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Barbora Benešová
Šárka Nečasová
Jan Scherz
Anja Schlömerkemper

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FLUID-RIGID BODY INTERACTION IN AN INCOMPRESSIBLE ELECTRICALLY CONDUCTING FLUID [1]

BARBORA BENEŠOVÁ[1], ŠÁRKA NEČASOVÁ[2], JAN SCHERZ[1,2,3], ANJA SCHLÖMERKEMPER[3]

[1] Department of mathematical analysis, Faculty of mathematics and physics, Charles University in Prague, Sokolovská 83, Prague 8, 18675, Czech Republic

[2] Mathematical Institute, Academy of Sciences, Žitná 25, Prague 1, 11567, Czech Republic

[3] Institute of Mathematics, University of Würzburg, Emil-Fischer-Str. 40, 97074 Würzburg, Germany

Abstract

We analyze a mathematical model that describes the interaction between an insulating rigid body and an incompressible electrically conducting fluid, in which the body is located. The model as well as the mathematical analysis involve the fields of fluid-structure interaction and of magnetohydrodynamics. Our main result is the proof of the existence of a weak solution to the corresponding system of partial differential equations. The proof relies on a use of a time discretization via the Rothe method to decouple the system. This allows us to deal with test functions, depending on the position of the moving body and therefore on the solution of the system, in the weak formulation of the induction equation. The proof moreover makes use of the Brinkman penalization in order to cope with the mechanical part of the problem.

1 Introduction

In this paper, we consider a system of partial differential equations describing the movement of an insulating rigid body through an electrically conducting incompressible fluid. We prove the existence of a weak solution to this system. The insulating solid interacts mechanically with the fluid, which is why the studied problem falls into the broad class of *fluid-structure interaction*. Since the electrically conducting fluid further interacts with electromagnetic fields, it also constitutes a problem of *magnetohydrodynamics*. While a number of mathematical works can be found in both those classes, a combination of the two seems to be missing. Possible applications we have in mind include the interaction of both extra- and intracellular fluids with cell membranes in an organism. Even though membranes of cells are rather deformable than rigid objects, the study of the rigid body case can serve as a first step towards understanding the real-world situation.

Fluid-structure interaction describes any interaction between a moving fluid and a rigid or deformable solid contained in the fluid or surrounding it. In our case, we deal with a rigid body moving inside of

[1]This paper has been uploaded to arXiv, `https://arxiv.org/abs/2203.05953`.

a *viscous incompressible* fluid, a situation, for which the existence theory of weak and strong solutions has been studied by many authors. For an introduction to the problem of a fluid coupled with a rigid body see [17, 39]. Let us mention that first results on the existence of weak solutions until the first collision go back to the works of Conca, Starovoitov and Tucsnak [6], Desjardins and Esteban [9], Gunzburger, Lee and Seregin [19], Hoffman and Starovoitov [25]. Further, the possibility of collisions in case of weak solutions has been addressed for example in the work of [37], where the global-in-time existence of a weak solution in two dimensions is shown, i.e. contacts between the bodies and the boundary as well as between the bodies themselves are possible subject to the condition of vanishing relative velocity and acceleration. For a global-in-time existence result for the three dimensional case we refer to e.g. [14]. The latter result has also counterparts for the compressible situation with the no-slip boundary condition [13] and with the Navier-slip boundary condition [32]. Finally, we also want to mention existence results on strong solutions, see e.g. [18, 41, 43].

Magnetohydrodynamics stands for the interaction of electrically conducting fluids with electromagnetic fields, see e.g. [5, 29]. This interplay is described mathematically by a coupling of the fluid equations with the Maxwell system [30]. In this coupling, the resulting equations are further simplified subject to certain physical assumptions, which is referred to as the *magnetohydrodynamic approximation*, c.f. [8, 12] and which allows one to reduce the electromagnetic part of the system to a problem for only the magnetic induction. In contrast to the fluid, in the present work, the rigid body is assumed to be non-conducting and non-magnetic and hence is not influenced by electromagnetic fields, nonetheless the electromagnetic trespass it. Thus, it may be viewed as vacuum from the electromagnetic point of view. For a result giving the existence of weak solutions to the magnetohydrodynamic model for an incompressible fluid we refer to [20]. A corresponding investigation of the compressible case can be found in [38]. In [3], in addition to the electric conductivity, the fluid is assumed to be thermally conductive. The combination of an insulating rigid but also immovable object with an electrically conducting incompressible fluid has been considered in [22] in two and in [23] in three dimensions.

The model considered in the present work (see Section 1.1) is an extension of the model studied in the latter two articles. The novelty lies in the fact that the solid in our model is not chosen as a fixed but as a freely moving rigid object, which causes various mathematical problems outlined below. In this article, we thus investigate the setting of a rigid solid which is neither electrically conducting nor magnetic and which is moving in an incompressible electrically conducting fluid. In particular, the electromagnetic fields influence indirectly the movement of the solid body through the motion of the surrounding fluid. The setting we study can serve as a basis towards the study of more sophisticated systems involving for example a compressible fluid, an electrically conducting magnetic body or different types of boundary conditions.

Our main result is the proof of the existence of weak solutions to the aforementioned model. In the weak formulation of the system we consider test functions which depend on the position of the solid body and therefore, since we work on a moving domain, also on the solution of the system itself. While in problems restricted to the interaction between fluids and solids the choice of such test functions is standard and the difficulties, which result from this choice, are well-investigated, this is not the case for magnetohydrodynamical problems in moving domains.

In our specific scenario, for the mechanical part we can rely on the Brinkman penalization method to overcome the problems arising from the test functions chosen in the momentum equation, c.f. (24) below. This method, in which the rigid body is approximated by rigidly moving, but permeable objects with vanishing permeability, has already been analyzed in detail in [4]. However, for the problem resulting from the solution-dependent test functions in the induction equation, c.f. (25) below, no such penalization method appears to be available, since it would require to approximate the non-conducting solid by solids with vanishing conductivity, which does not seem to be possible. In order to deal with this problem we thus decided to discretize the system in the time variable via the Rothe method, c.f. [36, Section 8], and regularize it. Thereby the system is decoupled, which allows us to first determine the position of the body at a specific discrete time and subsequently solve the induction equation at that time by classical arguments. This procedure constitutes the main novelty in our proof. Test functions which depend on the position of the non-conducting solid body are also considered for example in the proofs of [22, Theorem 2.1] and [23, Theorem 2.3], where the solid, however, is immovable and hence the test functions do not cause any difficulties, since they are given a priori. Moreover, our proof adopts various methods from [13], where the existence of weak solutions for an interaction problem between a rigid body and a compressible fluid is shown. Even though a different penalization method is applied there, we can still use several of the same arguments for the limit passage in the approximate system in our proof. A detailed description of the idea of our proof is given in Section 2.

The outline of the article is the following: In Section 1.1 the model is described. The corresponding weak formulation together with the main result follows in Section 1.2. The proof of the main result extends across Sections 2–6: In Section 2 the approximation to the original system is presented, to which the existence of a solution is shown in Section 3. Sections 4–6 deal with the limit passages required to return to the original equations. Finally, in Appendix 7 some auxiliary results are discussed.

1.1 Model description

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain occupied by a viscous nonhomogeneous incompressible fluid and a rigid body, let $T > 0$ and set $Q = Q(T) := (0,T) \times \Omega$. We denote the initial position of the rigid body by $S = S(0) \subset \Omega$ and we write $S(t) \subset \Omega$ for its position at any time $t \in [0,T]$, the movement of which can be expressed by means of some isometry. We further denote by $F(t) := \Omega \backslash \overline{S}(t)$ the domain filled with the fluid at time t, see Figure 1. Correspondingly, we also divide Q into the solid time-space domain

$$Q^S := \{(t,x) \in Q : \ x \in S(t)\}.$$

and its fluid counterpart $Q^F := Q \backslash \overline{Q}^S$, where $\overline{Q}^S$ denotes the closure of Q^S. We use this splitting of the domain to further split also any function f defined on Q into

$$f(t,x) = \begin{cases} f^F(t,x) & \text{for } (t,x) \in Q^F, \\ f^S(t,x) & \text{for } (t,x) \in Q^S \end{cases}$$

in order to distinguish between its fluid part f^F and its solid part f^S whenever it is necessary to stress the difference. The motion of both the fluid and the body is then described by the velocity field $u : Q \to \mathbb{R}^3$, the density $\rho : Q \to \mathbb{R}$ and, in case of the fluid, also by the pressure $p = p^F : Q^F \to \mathbb{R}$. The electromagnetic effects in the system are characterized in by the magnetic induction $B : Q \to \mathbb{R}^3$, the magnetic field $H : Q \to \mathbb{R}^3$, the electric field $E : Q \to \mathbb{R}^3$ and the electric current $j : Q \to \mathbb{R}^3$. The evolution of the system is described by the following equations:

$$\text{curl} H = j + J \qquad \text{in } Q^F, \tag{1}$$

$$\text{curl} H = 0 \qquad \text{in } Q^S, \tag{2}$$

$$\partial_t B + \text{curl} E = 0 \qquad \text{in } Q, \tag{3}$$

$$\text{div} E = 0 \qquad \text{in } Q^S, \tag{4}$$

$$\text{div} B = 0 \qquad \text{in } Q, \tag{5}$$

$$\nabla \cdot u = 0, \quad \partial_t \rho + u \cdot \nabla \rho = 0 \qquad \text{in } Q^F \tag{6}$$

$$\partial_t(\rho u) + \text{div}(\rho u \otimes u) + \nabla p = 2\nu \text{div} \mathbb{D}(u) + \rho g + \frac{1}{\mu} \text{curl} B \times B \qquad \text{in } Q^F, \tag{7}$$

