{G}, d_1)$ admits a quasi-isometric group embedding from each right-angled Artin group endowed with the word metric.*

*Proof.* Assume first that $M = \Sigma_g$ a closed surface of a positive genus. The $S^2$ case was already done in [35] and the case of surfaces with boundary will be discussed at the and of the proof. Let $n \in \mathbb{N}$. Let $D \subset \Sigma_g$ be a smoothly embedded open disc and $z_1, \ldots, z_n$ disjoint points in $D$. For each $i$, let $D_i \subset D$ be an embedded open disc centered at $z_i$ such that each short geodesic between two points in $D_i$ lies in $D_i$, and $D_i \cap D_j = \emptyset$ for each $i \neq j$. We denote by $\mathrm{Diff}(D; D_1, \ldots, D_n) < \mathcal{G}$ a group which contains all diffeomorphisms in $\mathcal{G}$ which are compactly supported in $D$ and act by identity on each $D_i$. This subgroup is equipped with the $L^1$-metric coming from $\mathcal{G}$.

**Lemma 1.** *Let $n > 4$. We identify $P_n$ with the pure mapping class group of the disc $D$ punctured at $z_1, \ldots, z_n$. Then the inclusion $P_n \to P_n(\Sigma_g)$ is a quasi-isometric embedding.*

*Proof.* Since $P_n$ is quasi-isometric to $B_n$ and $P_n(\Sigma_g)$ is quasi-isometric to $B_n(\Sigma_g)$, it is enough to show that the inclusion $B_n \to B_n(\Sigma_g)$ is a quasi-isometric embedding, where $B_n$ is identified with the mapping class group of the disc $D$ punctured at $z_1, \ldots, z_n$. By results of Goldberg [30] (c.f. Birman [5]) we have that $B_n$ is a subgroup of $B_n(\Sigma_g)$. In addition, the composition

$$B_n \xrightarrow{i} B_n(\Sigma_g) \xrightarrow{F} MCG_{g,n}$$

is injective, where $i$ is the inclusion, the map $F : B_n(\Sigma_g) \to MCG_{g,n}$ is the point pushing map and $MCG_{g,n}$ is the mapping class group of $\Sigma_g$ punctured at $z_1, \ldots, z_n$. Moreover, by a result of Hamenstadt [32, Theorem 2] the map $F \circ i : B_n \hookrightarrow MCG_{g,n}$ is a quasi-isometric embedding for $n > 4$. Since $F$ is Lipschitz, it follows follows that $i$ is a quasi-isometric embedding (see [10, Lemma 2.1]). □

Let us return to the proof of the corollary. Let

$$H : \mathrm{Diff}(D; D_1, \ldots, D_n) \to P_n < P_n(\Sigma_g),$$

where $H(\phi)$ is an element in $P_n(\Sigma_g)$ represented by a path $(\phi_t(z_1), \ldots, \phi_t(z_n))$ in the configuration space, and $\{\phi_t\}$ is any isotopy in $\mathcal{G}$ between the identity $\phi_0$ and $\phi := \phi_1$.

It follows from the results of Kim-Koberda [39] (c.f. Crisp-Wiest [18] and Benaim-Gambaudo [4]) that for each RAAG $\Gamma$ there exists $n$ and an embedding of $\Gamma$ into $\mathrm{Diff}(D; D_1, \ldots, D_n)$ whose composition with $H$ is a quasi-isometric embedding of $\Gamma$ into $P_n$. By Lemma 1, it gives as well a quasi-isometric embedding of $\Gamma$ into $P_n(\Sigma_g)$. Then in order to obtain a quasi-isometric embedding of $\Gamma$ into $\mathcal{G}$ it is enough to show that the map $H$ is large-scale Lipschitz whenever $n > 4$.

