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A forgotten theorem of Pełczyński: $(\lambda+)$-injective spaces need not be λ-injective–the case $\lambda \in (1, 2]$

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A FORGOTTEN THEOREM OF PEŁCZYŃSKI: $(\lambda+)$-INJECTIVE SPACES NEED NOT BE λ-INJECTIVE—THE CASE $\lambda \in (1,2]$

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ABSTRACT. Isbell and Semadeni [Trans. Amer. Math. Soc. 107 (1963)] proved that every infinite-dimensional 1-injective Banach space contains a hyperplane that is $(2+\varepsilon)$-injective for every $\varepsilon > 0$, yet is is *not* 2-injective and remarked in a footnote that Pełczyński had proved for every $\lambda > 1$ the existence of a $(\lambda + \varepsilon)$-injective $(\varepsilon > 0)$ that is not λ-injective. Unfortunately, no trace of the proof of Pełczyński's result has been preserved. In the present paper, we establish the said theorem for $\lambda \in (1, 2]$ by constructing an appropriate renorming of ℓ_∞. This contrasts (at least for real scalars) with the case $\lambda = 1$ for which Lindenstrauss [Mem. Amer. Math. Soc. 48 (1964)] proved the contrary statement.

1. Introduction

Let $\lambda \geqslant 1$. A Banach space X is λ-*injective* whenever for any pair of Banach spaces $E \subseteq F$, every bounded linear operator $T\colon E \to X$ may be extended to an operator $\widehat{T}\colon F \to X$ with norm at most $\lambda \cdot \|T\|$. Since every Banach space X embeds isometrically into $\ell_\infty(\Gamma)$ for some set Γ and the latter space is 1-injective, λ-injectivity is equivalent to the existence of a (bounded, linear) projection from $\ell_\infty(\Gamma)$ onto (any isometric copy of) X that has norm at most λ. We refer to [2, Section 2.5] for a modern exposition of injective spaces.

Let us say that a Banach space is $(\lambda+)$-*injective*, whenever it is $(\lambda + \varepsilon)$-injective for all $\varepsilon > 0$. Lindenstrauss proved that every $(1+)$-injective real Banach space is 1-injective ([7, Theorem 6.10]). (The case of complex scalars stubbornly remains open.) However, Isbell and Semadeni demonstrated that the statement of Lindenstrauss' theorem with 1 replaced by 2 is false ([6, Theorem 2]). They showed that the sought counterexample may always be arranged as a hyperplane in an arbitrary infinite-dimensional 1-injective space. Interestingly, in a footnote on p. 40 they announce the following result communicated to them by Pełczyński: Let $\lambda > 1$. Then there exists a $(\lambda+)$-injective space that is not λ-injective.

Semadeni informed the first-named author in a personal communication in 2016 that neither has he got any recollection of the result nor of the circumstances in which Pełczyński had communicated it to him. To the best of our knowledge, the result has not been

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rediscovered or reproved since. In the present paper we provide a proof in the restricted setting of $\lambda \in (1,2]$.

Theorem A. *Let $\lambda \in (1,2]$. If X is an infinite-dimensional 1-injective space, then X has a $(\lambda+)$-injective renorming that is not λ-injective.*

More specifically, for every $\lambda \in (1,2]$ there exists a functional $f \in \ell_\infty^*$ of norm one such that $\ker f$ *is $(\lambda+\varepsilon)$-complemented $(\varepsilon > 0)$, yet there is no minimal projection onto* $\ker f$ *of norm λ.*

The former part of Theorem A follows from the latter one as every 1-injective space X contains an isometric copy of ℓ_∞ (Lemma 2.5), which is thus 1-complemented therein. Moreover, ℓ_∞ is isomorphic to its hyperplanes, hence so is X. Consequently, every infinite-dimensional 1-injective Banach space contains a hyperplane whose projection constant is $\lambda \in (1,2]$, yet there is no minimal projection thereonto.