$$m \frac{d}{dt} V(t) = \frac{d}{dt} \int_{\overline{S}(t)} \rho u \, dx = \int_{\partial \overline{S}(t)} [2\nu \mathbb{D}(u) - p\text{Id}] \cdot \text{n} \, d\sigma + \int_{\overline{S}(t)} \rho g \, dx, \quad t \in [0,T] \tag{8}$$

$$\begin{aligned} \frac{d}{dt} (\mathbb{J}(t) w(t)) &= \frac{d}{dt} \int_{\overline{S}(t)} \rho (x - X) \times u \, dx \\ &= \int_{\partial \overline{S}(t)} (x - X) \times [2\nu \mathbb{D}(u) - p\text{Id}] \, \text{n} \, d\sigma + \int_{\overline{S}(t)} \rho (x - X) \times g \, dx, \quad t \in [0,T] \end{aligned} \tag{9}$$

supplemented by the relations

$$j = \sigma(E + u \times B) \quad \text{in } Q, \quad \sigma = \begin{cases} \sigma^F > 0 & \text{in } Q^F, \\ \sigma^S = 0 & \text{in } Q^S, \end{cases} \tag{10}$$

$$B = \mu H, \qquad \mu > 0 \quad \text{in } Q \tag{11}$$

and completed by the boundary and interface conditions

$$B(t) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad\qquad B^F(t) - B^S(t) = 0 \quad \text{on } \partial S(t), \tag{12}$$

$$E(t) \times \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad\qquad \big(E^F(t) - E^S(t)\big) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{13}$$

$$u(t) = 0 \quad \text{on } \partial\Omega, \qquad\qquad u^F(t) - u^S(t) = 0 \quad \text{on } \partial S(t). \tag{14}$$

The electromagnetic part (1)–(5) of the system is the reduced Maxwell system, c.f. [22], [23], while the mechanical part (6)–(9) consists of the balance of mass and momentum for the fluid and solid respectively, c.f. for example [14]. More precisely, in the Maxwell system we first have Ampère's law (1). We remark that, as in [22] and [23], this equation contains an additional source term J, which does not pose any further mathematical difficulties. On the non-conducting solid Ampère's law reduces to the condition (2). The system further contains the Maxwell-Faraday equation (3), Gauss's law (4) and Gauss's law for magnetism (5). We remark that, as in [22] and [23], these equations were here adjusted to the case that the solid is an insulator and further simplified by the magnetohydrodynamic approximation under some physical assumptions. A physical reasoning for the simplifications carried out due to the magnetohydrodynamic approximation can be found for example in [27, 28]. The reduced Maxwell system is supplemented by the relations (10), (11) and the boundary and interface conditions (12), (13). The equation (10) is known as Ohm's law in which the piecewise constant coefficient σ stands for the electrical conductivity, vanishing inside of insulators. It determines the effect of the fluid motion on the electromagnetic quantities. The constitutive relation (11) relates the magnetic induction to the magnetic field, see [26, Section 5.8]. In our case we want to point out that, in contrast to σ, the magnetic permeability μ is chosen as a constant value on the whole domain Q. As the magnetic permeability depends on the material, this is in general not physically accurate but a simplification required for the continuity of B across the interface stated in (12). The latter condition is needed to ensure that B is an element of some Sobolev space on Q, c.f. (21) below. The remaining conditions stated in (12) and (13) are standard.
Turning now to the mechanical part of the above system, we can identify the equations (6) and (7) as the incompressible Navier-Stokes equations made up of the continuity equation and incompressibility condition (6) and the momentum equation (7). In contrast to the purely mechanical case, (7) contains, in addition to the given external force g, a forcing term $\frac{1}{\mu}\mathrm{curl}B \times B$ - a reduced form of the Lorentz force - resulting from the electromagnetic interaction. Moreover, the operator $\mathbb{D}$ denotes the symmetric part of the gradient,

$$\mathbb{D}(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T.$$

and $\nu > 0$ is the viscosity coefficient. The relations (8) and (9) represent the balance of linear and angular momentum of the rigid body respectively. Indeed, since the body is rigid, its movement is characterized by its translational velocity V and its rotational velocity w; the overall velocity of the rigid body is determined as

$$u(t,x) = V(t) + w(t) \times (x - X(t)) \quad \text{on } Q^S.$$

The further notation in (8) and (9),

$$m := \int_{\overline{S}(t)} \rho(t,x)\, dx, \quad X(t) := \frac{1}{m}\int_{\overline{S}(t)} \rho(t,x)x\, dx,$$

$$\mathbb{J}(t)a \cdot b := \int_{\overline{S}(t)} \rho(t,x)\left[a \times (x - X(t))\right] \cdot \left[b \times (x - X(t))\right]\, dx, \quad a, b \in \mathbb{R}^3,$$

represent the mass, the center of mass and the inertia tensor of the rigid body respectively, while n denotes the outer unit normal vector on $\partial\Omega$ and $\partial S(t)$. The right-hand sides of (8) and (9) show that the movement of the rigid body is driven by the volume force g and the Cauchy stress $2\nu\mathbb{D}(u) - p\mathrm{Id}$, acting upon the interface of the body and the fluid via the integral over $\partial\overline{S}(t)$ in (8) and (9). Moreover, the fluid-structure interaction is incorporated in the second equation in (14). This relation, known as no-slip condition, imposes continuity of the velocity field across the interface between fluid and solid. Together with the no-slip boundary condition on $\partial\Omega$, i.e. the first equation in (14), this represents a standard set of boundary and transition conditions assumed commonly for the interaction between fluids and rigid bodies, c.f. for example [14], [37].

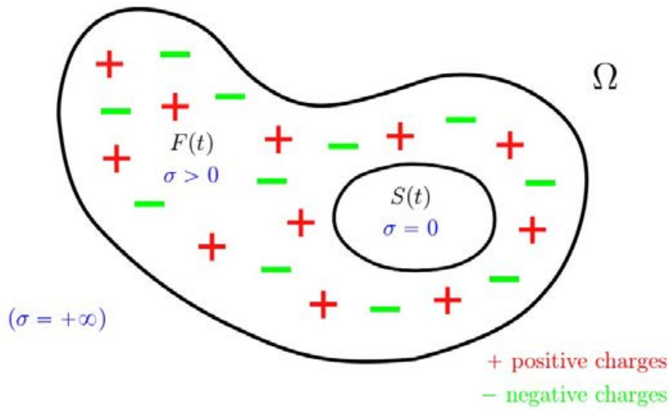

Figure 1: An insulating rigid body with domain $S(t)$ inside of an incompressible electrically conducting fluid with domain $F(t)$.

1.2 Weak formulation and main result

In order to define a weak formulation and state the main result, we first introduce some more notation. We use the standard Lebesgue-, Sobolev- and Bochner spaces and in addition the spaces

$$V^r(\Omega) := H^{r,2}_{\mathrm{div}}(\Omega) \quad \text{for } r \geqslant 0, \qquad V^r_0(\Omega) := \left\{ v \in H^{r,2}_{\mathrm{div}}(\Omega) : \ v|_{\partial\Omega} = 0 \right\} \quad \text{for } r > \frac{1}{2},$$

where $H^{r,2}_{\mathrm{div}}(\Omega)$ denotes the space of functions in $H^{r,2}(\Omega)$ which are in addition divergence-free. By S we denote a subset of Ω such that

$$S \text{ is open, bounded and connected, } S \neq \emptyset, \ |\partial S| = 0 \text{ and } \operatorname{dist}(S, \partial\Omega) > 0. \tag{15}$$

In the following we will describe the position of the rigid body through the use of a characteristic function. More precisely, for a function $\chi : \mathbb{R}^3 \to \{0, 1\}$ we write

$$S(\chi) := \left\{ x \in \mathbb{R}^3 : \ \chi(x) = 1 \right\}$$

and if $\chi(t) = \chi(t, \cdot)$ is a characteristic function for all $t \in [0, T]$, we write

$$Q^S(\chi, T) := \left\{ (t, x) \in [0, T] \times \mathbb{R}^3 : \ \chi(t, x) = 1 \right\} = \left\{ (t, x) \in [0, T] \times \mathbb{R}^3 : \ x \in S(\chi(t)) \right\}.$$

We further introduce the space of test functions

$$\mathcal{T}(\chi, T) := \left\{ \phi \in \mathcal{D}\left([0, T) \times \Omega\right) : \ \mathrm{div}\phi = 0, \ \mathbb{D}(\phi) = 0 \quad \text{on an open neighbourhood of } \overline{Q}^S(\chi, T) \right\},$$

meaning that for any $\phi \in \mathcal{T}(\overline{Q}^S(\chi, T))$ there is $\sigma > 0$ such that

$$\mathbb{D}(\phi) = 0 \quad \text{in } \left\{ (t, x) \in Q(T) : \ \operatorname{dist}\left((t, x), \overline{Q}^S(\chi, T) \right) < \sigma \right\}. \tag{16}$$

Similarly, we define

$$Y(\chi, T) := \left\{ b \in \mathcal{D}([0,T) \times \Omega) : \text{curl } b = 0 \quad \text{on an open neighbourhood of } \overline{Q}^S(\chi, T) \right\}. \tag{17}$$

We can now introduce the notion of weak solutions to (6)–(13) in the following way:

Definition 1.1. *Let $T > 0$, let $B_0, u_0 \in L^2(\Omega; \mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ and let $\rho_0 \in L^\infty(\Omega; \mathbb{R})$. Let further $S \subset \Omega$ satisfy (15) and let $\chi_0 \in L^\infty(\Omega; \mathbb{R})$ denote the characteristic function of S. Then the set of functions $\{\chi, \rho, u, B\}$ is called a weak solution of the problem (6)–(13) on $[0, T]$, if*

$$\chi \in C\left([0,T]; L^p(\Omega; \mathbb{R})\right) \quad \forall 1 \leqslant p < \infty, \tag{18}$$

$$\rho \in C\left([0,T]; L^p(\Omega; \mathbb{R})\right) \quad \forall 1 \leqslant p < \infty, \tag{19}$$

$$u \in L^\infty\left(0,T; L^2\left(\Omega; \mathbb{R}^3\right)\right) \bigcap L^2(0,T; V_0^1(\Omega)), \quad \mathbb{D}(u) = 0 \ \text{a.e. in } Q^S(\chi, T), \tag{20}$$

$$B \in L^\infty\left(0,T; L^2\left(\Omega; \mathbb{R}^3\right)\right) \bigcap L^2(0,T; V^1(\Omega)), \quad \operatorname{curl} B = 0 \ \text{a.e. in } Q^S(\chi, T), \quad B \cdot n = 0 \ \text{on } \partial\Omega, \tag{21}$$

and

$$-\int_0^T \int_\Omega \chi \partial_t \Theta dx dt - \int_\Omega \chi_0 \Theta(0,x)\, dx = \int_0^T \int_\Omega (\chi u) \cdot \nabla \Theta \, dx dt, \tag{22}$$

$$-\int_0^T \int_\Omega \rho \partial_t \psi dx dt - \int_\Omega \rho_0 \psi(0,x)\, dx = \int_0^T \int_\Omega (\rho u) \cdot \nabla \psi \, dx dt, \tag{23}$$

$$\begin{aligned} -\int_0^T \int_\Omega \rho u \cdot \partial_t \phi \, dx dt - \int_\Omega \rho_0 u_0 \cdot \phi(0,x)\, dx = \int_0^T \int_\Omega & (\rho u \otimes u) : \nabla \phi - 2\nu \mathbb{D}(u) : \nabla \phi \\ & + \rho g \cdot \phi + \frac{1}{\mu} (\operatorname{curl} B \times B) \cdot \phi \, dx dt, \end{aligned} \tag{24}$$

$$-\int_0^T \int_\Omega B \cdot \partial_t b \, dx dt - \int_\Omega B_0 \cdot b(0,x)\, dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B + \frac{1}{\sigma} J \right] \cdot \operatorname{curl} b \, dx dt \tag{25}$$

for all Θ, $\psi \in \mathcal{D}([0,T) \times \Omega)$, $\phi \in \mathcal{T}(\chi, T)$ and $b \in Y(\chi, T)$.

We can now state our main result:

Theorem 1.1. *Let $T > 0$, $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain. Let further χ_0, ρ_0, u_0, B_0 and S be as in Definition 1.1 and assume $\partial\Omega$, ∂S to be of class $C^2 \bigcap C^{0,1}$. Finally, assume $g, J \in L^\infty(Q; \mathbb{R}^3)$ and assume that σ, μ, ν, $\underline{\rho}$, $\overline{\rho}$ are positive constants with*

$$0 < \underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho} < \infty \quad \text{a.e. on } \Omega.$$

Then there exist $T' > 0$ and a weak solution to the problem (6)–(13) on $[0, T']$ in the sense of Definition 1.1, satisfying the energy inequality

$$\begin{aligned} &\int_\Omega \frac{1}{2} \rho(\tau) |u(\tau)|^2 + \frac{1}{2} |B(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu \left|\nabla u(t,x)\right|^2 + \frac{1}{\sigma\mu^2} \left|\operatorname{curl} B(t,x)\right|^2 \, dx dt \\ &\leqslant \int_\Omega \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{2} |B_0|^2 \, dx + \int_0^\tau \int_\Omega \rho(t,x) g(t,x) \cdot u(t,x) + \frac{1}{\sigma} J(t,x) \cdot \operatorname{curl} B(t,x) \, dx dt \end{aligned} \tag{26}$$

for almost all $\tau \in [0, T']$. Moreover, there is an isometry $X(s; t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$S(\chi(t)) = X(s; t, S(\chi(s))) \quad \forall s, t \in [0, T'] \tag{27}$$

and T' can be chosen such that

$$T' = \sup \left\{ \tau \in [0,T] : \ \operatorname{dist}(S(\chi(t)), \partial\Omega) > 0 \quad \forall t \leqslant \tau \right\}. \tag{28}$$

Remark 1.1. *Since the solution ρ to the continuity equation (23), given by Theorem 1.1, satisfies $\rho \in L^2((0,T) \times \Omega)$, the transport theorem by DiPerna and Lions [10] implies that ρ also solves the renormalized continuity equation*

$$\partial_t \beta(\rho) + u \cdot \nabla \beta(\rho) = 0 \quad \text{in } (0,T) \times \Omega \tag{29}$$

in the distributional sense for any bounded $\beta \in C^1(\mathbb{R})$ vanishing near 0 and such that also $(\beta'(1+|\cdot|))^{-1}$ is bounded.

Remark 1.2. *In [14], the existence of a global-in-time solution to a fluid-rigid body system is shown by extending the solution after a possible collision with the boundary by a solution to a problem in which the body is now considered as a fixed part of the exterior of the domain. With Theorem 1.1 at hand, the same method could probably also be applied here to infer global-in-time existence.*

The proof of Theorem 1.1 will be accomplished via an approximation method in Sections 3–6 and is outlined in the following section.

2 Approximate system

We introduce the approximate system to the original system of equations and we show the existence of a solution to this approximation. The limit of this solution then again is supposed to give us the solution to the original problem. More precisely, we will use three different levels of approximation, characterized through 3 parameters $\Delta t, \epsilon, \eta > 0$:

- On Δt-level, we have a time discretization by the Rothe method, c.f. [36, Section 8.2]. To this end, $\Delta t > 0$ is chosen in such a way that $\frac{T}{\Delta t} \in \mathbb{N}$ and we split up the interval $[0,T]$ into the discrete times $k\Delta t$, $k = 1, ..., \frac{T}{\Delta t}$.
- On the ϵ-level, we add several regularization terms to the system, which help us to solve the approximate system and pass to the limit as $\Delta t \to 0$.
- On the η-level we add a penalization term to the momentum equation, which guarantees us that after passing to the limit in $\eta \to 0$, the limit velocity will coincide - on the solid part of the domain - with the rigid velocity of the body.

We now introduce the full approximate system, containing all three levels of the approximation, and give a more detailed explanation afterwards: Assuming that for some discrete time $k\Delta t$, $k \in \{1, ..., \frac{T}{\Delta t}\}$ the solution at time $(k-1)\Delta t$, indexed by $k-1$, has already been found and defining the test function space

$$W^k(\Omega) := \left\{ b \in H^{2,2}(\Omega) : \ b \cdot \mathrm{n}|_{\partial\Omega} = 0, \ \mathrm{curl}\ b = 0 \text{ in } \left\{ x \in \Omega : \ \chi^k_{\Delta t}(x) = 1 \right\} \right\}, \tag{30}$$

we seek functions

$$\rho^k_{\Delta t} : \ \Omega \to \mathbb{R}, \quad u^k_{\Delta t}, \ B^k_{\Delta t} : \ \Omega \to \mathbb{R}^3, \quad \chi_{\Delta t,k} : \ [(k-1)\Delta t, k\Delta t] \times \Omega \to \mathbb{R},$$

satisfying the discrete system at time $k\Delta t$,

$$-\int_{(k-1)\Delta t}^{k\Delta t}\int_{\mathbb{R}^3}\chi_{\Delta t,k}\partial_t\Theta\ dxdt=\int_{\mathbb{R}^3}\chi_{\Delta t}^{k-1}\Theta((k-1)\Delta t,x)\ dx-\int_{\mathbb{R}^3}\chi_{\Delta t}^{k}\Theta(k\Delta t,x)\ dx$$
$$+\int_{(k-1)\Delta t}^{k\Delta t}\left(\chi_{\Delta t,k}\Pi_{\Delta t}^{k-1}\right)\cdot\nabla\Theta\ dxdt,\tag{31}$$

$$-\int_{\Omega}\frac{\rho_{\Delta t}^{k}-\rho_{\Delta t}^{k-1}}{\Delta t}\psi dx=\int_{\Omega}u_{\Delta t}^{k-1}\cdot\nabla\rho_{\Delta t}^{k}\psi+\epsilon\nabla\rho_{\Delta t}^{k}\cdot\nabla\psi\ dx,\tag{32}$$

$$-\int_{\Omega}\frac{\rho_{\Delta t}^{k}u_{\Delta t}^{k}-\rho_{\Delta t}^{k-1}u_{\Delta t}^{k-1}}{\Delta t}\cdot\phi\ dx=\int_{\Omega}\left[\operatorname{div}\left(\rho_{\Delta t}^{k}u_{\Delta t}^{k-1}\otimes u_{\Delta t}^{k}\right)-2\nu\operatorname{div}\left(\mathbb{D}u_{\Delta t}^{k}\right)+\epsilon\nabla u_{\Delta t}^{k}\nabla\rho_{\Delta t}^{k}\right]\cdot\phi\ dx$$
$$+\int_{\Omega}\epsilon\Delta u_{\Delta t}^{k}\Delta\phi+\left[\frac{1}{\eta}\rho_{\Delta t}^{k-1}\chi_{\Delta t}^{k}\left(u_{\Delta t}^{k-1}-\Pi_{\Delta t}^{k-1}\right)-\rho_{\Delta t}^{k-1}g_{\Delta t}^{k}\right]\cdot\phi\ dx$$
$$-\int_{\Omega}\frac{1}{\mu}\left(\operatorname{curl}B_{\Delta t}^{k-1}\times B_{\Delta t}^{k-1}\right)\cdot\phi\ dx,\tag{33}$$