This fact follows from our main result, Theorem 1. Let us prove it. Set

$$X(D_1, \ldots, D_n) := \{(x_1, \ldots, x_n) \in X_n(\Sigma_g) | \, x_i \in D_i\}.$$

Let $\{\phi_t\}$ be any isotopy in $\mathcal{G}$ between the identity $\phi_0 = \mathbf{1}$ and $\phi := \phi_1$. It follows from Theorem 1 that there exist positive constants $A$ and $B$ which depend only on $n$ such that

$$\int_{X(D_1,\ldots,D_n)} |[\lambda(x,\phi)]|_{P_n(\Sigma_g)} \, d\mu^{\otimes n}(x) \leq A \cdot l_1(\phi) + B\,.$$

Note that for each $x \in X(D_1, \ldots, D_n)$ we have $[\lambda(x,\phi)] = H(\phi)$. Thus

$$\int_{X(D_1,\ldots,D_n)} |[\lambda(x,\phi)]|_{P_n(\Sigma_g)} \, d\mu^{\otimes n}(x) = \mathrm{vol}(X(D_1, \ldots, D_n)) \cdot |H(\phi)|_{P_n(\Sigma_g)},$$

and we obtain the result, i.e., $H$ is large-scale Lipschitz:

$$|H(\phi)|_{P_n(\Sigma_g)} \leq \frac{A}{\mathrm{vol}(X(D_1,\dots,D_n))} \cdot l_1(\phi) + \frac{B}{\mathrm{vol}(X(D_1,\dots,D_n))}\,.$$

Assume now that $S$ is a surface with boundary. We can embed $S$ in a closed surface $S'$ such that the embedding is area preserving. This induces a monomorphism $\iota\colon \mathrm{Diff}_{c,0}(M_1,\sigma_1) \hookrightarrow \mathrm{Diff}_{c,0}(M_2,\sigma_2)$ (note that elements of $\mathrm{Diff}_{c,0}(M_1,\sigma_1)$ fix point-wise the neighbourhood of the boundary, so they can be extended by the identity). Since $\iota$ it Lipschitz, and the embedding of $\Gamma$ to $\mathrm{Diff}_{c,0}(M_2,\sigma_2)$ can be constructed such that it factors through $\mathrm{Diff}_{c,0}(M_1,\sigma_1)$, we get a quasi-isometric embedding to $\mathrm{Diff}_{c,0}(M_1,\sigma_1)$.

□

*Remark* 1.3.2. We note that Corollary 4 implies Corollary 2, providing the latter with a proof that does not use quasimorphisms.

Furthermore, note that Corollary 4 implies that for each $k \in \mathbb{N}$ there is a quasi-isometric embedding $i_k : \mathbb{Z}^k \to (\mathcal{G}, d_1)$. Moreover, there exists a $k$-tuple of autonomous Hamiltonian flows (one-parameter subgroups) $\{\{\phi_i^t\}_{t\in\mathbb{R}}\}_{1\leq i\leq k}$ which have disjoint supports, and $i_k(e_j) = \phi_j^1$ for each $1 \leq j \leq k$, where $e_j = (0,\dots,1,\dots,0)$ and 1 lies in the $j$-th entry. The above $k$-tuple of flows defines a homomorphism $j_k : \mathbb{R}^k \to (\mathcal{G}, d_1)$ such that $i_k = j_k|_{\mathbb{Z}^k}$. Since $\mathbb{R}^k$ is quasi-isometric to $\mathbb{Z}^k$, we obtain the following statement:

**Corollary 5.** *The metric group $(\mathcal{G}, d_1)$ admits a quasi-isometric embedding from $(\mathbb{R}^k, d)$ where $d$ is any metric on $\mathbb{R}^k$ induced by a vector-space norm.*

The following lemma allows us to prove that the quasi-isometric embeddings of $\mathbb{Z}^k, \mathbb{R}^k$ from Corollary 5 are in fact bi-Lipschitz embeddings.

**Lemma 2.** *Let $(G, d)$ be a metric group and let $A \subset V$ be a subgroup of a normed linear space. Then each Lipschitz quasi-isometric embedding $j : A \to G$ of metric groups is a bi-Lipschitz embedding. If $A$ is discrete and finitely generated then each homomorphism $j : A \to G$ is Lipschitz.*

*Proof.* Let $j : A \to G$ be a homomorphism. If $A$ is discrete and finitely generated, then $A \cong \mathbb{Z}^l$ for $l \in \mathbb{Z}_{\geq 0}$, the norm being equivalent to the standard norm on $\mathbb{Z}^l$. The upper bound $d(j(x), \mathbf{1}) \leq C|x|$ for all $x \in A$ is now immediate. Hence $j$ is Lipschitz.