The case of $\lambda = 2$ in Theorem A subsumes the conclusion of [6, Theorem 2] concerning 2-injectivity. Intriguingly, *en route* to the proof we rely on a result by Blatter and Chenney [5] from 1974, that had been obviously unavailable to Pełczyński in the 1960s, so his proof most likely must have been different. Furthermore, the complementary case of $\lambda \in (2,\infty)$ remains a mystery to us.

2. Preliminaries

We use standard notation from Banach space theory. All Banach spaces are considered under the fixed field of real or complex numbers. A *projection* is a bounded idempotent linear operator acting on a Banach space. We regard ℓ_∞ as the dual space of ℓ_1 and similarly ℓ_∞^n as $(\ell_1^n)^*$ $(n \in \mathbb{N})$. We shall require the fact that ℓ_∞^* decomposes canonically into $\ell_1 \oplus_1 c_0^\perp$. Elements of $c_0^\perp$ are called *singular* functionals and, of course, for every singular functional f one has $\langle f, x\rangle = 0$ $(x \in c_0)$. For a Banach space X and $x \in X$ we denote by $\widehat{x} = \kappa_X x \in X^{**}$, *i.e.*, $\langle \widehat{x}, f\rangle = \langle f, x\rangle$ $(x \in X^{**})$. More generally, for a bounded linear operator $T\colon X \to Y$ we denote by $\widehat{T}\colon X^{**} \to Y^{**}$ the second adjoint of T.

We record the following folk fact about norm-attainment of functionals in $c_0^\perp$. For the reader's convenience we include the proof.

Lemma 2.1. *Let $g \in c_0^\perp$, $\|g\| = 1$. Then g attains the norm if and only if there exists $y = (y_i)_{i=1}^\infty \in X = \ell_\infty$ such that*

$$\langle g, y\rangle = 1 \text{ and } \limsup_{j\to\infty} |y_j| = 1.$$

Moreover, since ℓ_∞/c_0 is non-reflexive, liftings of non-norm attaining functionals on ℓ_∞/c_0 to ℓ_∞ are functionals in $c_0^\perp$ that do not attain their norm.

Proof. If there is $y \in X$ with $\|y\| = 1$ such that $\langle g, y\rangle = 1$, then

$$1 = \|y\| \geqslant \limsup_{j\to\infty} |y_j|.$$

Since $g \in c_0^\perp$, $\limsup_{j\to\infty} |y_j| = 1$. Suppose that there exists $y \in X$ such that $\langle g, y\rangle = 1$ and $\limsup_{j\to\infty} |y_j| = 1$. Define $z = (z_j)_{j=1}^\infty \in X$ by $z_j = \operatorname{sgn}(y_j)$, whe $|y_j| > 1$ and $z_j = y_j$ otherwise. We *claim* that $y - z \in c_0$. If not, there exists an increasing sequence of natural numbers $(j_k)_{k=1}^\infty$ and $d > 0$ such that

$$\lim_{k\to\infty} |z_{j_k} - y_{j_k}| = d.$$

Without loss of generality, we may assume that $\{j_k \colon k \in \mathbb{N}\} \subset \{j \in \mathbb{N} \colon |y_j| > 1\}$. Consequently

$$d = \lim_{k\to\infty} |y_{j_k} - \operatorname{sgn}(y_{j_k})| = \lim_{k\to\infty} |y_{j_k}| - 1;$$

a contradiction. Since $y - z \in c_0$ and $g \in c_0^\perp$ one has $\langle g, y\rangle = \langle g, z\rangle = 1$. As $\|z\| = 1$, g attains the norm at z. □

For a Banach space X and a closed subspace Y of X we denote by $\mathcal{P}(X, Y)$ the (possibly empty) set of projectionS from X onto Y. In the case where this set is non-empty, we define the *projection constant*

$$\lambda(Y, X) = \inf\{\|P\| \colon P \in \mathcal{P}(X, Y)\}.$$

A projection $P \in \mathcal{P}(Y, X)$ is *minimal*, whenever $\|P\| = \lambda(Y, X)$. For a future reference, we record the following simple lemma.