$$-\int_{\Omega}\frac{B_{\Delta t}^{k}-B_{\Delta t}^{k-1}}{\Delta t}\cdot b\ dx=\int_{\Omega}\left[\frac{1}{\sigma\mu}\operatorname{curl}B_{\Delta t}^{k}-u_{\Delta t}^{k}\times B_{\Delta t}^{k-1}+\frac{\epsilon}{\mu^2}\left|\operatorname{curl}B_{\Delta t}^{k}\right|^2\operatorname{curl}B_{\Delta t}^{k}\right]\cdot\operatorname{curl}b\ dx$$
$$+\int_{\Omega}-\frac{1}{\sigma}J_{\Delta t}^{k}\cdot\operatorname{curl}b+\epsilon\operatorname{curl}\left(\operatorname{curl}B_{\Delta t}^{k}\right)\cdot\operatorname{curl}\left(\operatorname{curl}b\right)\ dx\tag{34}$$

for all $\Theta\in\mathcal{D}([(k-1)\Delta t,k\Delta t]\times\mathbb{R}^3)$, $\psi\in H^{1,2}(\Omega)$, $\phi\in V_0^2(\Omega)$ and $b\in W^k(\Omega)$. Here, the functions $\chi_{\Delta t}^{k}$ and $\Pi_{\Delta t}^{k-1}$, introduced in the equations (31) and (33), are defined by:

$$\chi_{\Delta t}^{k}:=\chi_{\Delta t,k}(k\Delta t),\quad\Pi_{\Delta t}^{k-1}=(u_G)_{\Delta t}^{k-1}+\omega_{\Delta t}^{k-1}\times\left(x-a_{\Delta t}^{k-1}\right)\tag{35}$$

and

$$(u_G)_{\Delta t}^{k-1}:=\frac{\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}u_{\Delta t}^{k-1}\ dx}{\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}\ dx},\tag{36}$$

$$\omega_{\Delta t}^{k-1}:=\left(I_{\Delta t}^{k-1}\right)^{-1}\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}\left(x-a_{\Delta t}^{k-1}\right)\times u_{\Delta t}^{k-1}\ dx,\tag{37}$$

$$I_{\Delta t}^{k-1}:=\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}\left(|x-a_{\Delta t}^{k-1}|^2\mathrm{id}-\left(x-a_{\Delta t}^{k-1}\right)\otimes\left(x-a_{\Delta t}^{k-1}\right)\right)\ dx,\tag{38}$$

$$a_{\Delta t}^{k-1}:=\frac{\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}x\ dx}{\int_{\mathbb{R}^3}\rho_{\Delta t}^{k}\chi_{\Delta t}^{k-1}\ dx}.\tag{39}$$

In order to keep the latter terms well-defined, we extend the functions $\rho_{\Delta t}^{l}$ by $\underline{\rho}$ and $u_{\Delta t}^{l}$ by 0 outside of Ω for any $l=0,...,k$. Moreover, the quantities $g_{\Delta t}^{k}$ and $J_{\Delta t}^{k}$ from (33) and (34) are defined in the following way: Since the given functions g and J are by assumption only L^∞-functions and thus not necessarily defined in the discrete times, we first need to introduce some mollifications. For example, we can define

$$g_\gamma(t):=\int_0^T\theta_\gamma\left(t+\xi_\gamma(t)-s\right)g(s)\ ds,\quad J_\gamma(t):=\int_0^T\theta_\gamma\left(t+\xi_\gamma(t)-s\right)J(s)\ ds,\quad\xi_\gamma(t):=\gamma\frac{T-2t}{T},$$

where $\theta_\gamma:\mathbb{R}\to\mathbb{R}$ is a mollifier. Then we choose $\gamma=(\gamma\Delta t)$, $\gamma(\Delta t)\to0$ for $\Delta t\to0$ and set

$$g_{\Delta t}^{k}:=g_{\gamma(\Delta t)}(k\Delta t),\quad J_{\Delta t}^{k}:=J_{\gamma(\Delta t)}(k\Delta t).\tag{40}$$

The idea behind the time discretization is to decouple the system, so that we can solve the equations one after another by using retarded functions in the coupling terms. In particular, it helps us with solving the induction equation, in which - even on approximation level - the test functions depend on the characteristic function of the rigid body and therefore on the solution of the overall system itself, c.f. (17). In our discretization, we will be able to first determine the position of the solid up to

a certain discrete time and subsequently choose the test functions for the induction equation at this specific time accordingly. The existence of the magnetic induction on the discrete level then follows by standard methods.
We also want to point out, that the function $\chi_{\Delta t,k}$ represents an exception in this system: It is the only function, which is immediately constructed as a time-dependent function. The reason for this is that we want it to take only the values 0 and 1 so that we are able to determine the position of the rigid body at any time. Inspired by [21], we can guarantee this by constructing $\chi_{\Delta t,k}$ by solving a classical transport equation on the small interval $[(k-1)\Delta t, k\Delta t]$, in case of a discrete transport equation we might lose the property.
Next we note that the mapping $\Pi^{k-1}_{\Delta t}$ is, by definition, a rigid velocity field with the translational velocity $(u_G)^{k-1}_{\Delta t}$ and the rotational velocity $w^{k-1}_{\Delta t}$. The constant terms $I^{k-1}_{\Delta t}$ and $a^{k-1}_{\Delta t}$ can be considered as a discrete version of the inertia tensor and the center of mass of the rigid body described by the characteristic function $\chi_{\Delta t}$ at time $k\Delta t$. Once we will have passed to the limit in the time discretization, we will see that the limit of $\overline{\Pi}'_{\Delta t}$ is actually the projection of the velocity onto a rigid velocity field. This comes into play in the penalization term from the η-level of the approximation mentioned above, namely the term

$$\frac{1}{\eta}\rho^{k-1}_{\Delta t}\chi^{k}_{\Delta t}\left(u^{k-1}_{\Delta t}-\Pi^{k-1}_{\Delta t}\right),$$

from (33). As mentioned before, we can use this term to infer that after letting $\eta \to 0$ the limit velocity coincides, in the solid area, with the velocity of the rigid body, which is what we require to obtain (22). This penalization method, which is known as Brinkman penalization, is discussed rigorously in [4]. Physically speaking, it describes an extension of the fluid into the solid region, i.e. the approximate body, while still moving via a rigid velocity field, is now permeable and the limit passage $\eta \to 0$ represents the process of letting the permeability vanish. This technique can be considered as an extension of the penalty method used in [2] for a fluid-structure interaction problem in which the movement of the solid is prescribed. It further finds use in [33], where the examined solid is additionally deformable and self-propelled and it is moreover of interest for finite element approaches to the problem, c.f. [7], [24]. There are also other penalization methods available as for example in [37], where an approach is used in which the solids are approximated by a fluid with viscosity rising to infinity.
Finally, it remains to discuss the various regularization terms from the ϵ-level of the approximation. In the continuity equation, the Laplacian of the density is added to the right-hand side, which allows us to show an upper bound for ρ as well as some bound away from 0. This is needed because such a bound cannot be guaranteed from the discrete version of the transport equation. In order to compensate for this term in the energy inequality, the term $\epsilon\nabla u^k_{\Delta t}\nabla\rho^k_{\Delta t}$ is added to the momentum equation. The second new term in this equation, $\epsilon\Delta^2 u^k_{\Delta t}$, is needed for passing to the limit in $\Delta t \to 0$. Moreover, we have two regularization terms in the induction equation, the 4-th curl of the magnetic induction and the term $\mathrm{curl}(|\mathrm{curl}B^k_{\Delta t}|^2\mathrm{curl}B^k_{\Delta t})$ which is also known as the 4-double-curl. The first one is used for the construction of $B^k_{\Delta t}$ via a weakly continuous coercive operator, while we require the latter one in the energy inequality: in the time-dependent version of the system, the mixed terms from the momentum and the induction equation cancel each other. On the discrete level this is not the case, as the involved functions are chosen from distinct discrete times. However, the 4-double-curl enables us to absorb the problematic terms into the positive left-hand side, so that we can get the uniform bounds needed for the limit passage as $\Delta t \to 0$. We also remark, that the 4-double-curl was chosen instead of the 4-Laplacian in order to allow us to apply the Helmholtz-decomposition [40, Theorem 4.2]. This is why the test functions $b \in W^k(\Omega)$, which are not divergence-free, can be used in (34). We complement the equations by the relations

$$\begin{gathered}\operatorname{div} u^k_{\Delta t} = \operatorname{div} B^k_{\Delta t} = 0 \quad \text{in } \Omega, \qquad \operatorname{curl} B^k_{\Delta t} = 0 \quad \text{in } \left\{x \in \Omega :\ \chi^k_{\Delta t}(x) = 1\right\}, \\ u^k_{\Delta t} = 0 \quad \text{on } \partial\Omega, \qquad B^k_{\Delta t}\cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \\ \rho^0_{\Delta t} = \rho_0, \qquad \chi_{\Delta t,k}(0) = \chi_0, \qquad u^0_{\Delta t} = u_0, \qquad B^0_{\Delta t} = B_0. \end{gathered} \tag{41}$$

In Section 3 we will show the existence of a solution to this approximate system. In Section 4 we will pass to the limit with respect to $\Delta t \to 0$, i.e. we will return from the discrete system back to a system depending on time. Afterwards, in Section 5, we let the regularization terms vanish from our approximation by letting $\epsilon \to 0$. Finally, in Section 6, we will obtain the desired solution to our original system by sending also the last parameter η to 0.

3 Existence of the approximate solution

In this section we prove the existence of a solution to the approximate system. To this end, we first introduce another function space for fixed discrete time indices k:

$$Y^k(\Omega) := \Big\{ b \in L^2\left(\Omega;\mathbb{R}^3\right) : \ b \cdot \mathrm{n}|_{\partial\Omega} = 0, \ \mathrm{div}\, b = 0 \text{ in } \Omega, \ \mathrm{curl}\, b = 0 \text{ in } \Big\{ x \in \Omega : \ \chi_{\Delta t}^k(x) = 1 \Big\}, \\ \mathrm{curl}\,(\mathrm{curl}\, b) \in L^2(\Omega) \Big\}.$$

While the more general space $W^k(\Omega)$ in (30), containing also functions which are not divergence-free, serves as a test function space for the induction equation at the discrete time $k\Delta t$, the space $Y^k(\Omega)$ will be the space in which we construct the magnetic induction at time $k\Delta t$. As for functions $b \in Y^k(\Omega)$ it holds $\mathrm{curl}(\mathrm{curl} b) = \Delta b$, both of these spaces can be equipped with the $H^{2,2}$-norm.