Suppose now that $A \subset V$ is a subgroup and $j : A \to G$ is a Lipschitz quasi-isometric embedding. Let $C \geq 1$ be such that

$$\frac{1}{C}|x| - B \leq d(j(x), \mathbf{1}) \leq C|x| \tag{1}$$

for a constant $B \geq 0$ and all $x \in A$. We claim that the inequality (1) holds with $B = 0$. Indeed, consider (1) for $x^m$ where $m \in \mathbb{Z}_{>0}$. We have

$$\frac{1}{C}m|x| - B \leq d(j(x^m), \mathbf{1}) = d(j(x)^m, \mathbf{1}) \leq m \cdot d(j(x), \mathbf{1}),$$

where the last inequality is due to the right-invariance of $d$. Dividing by $m$ yields

$$\frac{1}{C}|x| - B/m \leq d(j(x), \mathbf{1}),$$

which finishes the proof by taking limits as $m \to \infty$. □

This immediately implies the following strengthening of Corollary 5, since $j_k$ is evidently Lipschitz.

**Corollary 6.** *The metric group $(\mathcal{G}, d_1)$ admits a bi-Lipschitz embedding from $(\mathbb{R}^k, d)$ where $d$ is any metric on $\mathbb{R}^k$ induced by a vector-space norm.*

*Remark* 1.3.3. It should be possible to prove Corollary 6 by using quasimorphisms, as in [14].

*Remark* 1.3.4. Let $p \geq 1$. Note that since, by Jensen's (or Hölder's) inequality,

$$d_1 \leq d_p,$$

all the above results for $d_1$ continue to hold for $d_p$.

Recall that if $M$ has boundary then we assume that elements of $\mathrm{Diff}_{c,0}(M, \sigma_1)$ fix point-wise an open neighbourhood of $\partial M$. Thus if $(M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)$ is a measure preserving embedding of manifolds, then extending a diffeomorphism on $M_1$ by the identity to a diffeomorphim of $M_2$ gives a well defined monomorphism $\iota\colon \mathrm{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \mathrm{Diff}_{c,0}(M_2, \sigma_2)$.

We finish this section with a question.

*Question* 1. Let $(M_1, \sigma_1) \hookrightarrow (M_2, \sigma_2)$ be a measure preserving embedding of surfaces. Is the monomorphism $\iota\colon \mathrm{Diff}_{c,0}(M_1, \sigma_1) \hookrightarrow \mathrm{Diff}_{c,0}(M_2, \sigma_2)$ a quasi-isometric embedding?

*Remark* 1.3.5. Note that this question is motivated by our results since the statement holds for the compositions of $\iota$ with the embeddings of right-angled Artin groups and $(\mathbb{R}^k, d)$ from the proofs of Corollary 4 and Corollary 6.

## 1.4 Outline of the proof

In order to show Theorem 1, we define a Riemannian metric $g$ on $X_n(M)$ such that $d_g$, the geodesic metric on $X_n(M)$ induced by $g$, extends to the geodesic metric on the compactification $\overline{X}_n(M)$. This allows us to use the Švarc-Milnor lemma.

More precisely, Theorem 1 is a consequence of the following two propositions (we defer the proofs to Section 2).

**Proposition 1.1.** *Let $\lambda$ be a piecewise $C^1$ loop in $X_n(M)$ based at $q$. Let $S$ be a finite generating set of $P_n(M)$. The word norm of the class $[\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)$ with respect to $S$ satisfies*

$$|[\lambda]|_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,$$

*for constants $A_0, B_0 > 0$ depending only on $S$ and $n$.*

The next proposition says that the average length of loops $\lambda(x, \overline{\phi})$ is controlled by $l_1(\overline{\phi})$.