Lemma 2.2. *Let X be a Banach space and suppose that $Y \subset X$ is a complemented subspace. If $T \colon X \to X$ is a linear surjective isometry, then*

$$\lambda(Y, X) = \lambda(T(Y), X).$$

Moreover, there exists a minimal projection in $\mathcal{P}(X, Y)$ if and only if there exists a minimal projection in $\mathcal{P}(X, T(Y))$.

Proof. We define a transformation $\widehat{T} \colon \mathcal{P}(X, Y) \to \mathcal{P}(X, T(Y))$ by $\widehat{T}(P) = T \circ P \circ T^{-1}$ ($P \in \mathcal{P}(X, Y)$). Then $\widehat{T}$ is a surjective isometry from $\mathcal{P}(X, Y)$ onto $\mathcal{P}(X, T(Y))$, which proves the lemma. □

Let us now specify to self-isometries of ℓ_∞.

Lemma 2.3. *Let $X = \ell_\infty$. Suppose that $f \in X^*$ is a norm-one functional decomposed as $f = h + g$, where $h = (h_i)_{i=1}^\infty \in \ell_1$ and $g \in c_0^\perp$. Let $T \colon X \to X$ be defined as*

$$Tx = (\overline{\operatorname{sgn}(h_1)}x_1, \overline{\operatorname{sgn}(h_2)}x_2, \overline{\operatorname{sgn}(h_3)}x_1, \ldots) \quad (x \in X).$$

Then $\lambda(\ker f, X) = \lambda(T(X), X)$. Moreover, there exists a minimal projection in $\mathcal{P}(X, Y)$ if and only if there exists a minimal projection in $\mathcal{P}(X, T(Y))$.

Proof. Observe that T is a linear, surjective isometry. The result follows from Lemma 2.2. □

We shall also require the following consequence of [5, Theorem 2].

Lemma 2.4. *Let $h = (h_k)_{k=1}^\infty \in \ell_1$ be regarded a functional on ℓ_∞. If h is non-zero and $\|h\|_{\ell_\infty} < 1/2$, $\|f\|_{\ell_1} = 1$, then*

$$\lambda(\ker f, \ell_\infty) = 1 + \Big(\sum_{k=1}^\infty \frac{|h_k|}{1-2|h_k|}\Big)^{-1}.$$

In particular, for $n \in \mathbb{N}$, $n \geqslant 3$ and $h^{\underline{n}} = (1/n, \ldots, 1/n) \in \ell_1^n$, we have

$$\lambda(Y_n, \ell_\infty) = 2 - \frac{2}{n},$$

where $Y_n = \ker h^{\underline{n}} \subset \ell_\infty^n$ and the latter space is identified with a subspace of ℓ_∞.

1-injective Banach spaces are characterised as spaces isometric to the space $C(K)$ of continuous functions on a compact extremally disconnected space K ([2, Theorem 6.8.3]). (A topological space is *extremally disconnected* whenever open subsets thereof have open closures.) We have been unable to locate the following apparently folk lemma in the literature so we provide its proof and we are indebted to K.P. Hart for clarification regarding the proof. That every infinite-dimensional injective space contains an isomorphic copy of ℓ_∞ is well known and may be found, for example, in [1, Proposition 1.15].

Lemma 2.5. *Every infinite-dimensional 1-injective Banach space contains an isometric copy of ℓ_∞.*

Proof. Let K be an extremally disconnected space so that X is isometric to $C(K)$. By the Balcar–Franek theorem [3], K contains a copy of $\beta\mathbb{N}$ (since the minimal weight of an infinite compact extremally disconnected space is continuum). Let us denote by D the relatively discrete copy of the integers in the copy of $\beta\mathbb{N}$ in K. Thus, one may find disjoint open sets $(U_d)_{d\in D}$ in K such that $U_d \cap D = \{d\}$ $(d \in D)$. As K is extremally disconnected, the closure of $U = \bigcup_{d\in D} U_d$ is a clopen subset of K and $\overline{U} = \beta U$. The map $r\colon U \to D$ given by $r(x) = d$ if and only if $x \in U_d$ $(d \in D)$ is a retraction. Consequently, it extends to a retraction $\beta r\colon \overline{U} \to \beta D$.