Proposition 3.1. *Let all the assumptions of Theorem 1.1 be satisfied and $\Delta t > 0$. Let further $g_{\Delta t}^k$ and $J_{\Delta t}^k$ be given by (40) for any $k = 0, ..., \frac{T}{\Delta t}$ and assume in addition that*

$$\rho_0 \in H^{1,2}(\Omega), \quad u_0, \ B_0 \in H^{2,2}(\Omega).$$

Then, for all $k = 1, ..., \frac{T}{\Delta t}$, there exist functions $\chi_{\Delta t,k} \in C([(k-1)\Delta t, k\Delta t]; L^p_{loc}(\mathbb{R}^3))$, $1 \leqslant p < \infty$ and

$$\rho_{\Delta t}^k \in H^{1,2}(\Omega), \quad \underline{\rho} \leqslant \rho_{\Delta t}^k \leqslant \overline{\rho}, \qquad u_{\Delta t}^k \in V_0^2(\Omega), \quad B_{\Delta t}^k \in Y^k(\Omega) \tag{42}$$

which satisfy the variational equations (31)–(34) for all test functions $\Theta \in \mathcal{D}([(k-1)\Delta t, k\Delta t] \times \mathbb{R}^3)$, $\psi \in H^{1,2}(\Omega)$, $\phi \in V_0^2(\Omega)$ and $b \in W^k(\Omega)$.

Proof

We consider some discrete time index $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$ and assume that the proposition is already proved for all $l = 1, ..., k-1$.

Step 1: The existence of a solution $\rho_{\Delta t}^k \in H^{1,2}(\Omega)$ to (32) follows immediately from the Lax-Milgram Lemma. Further, we may test (32) by $\max\left\{\rho_{\Delta t}^k - \overline{\rho}, 0\right\}$ to see

$$\int_\Omega \frac{\left|\max\left\{\rho_{\Delta t}^k - \overline{\rho}, 0\right\}\right|^2}{\Delta t}\, dx \leqslant \int_\Omega \frac{\left(\rho_{\Delta t}^{k-1} - \overline{\rho}\right)\max\left\{\rho_{\Delta t}^k - \overline{\rho}, 0\right\}}{\Delta t}\, dx \leqslant 0,$$

c.f. [34, Section 7.6.5]. Arguing similarly for the lower bound, we arrive at the estimates for $\rho_{\Delta t}^k$ in (42).

Step 2: As in [21], we consider the initial value problem

$$\frac{\partial X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x)}{\partial t} = \Pi_{\Delta t}^{k-1}\left(X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x)\right), \qquad X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;s,x) = x, \quad x \in \mathbb{R}^3, \quad s,t \in \mathbb{R}, \tag{43}$$

where t represents the time variable and s the initial time. Since $\Pi_{\Delta t}^{k-1}$ is constant in time and a rigid velocity field by (35), it is in particular Lipschitz-continuous. Then by the theory of ordinary differential equations, (43) defines a unique mapping

$$X_{\Delta t}^{\Pi_{\Delta t}^{k-1}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3. \tag{44}$$

We set

$$\chi_{\Delta t,k}(t,x) := \chi_{\Delta t,k-1}\left((k-1)\Delta t, X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(t;(k-1)\Delta t,x)\right) \quad \text{for } t \in [(k-1)\Delta t, k\Delta t] \tag{45}$$

and infer from [10, Theorem III.2] that this is the (unique renormalized) solution to the transport equation (31).

Step 3: We consider the operator

$$A: V_0^2(\Omega) \to \left(V_0^2(\Omega)\right)^*, \quad \langle Au, v\rangle_{(V_0^2(\Omega))^* \times V_0^2(\Omega)} := \int_\Omega \left(\frac{\rho_{\Delta t}^k u}{\Delta t}\right)\cdot v + \operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1} \otimes u\right)\cdot v$$
$$+2\nu\mathbb{D}(u) : \nabla v + \epsilon\left(\nabla u \nabla \rho_{\Delta t}^k\right)\cdot v + \epsilon(\Delta u)\cdot(\Delta v)\, dx$$

for $u, v \in V_0^2(\Omega)$. Because of the regularization term, A is coercive on $V_0^2(\Omega)$. Further, the bilinear form $\langle A\cdot,\cdot\rangle_{(V_0^2(\Omega))^*, V_0^2(\Omega)}$ is bounded on $V_0^2(\Omega)$ and hence the Lax-Milgram Lemma again implies the existence of $u_{\Delta t}^k \in V_0^2(\Omega)$ satisfying (33).

Step 4: We introduce

$$\tilde{A}: Y^k(\Omega) \to \left(Y^k(\Omega)\right)^*,$$
$$\left\langle \tilde{A}(B), b\right\rangle_{(Y^k(\Omega))^* \times Y^k(\Omega)} := \int_\Omega \frac{B}{\Delta t}\cdot b + \epsilon \operatorname{curl}(\operatorname{curl} B)\cdot \operatorname{curl}(\operatorname{curl} b)$$
$$+\left[\frac{1}{\sigma\mu}\operatorname{curl} B + \frac{\epsilon}{\mu^2}\left|\operatorname{curl} B\right|^2 \operatorname{curl} B\right]\cdot \operatorname{curl} b\, dx.$$

Clearly, $\tilde{A}$ is coercive:

$$\left\langle \tilde{A}(B), B\right\rangle_{(Y^k)^* \times Y^k} \geqslant \frac{1}{\Delta t}\|B\|_{L^2(\Omega)}^2 + \epsilon\|\Delta B\|_{L^2(\Omega)}^2 \geqslant c\|B\|_{H^{2,2}(\Omega)}^2 = c\|B\|_{Y^k}^2.$$

Further, if $B_n \rightharpoonup B$ in $Y^k(\Omega)$, then from the Rellich-Kondrachov embedding we know $B_n \to B$ in $W^{1,4}(\Omega)$, which again gives us weak continuity of $\tilde{A}$. Coercivity and weak continuity imply surjectivity of $\tilde{A}$ (see for example [16, Theorem 1.2]) and so we infer the existence of a solution $B_{\Delta t}^k \in Y^k(\Omega)$ to (34) for all $b \in Y^k$ and by the Helmholtz-decomposition [40, Theorem 4.2] even for all $b \in W^k(\Omega)$.

□

Remark 3.1. *For any fixed $s, t \in \mathbb{R}$ the mapping (44) is an isometry. Indeed, from $\Pi_{\Delta t}^{k-1}$ being a rigid velocity field and the ordinary differential equation (43), it follows that*

$$\frac{\partial}{\partial t}\left|X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x) - X_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,y)\right|^2 = 0 \tag{46}$$

for any $x, y \in \mathbb{R}^3$.

In the remainder of this section we derive an energy inequality for our discrete solution. To this end we extend, without loss of generality,

$$u_{\Delta t}^l(x) = 0, \quad \rho_{\Delta t}^l(x) = \underline{\rho}, \qquad \forall x \in \mathbb{R}^3\backslash\Omega,\ l = 0, ..., \frac{T}{\Delta t}. \tag{47}$$

We fix some $k \in \{1, ..., \frac{T}{\Delta t}\}$. For arbitrary $l \leqslant k$ we test the continuity equation (32) at the discrete time $l\Delta t$ by $\frac{1}{2}|u_{\Delta t}^l|^2$ and subtract the result from the momentum equation (33), also at time $l\Delta t$, tested by $u_{\Delta t}^l$. This yields

$$\int_\Omega \frac{1}{2\Delta t}\rho_{\Delta t}^l |u_{\Delta t}^l|^2 - \frac{1}{2\Delta t}\rho_{\Delta t}^{l-1}|u_{\Delta t}^{l-1}|^2 + 2\nu|\nabla u_{\Delta t}^l|^2 + \frac{1}{\eta}\rho_{\Delta t}^{l-1}\chi_{\Delta t}^l(u_{\Delta t}^{l-1} - \Pi_{\Delta t}^{l-1})\cdot u_{\Delta t}^l + \epsilon|\Delta u_{\Delta t}^l|^2\, dx$$
$$\leqslant \int_\Omega \rho_{\Delta t}^{l-1} g_{\Delta t}^l \cdot u_{\Delta t}^l + \frac{1}{\mu}(\operatorname{curl} B_{\Delta t}^{l-1} \times B_{\Delta t}^{l-1})\cdot u_{\Delta t}^l\, dx. \tag{48}$$

Next, we test the magnetic equation (34) at time $l\Delta t$ by $\frac{1}{\mu}B^l_{\Delta t}$ and estimate

$$\frac{1}{2\mu\Delta t}\left[\|B^l_{\Delta t}\|^2_{L^2(\Omega)} - \|B^{l-1}_{\Delta t}\|^2_{L^2(\Omega)}\right] + \frac{1}{\mu}\int_\Omega \frac{1}{\sigma\mu}|\mathrm{curl} B^l_{\Delta t}|^2 \, dx$$
$$\leqslant -\int_\Omega \frac{\epsilon}{\mu^3}|\mathrm{curl} B^l_{\Delta t}|^4 + \frac{\epsilon}{\mu}\left|\Delta B^l_{\Delta t}\right|^2 - \frac{1}{\mu}(u^l_{\Delta t}\times B^{l-1}_{\Delta t})\cdot \mathrm{curl} B^l_{\Delta t} - \frac{1}{\sigma\mu}J^l_{\Delta t}\cdot \mathrm{curl} B^l_{\Delta t}\, dx. \tag{49}$$