**Proposition 1.2.** *Let $\overline{\phi} = \{\phi_t\}$ be an isotopy in $\mathcal{G}$ such that $\phi_0 = 1$. There exist constants $A_1, B_1 > 0$ depending only on $n$, such that*

$$\int_{X_n(M)\setminus Z \cup (\phi_1)^{-1}(Z)} l_g(\lambda(x, \overline{\phi}))\, d\mu^{\otimes n}(x) \leq A_1 \cdot l_1(\overline{\phi}) + B_1.$$

To show this proposition, we first compare a metric $g$ to an auxiliary metric $g_0$. The metric $g_0$ does not extend to a metric on the compactification $\overline{X}_n(M)$, but it is relatively easy to show [43, Lemma 5.2], that the inequality from Proposition 1.2 holds for $g_0$.

# 2 Proofs.

## 2.1 Compactification of the configuration space

In this section $M$ is a compact $m$-dimensional manifold. Below we describe the compactification of $X_n(M)$ using an embedding of the configuration space to a high dimensional Euclidean space. We follow closely the construction given in [52].

Let us start with describing a family of maps on $X_n(\mathbb{R}^d)$. Let $\pi_{ij}\colon X_n(\mathbb{R}^d) \to S^{d-1}$ by defined by the formula

$$\pi_{ij}(x) = \frac{x_i - x_j}{|x_i - x_j|},$$

where $0 < i < j < n+1$ and $x = (x_1, \ldots, x_n) \in X_n(\mathbb{R}^d)$. Let $s_{ijk}\colon X_n(\mathbb{R}^d) \to [0, \infty]$ be defined by the formula

$$s_{ijk}(x) = \frac{|x_i - x_j|}{|x_i - x_k|},$$

where $0 < i < j < k < n+1$ and $[0, \infty]$ is the one point compactification of $[0, \infty)$. Let $\iota\colon X_n(\mathbb{R}^d) \hookrightarrow (\mathbb{R}^d)^n$ be the standard inclusion and let $A_n[\mathbb{R}^d] = (\mathbb{R}^d)^n \times (S^{n-1})^{\binom{n}{2}} \times [0, \infty]^{\binom{n}{3}}$. We consider the embedding of $X_n(\mathbb{R}^d)$ to the ambient space $A_n[\mathbb{R}]$ given by the product map

$$\alpha_n = \iota \times (\pi_{ij}) \times (s_{ijk})\colon X_n(\mathbb{R}^d) \to A_n[\mathbb{R}^d].$$

Suppose $M$ is a submanifold of $\mathbb{R}^d$. Let $\overline{X}_n(M)$ be the closure of the image $\alpha_n(X_n(M))$ in $A_n[\mathbb{R}^d]$. Then $\overline{X}_n(M)$ is a manifold with boundary, and the inclusion $X_n(M) \hookrightarrow \overline{X}_n(M)$ is a homotopy equivalence [52, Theorem 4.4 and Corollary 4.5].

## 2.2 Two metrics on $X_n(M)$ and proof of Proposition 1.1

On $[0, \infty]$ we introduce the structure of a smooth manifold by a 1-map atlas $e^{-x}\colon [0, \infty] \to [0, 1]$. Let $g_{exp}$ be the Riemannian metric on $[0, \infty]$ given by the pull-back of the Euclidean metric from $[0, 1]$. In particular, we have that $|dx|_{g_{exp}} = e^{-x}$, where $dx$ is a standard 'unit' vector field on $[0, \infty)$.

Let $euc$ be the metric on $A_n[\mathbb{R}^d]$ given by the product of standard metrics on $\mathbb{R}^d$, $S^{d-1}$ and $([0, \infty], g_{exp})$. The first metric on $X_n(\mathbb{R}^d)$ we want to consider is defined to be the pull-back of $euc$ to $X_n(\mathbb{R}^d)$ by the map $\alpha_n$:

$$g = \alpha_n^*(euc).$$

Now, since we regard $M$ as a submanifold of $\mathbb{R}^d$, $g$ induces a Riemannian metric on $X_n(M)$.

The second metric is defined as follows. Let $x = (x_1, \ldots, x_n) \in X_n(\mathbb{R}^d)$. Let $d(x)$ denote the minimal distance between the points in $x$, that is:

$$d(x) = \min\{|x_i - x_j| \ : \ i \neq j\},$$

where $|x - y|$ denotes the standard Euclidean distance between vectors $x, y \in \mathbb{R}^d$.