The retraction $p = \beta r$ yields a contractive linear projection $P\colon C(K) \to C(K)$ given by $Pf = f \circ p$ $(f \in C(K))$ with range isometric to $C(\beta\mathbb{N}) \equiv \ell_\infty$. □

3. Proof of Theorem A

In order to prove Theorem A we show that for any $\lambda \in (1,2)$ there exists a subspace W_λ of ℓ_∞, such that $\lambda(W_\lambda, \ell_\infty) = \lambda$ and for every $P \in \mathcal{P}(\ell_\infty, W_\lambda)$ one has $\|P\| > \lambda$.

We first consider the case of $\lambda \in (1,2)$ and the case $\lambda = 2$ will be treated separately.

Theorem 3.1. *Let $X = \ell_\infty$. For every $\lambda \in (1,2)$ there exists a functional $f \in X^*$ of norm one such that $\lambda(\ker f, X) = \lambda$, yet there is no minimal projection onto* $\ker f$.

Proof. First we consider the real case. Let $f \in X^*$, $\|f\| = 1$ be such that $f = h + g$, where $g \in c_0^\perp$, $h \in \ell_1$, $1 = \|h\| + \|g\|$, and $2|h_i| < \|h\|_\infty$ $(i \in \mathbb{N})$. Set $Y = \ker f$. By [4, Theorem

2.4],

$$\lambda(Y,X) = 1 + \Big(\|g\| + \sum_{i=1}^{\infty} \frac{|h_i|}{1-2|h_i|}\Big)^{-1}. \tag{1}$$

Let $h^{\underline{n}} = (1/n,\ldots,1/n) \in \ell_1^n$ and $Y_n = \ker h^{\underline{n}}$. By Lemma 2.4, $\lambda(Y_n,\ell_\infty^n) = 2 - \frac{2}{n}$. Fix $a \in [1/(n-1),1]$ and let $Y_{a,n}$ be the kernel of

$$h_{a,n} = \big(\frac{1}{1+(n-1)a}, \frac{a}{1+(n-1)a}, \ldots, \frac{a}{1+(n-1)a}\big) \in \ell_1^n.$$

By Lemma 2.4,

$$\lambda(Y_{a,n},\ell_\infty^n) = 1 + \big(\frac{1}{1+(n-1)a-2} + \frac{(n-1)a}{1+(n-3)a}\big)^{-1}.$$

Fix $g \in c_0^\perp$, $\|g\| = 1$ such that g does not attain its norm. For any $b \in [0,1]$ define

$$f_{n,a,b} = (1-b)g + bh_{a,n}. \tag{2}$$

By [4, Theorem 3.4], there is no minimal projection onto the kernel of $f_{n,a,b}$.

Let us $\varepsilon > 0$. We *claim* that for any $\lambda \in (1+\varepsilon, 2-\varepsilon)$ there is a subspace W_λ of ℓ_∞ such that $\lambda(W_\lambda,\ell_\infty) = \lambda$ and there is no minimal projection onto W_λ. Fix $n \in \mathbb{N}$ such that $\lambda(Y_n) = 2 - 2/n > 2 - \varepsilon$ and observe that the function

$$g(a) = 1 + \Big(\frac{1}{1+(n-1)a-2} + \frac{(n-1)a}{1+(n-3)a}\Big)^{-1}$$

is continuous, $g(1) = 2 - 2/n$, and $g(\frac{1}{n-1}) = 1$. Hence for $b \in (0,1)$ close to 1, there is $a \in [1/(n-1),1]$ with $\lambda(\ker f_{n,a,b}, X) = \lambda$.

Since $\varepsilon > 0$ was arbitrary, for any $\lambda \in (1,2)$ there exists W_λ such that $\lambda(W_\lambda, X) = \lambda$ and there is no minimal projection onto W_λ, as required, so the proof in the real case is complete.