Adding this to (48) and summing over all $l \leqslant k$, we infer

$$\frac{1}{2\Delta t}\underline{\rho}\|u^k_{\Delta t}\|^2_{L^2(\Omega)} - \frac{1}{2\Delta t}\overline{\rho}\|u^0_{\Delta t}\|^2_{L^2(\Omega)} + \sum_{l=1}^k \left(2\nu\|\nabla u^l_{\Delta t}\|^2_{L^2(\Omega)} + \epsilon\|\Delta u^l_{\Delta t}\|^2_{L^2(\Omega)}\right)$$
$$+\frac{1}{2\mu\Delta t}\left(\|B^k_{\Delta t}\|^2_{L^2(\Omega)} - \|B^0_{\Delta t}\|^2_{L^2(\Omega)}\right) + \sum_{l=1}^k\left(\frac{1}{\sigma\mu}\|\mathrm{curl} B^l_{\Delta t}\|^2_{L^2(\Omega)} + \frac{\epsilon}{\mu^3}\|\mathrm{curl} B^l_{\Delta t}\|^4_{L^4(\Omega)} + \frac{\epsilon}{\mu}\left\|\Delta B^l_{\Delta t}\right\|^2_{L^2(\Omega)}\right)$$
$$\leqslant \sum_{l=1}^k \int_\Omega -\frac{1}{\eta}\rho^{l-1}_{\Delta t}\chi^l_{\Delta t}(u^{l-1}_{\Delta t} - \Pi^{l-1}_{\Delta t})\cdot u^l_{\Delta t} + \rho^{l-1}_{\Delta t} g^l_{\Delta t}\cdot u^l_{\Delta t} + \frac{1}{\mu}(\mathrm{curl} B^{l-1}_{\Delta t}\times B^{l-1}_{\Delta t})\cdot u^l_{\Delta t}$$
$$+\frac{1}{\mu}(u^l_{\Delta t}\times B^{l-1}_{\Delta t})\cdot \mathrm{curl} B^l_{\Delta t} + \frac{1}{\sigma\mu}J^l_{\Delta t}\cdot \mathrm{curl} B^l_{\Delta t} dx. \tag{50}$$

In order to estimate the right-hand side here, we need

$$\|\Pi^{l-1}_{\Delta t}\|_{L^2(\Omega)} \leqslant c\|u^{l-1}_{\Delta t}\|_{L^2(\Omega)}, \tag{51}$$

which can be proved in the following way: We distinguish between two cases, the first one being $\mathrm{supp}\chi^{l-1}_{\Delta t}\bigcap\Omega = \varnothing$. Then, as $u^{l-1}_{\Delta t} = 0$ outside of Ω, both sides of (51) are equal to zero and so the inequality is trivially satisfied. For the second case, $\mathrm{supp}\chi^{l-1}_{\Delta t}\bigcap\Omega \neq \varnothing$, we note that from (44) being an isometry and by (45) it follows the existence of a compact set K independent of l and Δt such that in this case $\mathrm{supp}\chi^{l-1}_{\Delta t}\subset K$. This allows us to reduce the integrals over $\mathbb{R}^3$ in (36) - (39) to integrals over K. Moreover, as $\rho^l_{\Delta t}\geqslant \underline{\rho}$ was extended by $\underline{\rho}$ on $\mathbb{R}^3\backslash\Omega$, we know

$$\int_{\mathbb{R}^3}\rho^l_{\Delta t}\chi^{l-1}_{\Delta t}\, dx \geqslant \underline{\rho}|S| > 0. \tag{52}$$

Thus we can estimate

$$\left|a^{l-1}_{\Delta t}\right| = \left|\frac{\int_{\mathbb{R}^3}\rho^l_{\Delta t}\chi^{l-1}_{\Delta t}x\, dx}{\int_{\mathbb{R}^3}\rho^l_{\Delta t}\chi^{l-1}_{\Delta t}l\, dx}\right| \leqslant c\left|\int_K \rho^l_{\Delta t}x\, dx\right| \leqslant c \tag{53}$$

with c independent of l and Δt. By similar computations, c.f. also [4, Section 3.2] and the proof of [33, Lemma 4], we obtain

$$\left|(u_G)^{l-1}_{\Delta t}\right| \leqslant c\left\|u^{l-1}_{\Delta t}\right\|_{L^2(\Omega)}, \quad \left|\omega^{l-1}_{\Delta t}\right| \leqslant c\left\|u^{l-1}_{\Delta t}\right\|_{L^2(\Omega)}, \quad v\cdot\left(I^{l-1}_{\Delta t}v\right) \geqslant c|u|^2 \quad \forall v\in\mathbb{R}^3, \tag{54}$$

where the last inequality uses that since (44) is an isometry, one can find for any $\Delta t > 0$, $l = 1, ..., \frac{T}{\Delta t}$ some ball $B_r(l,\Delta t)\subset\mathbb{R}^3$ with radius $r > 0$ independent of l and Δt such that $B_r(l,\Delta t)\subset S(\chi^{l-1}_{\Delta t})$. Thus, (51) is also satisfied in the second case. Now, exploiting (51) and applying Young's inequality, the right-hand side of (50) can be bounded by

$$\sum_{l=1}^k\left[\frac{\overline{\rho}}{2\eta}\|u^{l-1}_{\Delta t}\|^2_{L^2(\Omega)} + \frac{\overline{\rho}}{2\eta}\|u^l_{\Delta t}\|^2_{L^2(\Omega)} + \frac{c^2\overline{\rho}}{2}\|g\|^2_{L^\infty(\Omega)} + \frac{\overline{\rho}}{2}\|u^l_{\Delta t}\|^2_{L^2(\Omega)}\right.$$
$$+\frac{1}{\mu}\|\mathrm{curl} B^{l-1}_{\Delta t}\|_{L^4(\Omega)}\|B^{l-1}_{\Delta t}\|_{L^4(\Omega)}\|u^l_{\Delta t}\|_{L^2(\Omega)} + \frac{c^2}{2\sigma\mu}\|\mathrm{curl} J\|^2_{L^\infty(\Omega)} + \frac{1}{2\sigma\mu}\|B^l_{\Delta t}\|^2_{L^2(\Omega)}$$
$$\left.+\frac{1}{\mu}\|\mathrm{curl} B^l_{\Delta t}\|_{L^4(\Omega)}\|B^{l-1}_{\Delta t}\|_{L^4(\Omega)}\|u^l_{\Delta t}\|_{L^2(\Omega)}\right]. \tag{55}$$

Using the Poincaré-type estimate

$$\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)} \leqslant c\left\|\operatorname{curl} B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)},$$

c.f. [1, Corollary 3.4], we further estimate

$$\begin{aligned}
\frac{1}{\mu}\|\operatorname{curl} B_{\Delta t}^{l}\|_{L^4(\Omega)}\|B_{\Delta t}^{l-1}\|_{L^4(\Omega)}\|u_{\Delta t}^{l}\|_{L^2(\Omega)} &\leqslant \frac{c^2\mu}{\epsilon}\|u_{\Delta t}^{l}\|_{L^2(\Omega)}^2+\frac{\epsilon}{8\mu^3}\|\operatorname{curl} B_{\Delta t}^{l-1}\|_{L^4(\Omega)}^4+\frac{\epsilon}{8\mu^3}\|\operatorname{curl} B_{\Delta t}^{l}\|_{L^4(\Omega)}^4,\\
\frac{1}{\mu}\|\operatorname{curl} B_{\Delta t}^{l-1}\|_{L^4(\Omega)}\|B_{\Delta t}^{l-1}\|_{L^4(\Omega)}\|u_{\Delta t}^{l}\|_{L^2(\Omega)} &\leqslant \frac{c^2\mu}{\epsilon}\|u_{\Delta t}^{l}\|_{L^2(\Omega)}^2+\frac{\epsilon}{4\mu^3}\|\operatorname{curl} B_{\Delta t}^{l-1}\|_{L^4(\Omega)}^4.
\end{aligned}$$

Consequently, we can absorb several quantities from (55), including - for Δt sufficiently small - the L^2-norm of $u_{\Delta t}^k$, into the left-hand side of (50) and obtain

$$\begin{aligned}
&\frac{\underline{\rho}}{4\Delta t}\left\|u_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2+\sum_{l=1}^{k}\left(2\nu\left\|\nabla u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\epsilon\left\|\Delta u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right)+\frac{1}{4\mu\Delta t}\left\|B_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2\\
&+\sum_{l=1}^{k}\left(\frac{1}{\sigma\mu}\left\|\operatorname{curl} B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\epsilon}{2\mu^3}\left\|\operatorname{curl} B_{\Delta t}^{l}\right\|_{L^4(\Omega)}^4+\frac{\epsilon}{\mu}\left\|\Delta B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right)\\
\leqslant\ &\frac{\overline{\rho}}{2\Delta t}\left\|u_{\Delta t}^{0}\right\|_{L^2(\Omega)}^2+\frac{1}{2\mu\Delta t}\left\|B_{\Delta t}^{0}\right\|_{L^2(\Omega)}^2+\frac{Tc^2\overline{\rho}}{2\Delta t}\|g\|_{L^\infty(\Omega)}^2+\frac{Tc^2}{2\sigma\mu\Delta t}\|\operatorname{curl} J\|_{L^\infty(\Omega)}^2+\frac{\epsilon}{8\mu^3}\left\|\operatorname{curl} B_{\Delta t}^{0}\right\|_{L^4(\Omega)}^4\\
&+\frac{\epsilon}{4\mu^3}\left\|\operatorname{curl} B_{\Delta t}^{0}\right\|_{L^4(\Omega)}^4+\frac{\overline{\rho}}{2\eta}\left\|u_{\Delta t}^{0}\right\|_{L^2(\Omega)}^2+\sum_{l=1}^{k-1}\left[\frac{\overline{\rho}}{\eta}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\overline{\rho}}{2}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{2c^2\mu}{\epsilon}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right.\\
&\left.+\frac{1}{2\sigma\mu}\left\|B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right].
\end{aligned}\tag{56}$$

Hence, from the discrete Gronwall estimate (c.f. [36, (1.67)]), we infer the bound

$$\begin{aligned}
&\left\|u_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2+\Delta t\sum_{l=1}^{k}\left(\left\|\nabla u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\left\|\Delta u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right)+\left\|B_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2\\
&+\Delta t\sum_{l=1}^{k}\left(\left\|\operatorname{curl} B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\left\|\operatorname{curl} B_{\Delta t}^{l}\right\|_{L^4(\Omega)}^4+\left\|\Delta B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\right)\\
\leqslant\ & c(u^0,B^0,\overline{\rho},\underline{\rho},c,g,J,\sigma,\mu,\nu,\epsilon,\eta,T)\quad \forall k=1,\dots,\frac{T}{\Delta t}
\end{aligned}\tag{57}$$

uniformly in Δt and k.