Note that on $X_n(\mathbb{R}^d)$ we have the Euclidean metric restricted from $(\mathbb{R}^d)^n$. We rescale this metric by the factor $\frac{1}{d}$, i.e. we define a metric $g_0$ on $X_n(\mathbb{R}^d)$ by

$$|v|_{g_0} = \frac{|v|}{d(x)},$$

where $v \in T_x(X_n(\mathbb{R}^d)) = (\mathbb{R}^d)^n$ and $|v|$ is the Euclidean length of $v \in (\mathbb{R}^d)^n$. One should note that $d(x)$, and consequently $g_0$, is continuous, but not differentiable. A manifold with such a metric is called a $C^0$-Riemannian manifold. On a $C^0$-Riemannian manifold one defines the lengths of paths and a geodesic metric in the same way as in the smooth case.

Again, $g_0$ restricts to a metric on $X_n(M)$.

*Proof of Proposition 1.1.* We need to show that

$$|[\lambda]|_{P_n(M)} \leq A_0 \cdot l_g(\lambda) + B_0,$$

where $\lambda$ be a piecewise $C^1$ loop in $X_n(M)$ based at $q$, $[\lambda] \in \pi_1(X_n(M), q) \cong P_n(M)$ and $|[\lambda]|_{P_n(M)}$ is the word norm of $[\lambda]$.

By [52, Theorem 4.4] $\overline{X}_n(M)$ is a manifold with corners and $X_n(M)$ is its interior. Moreover, the coordinate charts on $\overline{X}_n(M)$ are defined in such a way that the embedding of $\overline{X}_n(M)$ into $A_n[\mathbb{R}^d]$ is smooth. Thus one can restrict the metric *euc* to $\overline{X}_n(M)$. In other words, $g = \alpha_n^*(euc)$ extends to the compactification $\overline{X}_n(M)$. Now the proposition follows directly from the Švarc-Milnor lemma [22, 54, 44]. More precisely, consider the following formulation of this result [15, Proposition 8.19].

**Lemma 3.** *Let a group $\Gamma$ act properly and discontinuously by isometries on a proper length space $X$. If the action is cocompact, then $\Gamma$ is finitely generated and for any choice of base-point $x_0 \in X$, the map $\Gamma \to X$, $h \mapsto h \cdot x_0$, is a quasi-isometry.*

Denote by $g_c$ the metric on $\overline{X}_n(M)$ restricted from *euc*. We apply this result to $X = \widetilde{\overline{X}}_n(M)$, $\Gamma = \pi_1(\overline{X}_n(M), \pi(x_0)) \cong \pi_1(X_n(M), \pi(x_0))$, and the length space structure on $X$ being induced by the lift $\widetilde{g_c} = \pi^* g_c$ to $X$ of $g_c$ on $K = X/\Gamma = \overline{X}_n(M)$ by the natural projection map $\pi : X \to K$. Note that $d_{\widetilde{g_c}}(x_0, h \cdot x_0) \leq l_{g_c}(\lambda)$ where $\lambda$ is any $C^1$ loop in $X_n(M)$ based at $\pi(x_0)$ representing the element $h$. Moreover, since $g_c$ is an extension of $g$, we have $l_{g_c}(\lambda) = l_g(\lambda)$. □

## 2.3 Proof of Proposition 1.2

We begin with the proof of the main technical result:

**Lemma 4.** *There exists $C > 0$ depending only on $n$, such that $|v|_g \leq C \cdot |v|_{g_0}$ for every $v \in T(X_n(M))$.*

*Proof.* Since $\alpha_n$ is a product map, in order to get a bound on the norm $|v|_g$, we need to bound norms of vectors $D\pi_{ij}(v)$ and $Ds_{ijk}(v)$. Let $x = (x_1, \ldots, x_n) \in X_n(M)$ and let $v = (v_1, \ldots, v_n) \in T_x(X_n(M))$. There exists $\epsilon > 0$ such that $x + tv \in X_n(\mathbb{R}^d)$ for every $t < \epsilon$. Note that even though the path $\{x + tv\}_{t \in [0,\epsilon]}$ is not contained in $X_n(M)$, it still represents the tangent vector $v \in T_x(X_n(M)) < T_x(X_n(\mathbb{R}^d))$. We have

$$\begin{aligned} D\pi_{ij}(v) &= \frac{d}{dt}_{|t=0} \pi_{ij}(x + tv) \\ &= \frac{d}{dt}_{|t=0} \frac{x_i - x_j + t(v_i - v_j)}{|x_i - x_j + t(v_i - v_j)|} \\ &= \frac{v_i - v_j}{|x_i - x_j|} - \frac{(x_i - x_j)\langle v_i - v_j, x_i - x_j \rangle}{|x_i - x_j|^3}. \end{aligned}$$