The complex case requires an extra adjustment. Reasoning as in [4, Lemma 2.4], we can show that for any $P \in \mathcal{P}(X,Y)$ there exists $y = (y_i)_{i=1}^\infty \in X$ such that $\langle f, y\rangle = 1$, $Px = x - \langle f, x\rangle y$ $(x \in X)$, and

$$\|P\| = \sup\{|1 - h_j y_j| + |y_j|(1-|h_j|)\colon j \in \mathbb{N}\}. \tag{3}$$

Applying Lemmata 2.2–2.3, we may assume that $h_i \geqslant 0$ $(i \in \mathbb{N})$. Pick $y \in X$ with $\langle f, y\rangle = 1$. Let $h^{\underline{n}}$ be as in the statement of Lemma 2.4. Moreover, let $g \in c_0^\perp$ be a norm-one functional that does not attain its norm (see Lemma 2.1 for details). Finally, let

- $f_{n,a,b} = (1-b)g + bh_{a,n}$, and
- $P \in \mathcal{P}(X, \ker f_{n,a,b})$.

Then

$$Px = P_w x = x - \langle f_{n,a,b}, x\rangle w,$$

where $w \in X$ satisfies $\langle f_{n,a,b}, w\rangle = 1$. We set

$$y = (\operatorname{Re} w_1, \operatorname{Re} w_2, \ldots).$$

Then $\langle f, w\rangle = 1$. Moreover, it follows from (3) that $\|P_w\| \geqslant \|P_y\|$. We may now repeat the reasoning from the real case, to arrive at the conclusion. □

Let us now consider the case of $\lambda = 2$, which is parallel to [6, Theorem 2].

Theorem 3.2. *Let $g \in c_0^{\perp}$, $\|g\| = 1$ and let $Y = \ker g$. Then $\lambda(Y, X) = 2$. Moreover, Y is 2-injective if and only if g attains its norm.*

Proof. It is well-known that when W is a hyperplane in a Banach space Z then $\lambda(W, Z) \leqslant 2$. Consequently, we need to show that $\lambda(Y, X) \geqslant 2$.

Let $P \in \mathcal{P}(X, Y)$. Then there exists $y \in X$ with $\langle g, y\rangle = 1$ such that $Px = x - \langle g, x\rangle y$ ($x \in X$). Since $\langle g, y\rangle = 1$ and $g \in c_0^{\perp}$, we have $\limsup_{j\to\infty} |y_j| \geqslant 1$.

Fix $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $|y_j| > 1 - \varepsilon$. Take $x = (x_j)_{j=1}^{\infty} \in X$ of norm one such that $\langle g, x\rangle > 1 - \varepsilon$. Since $g \in c_0^{\perp}$, we may assume that $x_i = 0$ for $i \leqslant j$. Let $w = (w_1, ..., w_j) \in \ell_\infty^j$ be chosen so that $\|w\| = 1$ and $w_j = -\mathrm{sgn}(\langle g, x\rangle)y_j$. Let $x^1 = w + x$. We note that $\|x^1\| = 1$ and

$$\|P\| \geqslant |(Px^1)_j| = | - \mathrm{sgn}(\langle g, x\rangle)y_j - \langle g, x\rangle y_j| = 1 + |\langle g, x\rangle||y_j| \geqslant 1 + (1 - \varepsilon)(1 - \varepsilon),$$

which shows that indeed $\lambda(Y, X) = 2$.

- If g does not attain the norm and let $P \in \mathcal{P}(X, Y)$ be as above. Since g does not attain the norm, $\langle g, y\rangle > 1$, so by Lemma 2.1, $\limsup_{j\to\infty} |y_j| = d > 1$. Reasoning as above, we conclude that for any $\varepsilon > 0$ we have
$$\|P\| \geqslant 1 + (1 - \varepsilon)(d - \varepsilon).$$
Consequently, $\|P\| \geqslant 1 + d > 2$, which proves the claim.
- If g attains the norm, we may take $y \in X$, with $\|y\| = 1$ such that $\langle g, y\rangle = 1$. Again, let $Px = x - \langle f, x\rangle y$. Obviously, $P \in \mathcal{P}(X, Y)$ and
$$\|P\| \leqslant 1 + \|g\|\|y\| = 2.$$
Hence $\lambda(Y, X) = 2 = \|P\|$, as required.

□

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