4 Limit passage with respect to $\Delta t \to 0$

We now want to pass to the limit in the time discretization, i.e. $\Delta t \to 0$. To do so, we introduce piecewise constant as well as piecewise affine interpolants of our functions defined so far only in the discrete time points. Namely, for the time-independent quantities $f_{\Delta t}^k$ defined for $k=0,\dots,\frac{T}{\Delta t}$ we set

$$f_{\Delta t}(t) := \left(\frac{t}{\Delta t}-(k-1)\right)f_{\Delta t}^{k}+\left(k-\frac{t}{\Delta t}\right)f_{\Delta t}^{k-1}\qquad \text{for } (k-1)\Delta t<t\leqslant k\Delta t,\quad k=1,\dots,\frac{T}{\Delta t},\tag{58}$$

$$\overline{f}_{\Delta t}(t) := f_{\Delta t}^{k}\qquad \text{for } (k-1)\Delta t<t\leqslant k\Delta t,\quad k=0,\dots,\frac{T}{\Delta t},\tag{59}$$

$$\overline{f}'_{\Delta t}(t) := f_{\Delta t}^{k-1}\qquad \text{for } (k-1)\Delta t<t\leqslant k\Delta t,\quad k=1,\dots,\frac{T}{\Delta t}.\tag{60}$$

We will use the same notation also for the interpolation of the discrete momentum function $(\rho u)_{\Delta t}^k := \rho_{\Delta t}^k u_{\Delta t}^k$, $k=0,\dots,\frac{T}{\Delta t}$. Regarding the solution to the transport equation on $[0,T]$, we glue together the already time-dependent functions $\chi_{\Delta t,k}$, defined on the intervals $[(k-1)\Delta t, k\Delta t]$. More specifically, we set

$$\chi_{\Delta t}(t) := \chi_{\Delta t,k}(t) \qquad \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}.$$

By the construction of $\chi_{\Delta t,k}$ in Proposition 3.1 it holds $\chi_{\Delta t} \in C([0,T]; L^p_{\text{loc}}(\mathbb{R}^3))$, $1 \leqslant p < \infty$, and $\chi_{\Delta t}$ is the solution to

$$-\int_0^T \int_{\mathbb{R}^3} \chi_{\Delta t} \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} \left(\chi_{\Delta t} \overline{\Pi}'_{\Delta t}\right) \cdot \nabla\Theta\ dx dt \tag{61}$$

for any $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$. According to the transport theory by DiPerna and Lions, c.f. [10, Thoerem III.2], this solution is unique and can be represented by

$$\chi_{\Delta t}(t,x) := \chi_0 \left(X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}(t;0,x) \right) \quad \text{for } t \in [0,T]. \tag{62}$$

Here $X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}$ denotes the unique solution to the initial value problem

$$\frac{\partial X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}(s;t,x)}{\partial t} = \overline{\Pi}'_{\Delta t} \left(t, X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}(s;t,x) \right), \qquad X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}(s;s,x) = x, \quad x \in \mathbb{R}^3, \quad s,t \in [0,T], \tag{63}$$

given by the Carathéodory theorem [36, Theorem 1.45]. By the uniqueness of this solution, the function $X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}$ can also be written as a composition of the mappings (44). In particular, by the corresponding property of those functions (c.f. Remark 3.1), the mapping

$$x \to X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}(s;t,x), \quad s,t \in [0, k\Delta t] \tag{64}$$

is an isometry from $\mathbb{R}^3$ to $\mathbb{R}^3$. The a-priori estimate (57) translates to the following uniform bounds for the above defined interpolants

$$\|u_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}'_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c, \tag{65}$$

$$\|u_{\Delta t}\|_{L^2(0,T;H^{2,2}(\Omega))} + \|\overline{u}_{\Delta t}\|_{L^2(0,T;H^{2,2}(\Omega))} + \|\overline{u}'_{\Delta t}\|_{L^2(0,T;H^{2,2}(\Omega))} \leqslant c, \tag{66}$$

$$\|B_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{B}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c, \tag{67}$$

$$\|B_{\Delta t}\|_{L^2(0,T;H^{2,2}(\Omega))} + \|\overline{B}_{\Delta t}\|_{L^2(0,T;H^{2,2}(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^2(0,T;H^{2,2}(\Omega))} \leqslant c, \tag{68}$$

$$\|\operatorname{curl} B_{\Delta t}\|_{L^4((0,T)\times\Omega)} + \|\operatorname{curl} \overline{B}_{\Delta t}\|_{L^4((0,T)\times\Omega)} + \left\|\operatorname{curl} \overline{B}'_{\Delta t}\right\|_{L^4((0,T)\times\Omega)} \leqslant c. \tag{69}$$

These bounds allow us to find functions

$$B \in \left\{ b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;V^2(\Omega)\right) :\ b \cdot \mathrm{n}|_{\partial\Omega} = 0 \right\} \tag{70}$$

$$u \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;V_0^2(\Omega)\right) \tag{71}$$

such that for selected non-relabeled subsequences

$$\overline{B}'_{\Delta t},\ \overline{B}_{\Delta t},\ B_{\Delta t} \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad \overline{B}'_{\Delta t},\ \overline{B}_{\Delta t},\ B_{\Delta t} \rightharpoonup B \quad \text{in } L^2\left(0,T;H^{2,2}(\Omega)\right), \tag{72}$$

$$\overline{u}'_{\Delta t},\ \overline{u}_{\Delta t},\ u_{\Delta t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad \overline{u}'_{\Delta t},\ \overline{u}_{\Delta t},\ u_{\Delta t} \rightharpoonup u \quad \text{in } L^2\left(0,T;H^{2,2}(\Omega)\right). \tag{73}$$

The equality between the weak limits of $\overline{B}'_{\Delta t}$, $\overline{B}_{\Delta t}$, $B_{\Delta t}$ and $\overline{u}'_{\Delta t}$, $\overline{u}_{\Delta t}$, $u_{\Delta t}$ is given by Lemma 7.1. The inclusions (70) and (71) follow from the fact that $\overline{u}_{\Delta t}(t) \in V_0^2$ for any $t \in [0,T]$ and $\overline{B}_{\Delta t}(t) \in Y^k$ for any $t \in ((k-1)\Delta t, k\Delta t]$. Moreover, for the discretized external forces $\overline{J}_{\Delta t}$ and $\overline{g}_{\Delta t}$ it follows

$$\overline{J}_{\Delta t} \to J \quad \text{in } L^p((0,T)\times\Omega), \qquad \overline{g}_{\Delta t} \to g \quad \text{in } L^p((0,T)\times\Omega) \qquad \forall 1 \leqslant p < \infty,$$

directly from their definition in (40), c.f. [36, Lemma 8.7].

4.1 Characteristic function

The fact that it still holds $\overline{\Pi}^{l-1}_{\Delta t} = 0$ whenever $\text{supp}\chi^l_{\Delta t} \bigcap \Omega = \varnothing$ and (53), (54), (57) in the other case imply the condition (209) from Lemma 7.2 and in particular we get the existence of a function $\Pi \in L^\infty(0,T; W^{1,\infty}_{\text{loc}}(\mathbb{R}^3))$ such that

$$\overline{\Pi}'_{\Delta t} \overset{*}{\rightharpoonup} \Pi \quad \text{in } L^\infty\left(0,T; W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)\right), \qquad \Pi(t,x) = v(t) + w(t) \times x, \qquad v, w \in L^\infty(0,T). \tag{74}$$

In (119) we will characterize the limit function Π more specifically through the density, the velocity and the characteristic function, similar to (35). By the relations (61) and (63) we also have the conditions (210) and (211) of Lemma 7.2 which, in combination with Remark 7.1, implies that

$$\chi_{\Delta t} \to \chi \quad \text{in } C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \qquad \chi(t,x) = \chi_0\left(X^\Pi(t;0,x)\right), \tag{75}$$

$$X^{\overline{\Pi}'_{\Delta t}}_{\Delta t} \to X^\Pi \quad \text{in } C\left([0,T] \times [0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{76}$$

where χ and X^Π are the unique solutions of

$$-\int_0^T \int_{\mathbb{R}^3} \chi \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\, dx = \int_0^T \int_{\mathbb{R}^3} (\chi \Pi) \cdot \nabla \Theta \, dx dt \quad \forall \Theta \in \mathcal{D}\left([0,T) \times \mathbb{R}^3\right), \tag{77}$$

$$\frac{dX^\Pi(s;t,x)}{dt} = \Pi\left(t, X^\Pi(s;t,x)\right), \qquad X^\Pi(s;s,x) = x \tag{78}$$

respectively. From (75) it also follows that

$$\overline{\chi}_{\Delta t},\ \overline{\chi}'_{\Delta t} \to \chi \quad \text{in } C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \tag{79}$$

which is obtained in the same way as the similar statement in [36, Lemma 8.7].