By definition $D\pi_{ij}(v)$ is a vector in $\mathbb{R}^d$ tangent to a $(d-1)$-dimensional sphere at point $\pi_{ij}(x)$. By $|D\pi_{ij}(v)|$ we denote its Euclidean length.

After applying inequalities $\langle v_i - v_j, x_i - x_j\rangle \le |v_i - v_j||x_i - x_j|$ and $|v_i - v_j| \le |v_i| + |v_j|$, we obtain:

$$
\begin{aligned}
|D\pi_{ij}(v)| &\le 2\frac{1}{|x_i - x_j|}(|v_i| + |v_j|)\\
&\le \frac{2}{d(x)}(|v_i| + |v_j|)\\
&\le \frac{2}{d(x)}(\sum_i |v_i|).
\end{aligned}
$$

Similarly, for $s_{ijk}$ we compute (already in the map):

$$
\begin{aligned}
De^{-s_{ijk}}(v) &= \frac{d}{dt}_{|t=0} e^{-s_{ijk}(x+tv)}\\
&= \frac{d}{dt}_{|t=0} e^{-\frac{|x_i - x_j + t(v_i - v_j)|}{|x_i - x_k + t(v_i - v_k)|}}\\
&= \Big[\frac{\langle v_i - v_j, x_i - x_j\rangle}{|x_i - x_j||x_i - x_k|} - \frac{|x_i - x_j|\langle x_i - x_k, v_i - v_k\rangle}{|x_i - x_k|^3}\Big]e^{-\frac{|x_i - x_j|}{|x_i - x_k|}}.
\end{aligned}
$$

By definition $|Ds_{ijk}(v)|_{g_{exp}} = |De^{-s_{ijk}}(v)|$. Using similar inequalities as before we get:

$$
|Ds_{ijk}(v)|_{g_{exp}} \le \Big[\frac{|v_i| + |v_j|}{|x_i - x_k|} + \frac{(|v_i| + |v_k|)|x_i - x_j|}{|x_i - x_k|^2}\Big]e^{-\frac{|x_i - x_j|}{|x_i - x_k|}}.
$$

Applying the inequality $e^{-x} \le \frac{1}{x}$ to the last term we obtain:

$$
\begin{aligned}
|Ds_{ijk}(v)|_{g_{exp}} &\le \Big[\frac{|v_i| + |v_j|}{|x_i - x_k|} + \frac{(|v_i| + |v_k|)|x_i - x_j|}{|x_i - x_k|^2}\Big]\frac{|x_i - x_k|}{|x_i - x_j|}\\
&= \frac{|v_i| + |v_j|}{|x_i - x_j|} + \frac{|v_i| + |v_k|}{|x_i - x_k|}\\
&\le \frac{1}{d(x)}(2|v_i| + |v_j| + |v_k|)\\
&\le \frac{2}{d(x)}(\sum_i |v_i|).
\end{aligned}
$$

Finally, since $\sum_i |v_i| \le \sqrt{n}|v|$, we get:

$$
|D\pi_{ij}(v)| \le \frac{\sqrt{n}}{d(x)}|v|,
$$

$$
|Ds_{ijk}(v)|_{g_{exp}} \le \frac{2\sqrt{n}}{d(x)}|v|.
$$

We assume that $M$ is compact, therefore there exists $A > 0$ such that $d(x) \le A$ for every $x \in X_n(M)$. Now we can bound the pulled-back metric $|\cdot|_g$:

$$
\begin{aligned}
|v|_g^2 &= |D\iota(v)|^2 + \sum_{i,j} |D\pi_{ij}(v)|^2 + \sum_{i,j,k} |Ds_{ijk}(v)|^2_{g_{exp}}\\
&\le \frac{A^2}{d(x)^2}|v|^2 + \sum_{i,j} \frac{n}{d(x)^2}|v|^2 + \sum_{i,j,k} \frac{4n}{d(x)^2}|v|^2\\
&= C\frac{|v|^2}{d(x)^2} = C \cdot |v|^2_{g_0},
\end{aligned}
$$

for $C = A^2 + n\binom{n}{2} + 4n\binom{n}{3}$. $\square$

*Remark* 2.3.1. In the proof of Lemma 4, the inequality $e^{-x} < \frac{1}{x}$ is used to show the bound on the length of $Ds_{ijk}$. Indeed, the choice of the function $e^{-x}\colon [0,\infty] \to [0,1]$ to define a smooth structure and the metric on $[0,\infty]$ is not completely arbitrary. For a different identification, e.g. $ln(\frac{x+e}{x+1})\colon [0,\infty] \to [0,1]$, we get a different smooth structure on $[0,\infty]$ (in the sense, that the identity map is not smooth) and a metric which is not equivalent to $g_{exp}$. Then after pull-back by $\alpha_n$ we get a metric on $X_n(M)$ which is not equivalent to $g$ and for this metric Lemma 4 might not hold.

*Proof of Proposition 1.2.* We need to show, that

$$\int_{X_n(M)\setminus Z\cup(\phi_1)^{-1}(Z)} l_g(\lambda(x,\overline{\phi}))\, d\mu^{\otimes n}(x) \leq A_1 \cdot l_1(\overline{\phi}) + B_1,$$

where $\overline{\phi} = \{\phi_t\}$ is an isotopy in $\mathcal{G}$ such that $\phi_0 = 1$.

A similar inequality was proven in [43] for a metric which is equivalent to $g_0$. Let us first describe this metric. By $d_M$ denote the geodesic metric on $M$ induced by the restriction of the standard Riemannian metric on $\mathbb{R}^d$. Let $x \in X_n(M)$ and $v \in T_x(X_n(M))$. We define

$$|v|_{g_b} = \frac{|v|}{d_M(x)},$$

where $d_M(x) = \min\{d_M(x_i,x_j)\ :\ i \neq j\}$ and $|v|$ is the Euclidean length of $v$ seen as a vector in $(\mathbb{R}^d)^n$. The difference between the function $d$ used to define $g_0$ and $d_M$ is that in $d$ we measure the distance between points $x_i$ and $x_j$ in the ambient space $\mathbb{R}^d$ and in $d_M$ inside $M$. Since $M$ is compact, clearly $d$ and $d_M$ are equivalent and consequently $g_b$ and $g_0$ are equivalent.

It follows from [43, Lemma 5.2] that

$$\int_{X_n(M)\setminus Z\cup(\phi_1)^{-1}(Z)} l_{g_b}(\{\phi_t \cdot x\})\, d\mu^{\otimes n}(x) \leq C' \cdot l_1(\overline{\phi}).$$

Recall that $\{\phi_t \cdot x\}$ is a path in $X_n(M)$ ($\phi_t$ acts on $x \in X_n(M)$ component-wise). Since $g_0$ and $g_b$ are equivalent, this inequality holds as well for $g_0$ (with a possibly different constant).

Let us now focus on the metric $g$. Since the geodesic metric $d_g$ defined by $g$ extends to the compactification of $X_n(M)$, the diameter of $(X_n(M), d_g)$ is finite. We can choose the system of short paths $\gamma(x)$ such that $l_g(\gamma(x)) \leq D$ for some $D > 0$ and every $x \in X_n(M) \setminus Z$.

Thus for every $x \in X_n(M) \setminus Z \cup (\phi_1)^{-1}(Z)$ we have

$$l_g(\lambda(x,\overline{\phi})) \leq l_g(\gamma(x)) + l_g(\{\phi_t \cdot x\}) + l_g(\gamma(\phi_1(x))) \leq 2D + l_g(\{\phi_t \cdot x\}).$$

Finally, due to Lemma 4, we have $l_g(\{\phi_t \cdot x\}) \leq C \cdot l_{g_0}(\{\phi_t \cdot x\})$ and the proposition follows. $\square$

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