4.2 Induction equation

In the passage to the limit of the induction equation (83) below, we consider test functions from the space $Y(\chi,T)$ which are curl-free in a neighbourhood of the solid region in the limit. To see that this is possible, let us choose an arbitrary $\gamma > 0$ and denote by $S_\gamma(\chi(t))$ and $S^\gamma(\chi(t))$ the γ-neighbourhood and the "γ-kernel" of $S(\chi(t))$ respectively, i.e.

$$S^\gamma(\chi(t)) := \left\{x \in \mathbb{R}^3 : \ \text{dist}\left(x, S(\chi(t))\right) < \gamma\right\}, \quad S_\gamma(\chi(t)) := \left\{x \in S(\chi(t)) : \ \text{dist}\left(x, \partial S(\chi(t))\right) > \gamma\right\}.$$

From the uniform convergence (76) of $X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}$ and the relation (62) between $X^{\overline{\Pi}'_{\Delta t}}_{\Delta t}$ and the characteristic functions $\chi_{\Delta t}$, it follows the existence of some $\delta(\gamma) > 0$ such that

$$S_\gamma(\chi(t)) \subset S\left(\overline{\chi}_{\Delta t}(t)\right) \subset S^\gamma\left(\chi(t)\right) \qquad \forall t \in [0,T],\ \Delta t < \delta(\gamma). \tag{80}$$

Now we fix an arbitrary function $b \in Y(\chi,T)$, hence there exists some $\gamma > 0$ such that b is curl-free in $S^\gamma(\chi(t))$ for any $t \in [0,T]$. Then, by the second inclusion in (80), b is also curl-free in $S\left(\overline{\chi}_{\Delta t}(t)\right)$ for any $\Delta t < \delta(\gamma)$. In other words,

$$b(t) \in W^k \quad \forall t \in ((k-1)\Delta t, k\Delta t],\ k = 1, ..., \frac{T}{\Delta t},\ \Delta t < \delta(\gamma), \tag{81}$$

so we may use $b(t)$ as a test function for the discrete induction equation.
Next we take an arbitrary interval $I \subset (0,T)$ and an arbitrary open ball $U \subset \mathbb{R}^3$ such that $\overline{I \times U} \subset Q^S(\chi,T) \bigcap Q$. The first inclusion in (80) implies

$$\text{curl} B = \lim_{\Delta t \to 0} \text{curl} \overline{B}_{\Delta t} = 0 \quad \text{a.e. in } \overline{I \times U} \text{ and thus in } Q^S(\chi,T) \bigcap Q. \tag{82}$$

Now we take $b \in Y(\chi, T)$ and $\Delta t > 0$ sufficiently small such that (81) holds true. For fixed k we test the discrete induction equation (34) by $b(t)$, $t \in ((k-1)\Delta t, k\Delta t]$, integrate over this interval and then sum up over all k to see

$$\int_0^T \int_\Omega \partial_t B_{\Delta t} \cdot b \; dxdt = \sum_{k=1}^{\frac{T}{\Delta t}} \int_{(k-1)\Delta t}^{k\Delta t} \int_\Omega \frac{B^k_{\Delta t} - B^{k-1}_{\Delta t}}{\Delta t} \cdot b \; dxdt$$
$$= \int_0^T \int_\Omega \left(-\frac{1}{\sigma\mu} \mathrm{curl} \overline{B}_{\Delta t} + \overline{u}_{\Delta t} \times \overline{B}'_{\Delta t} + \frac{1}{\sigma} \overline{J}_{\Delta t} - \frac{\epsilon}{\mu^2} |\mathrm{curl} \overline{B}_{\Delta t}|^2 \mathrm{curl} \overline{B}_{\Delta t} \right) \cdot \mathrm{curl} b \; dxdt$$
$$- \int_0^T \int_\Omega \epsilon \mathrm{curl} \left(\mathrm{curl} \overline{B}_{\Delta t} \right) : \mathrm{curl} \left(\mathrm{curl} b \right) \; dxdt. \tag{83}$$

An interpolation between $L^\infty(L^2)$ and $L^2(L^\infty)$ together with the estimates (65)–(68) leads to the bounds

$$\epsilon^{\frac{1}{4}} \left\| \overline{u}_{\Delta t} \right\|_{L^4(0,T;L^4(\Omega))} \leqslant c, \quad \epsilon^{\frac{1}{4}} \left\| \overline{B}'_{\Delta t} \right\|_{L^4(0,T;L^4(\Omega))} \leqslant c. \tag{84}$$

Together with the bound (69) this implies the existence of functions $z \in L^{\frac{4}{3}}((0,T) \times \Omega)$ and $z_1, z_2 \in L^2((0,T) \times \Omega)$ such that for chosen subsequences

$$\epsilon \left| \mathrm{curl} \overline{B}_{\Delta t} \right|^2 \mathrm{curl} \overline{B}_{\Delta t} \rightharpoonup \epsilon z \quad \text{in } L^{\frac{4}{3}}\left((0,T) \times \Omega\right), \tag{85}$$
$$\overline{u}_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_1 \quad \text{in } L^2\left((0,T) \times \Omega\right), \tag{86}$$
$$\mathrm{curl} \overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_2 \quad \text{in } L^2\left((0,T) \times \Omega\right). \tag{87}$$

With these convergences at hand we can pass to the limit in (83) and obtain

$$- \int_0^T \int_\Omega B \cdot \partial_t b \; dxdt - \int_\Omega B_0 \cdot b(0,x) \; dx$$
$$= \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \mathrm{curl} B + z_1 + \frac{1}{\sigma} J - \frac{\epsilon}{\mu^2} z \right] \cdot \mathrm{curl} b - \epsilon \mathrm{curl} \left(\mathrm{curl} B \right) : \mathrm{curl} \left(\mathrm{curl} b \right) \; dxdt \tag{88}$$

for all $b \in Y(\chi, T)$. The main difficulty of this section is now to identify the limit functions z_1 and z_2. The limit function z does not need to be identified, as it will vanish from the equation when we pass to the limit with $\epsilon \to 0$. We first note that

$$z_1 \cdot \mathrm{curl} b = 0 = (u \times B) \cdot \mathrm{curl} b \quad \text{a.e. in } Q^S(\chi, T) \bigcap Q, \tag{89}$$
$$z_2 = 0 = \mathrm{curl} B = \mathrm{curl} B \times B \quad \text{a.e. in } Q^S(\chi, T) \bigcap Q, \tag{90}$$

where (89) follows directly from $b \in Y(\chi, T)$ and (90) follows in the same way as (82). Hence it suffices to identify z_1 and z_2 in the fluid region. In order to do so, we choose an arbitrary interval $I = (a, d) \subset (0, T)$ and an arbitrary open ball $U \subset \Omega$ with $\overline{I \times U} \subset Q^F(\chi, T) := Q \backslash \overline{Q}^S(\chi, T)$. In (81) we have seen that, for any sufficiently small $\Delta t > 0$, functions from $\mathcal{D}(I \times U)$ are admissible test functions in (83). By a density argument, (83) may thus also be tested by any $b \in L^4(a, d; H^{2,2}_0(U))$, extended by 0 outside of $(a, d) \times U$. This, together with the $L^4((0,T) \times \Omega)$-bound of $\overline{u}_{\Delta t}$ in (84), leads to the dual estimate

$$\left\| \frac{\overline{B}_{\Delta t}(\cdot) - \overline{B}_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^{\frac{4}{3}}(a,d;H^{-2,2}(U))} = \left\| \partial_t B_{\Delta t} \right\|_{L^{\frac{4}{3}}(a,d;H^{-2,2}(U))} \leqslant c. \tag{91}$$

From this estimate, we can now derive a corresponding estimate for the retarded interpolant $\overline{B}'_{\Delta t}$. Indeed, for arbitrary $b \in L^4(a + \Delta t, d; H^{2,2}_0(U))$, we infer

$$\int_{a+\Delta t}^d \int_U \frac{\overline{B}'_{\Delta t}(t) - \overline{B}'_{\Delta t}(t - \Delta t)}{\Delta t} \cdot b(t) \; dxdt$$
$$= \int_a^{d-\Delta t} \int_U \frac{\overline{B}_{\Delta t}(t) - \overline{B}_{\Delta t}(t - \Delta t)}{\Delta t} \cdot b(t + \Delta t) \; dxdt \leqslant c \|b\|_{L^4\left(a+\Delta t, d; H^{2,2}_0(U)\right)}, \tag{92}$$

so it holds

$$\left\| \frac{\overline{B}'_{\Delta t}(\cdot) - \overline{B}'_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^{\frac{4}{3}}(a+\Delta t, d; H^{-2,2}(U))} \leqslant c. \tag{93}$$

This gives us the conditions for the discrete Aubin-Lions Lemma [11, Theorem 1], which yields

$$\overline{B}'_{\Delta t} \to B \quad \text{in } L^2(I; H^{-1,2}(U)). \tag{94}$$

By the arbitrary choice of $\overline{I \times U}$ this is sufficient to infer

$$z_1 = u \times B \quad \text{a.e. in } Q^F(\chi, T), \quad z_2 = \text{curl} B \times B \quad \text{a.e. in } Q^F(\chi, T). \tag{95}$$

4.3 Continuity equation

We test the discrete continuity equation (32) by $\rho^k_{\Delta t}$, apply Young's inequality and sum over all $k = 1, ..., l$, $l \in \left\{1, ..., \frac{T}{\Delta t}\right\}$ which leads to

$$\|\rho^l_{\Delta t}\|^2_{L^2(\Omega)} + 2\Delta t \sum_{k=1}^{l} \epsilon \|\nabla \rho^k_{\Delta}\|^2_{L^2(\Omega)} \leqslant \|\rho_0\|^2_{L^2(\Omega)} \quad \forall l \in \left\{1, ..., \frac{T}{\Delta t}\right\}. \tag{96}$$

Hence, $\overline{\rho}_{\Delta t}$ is bounded in $L^2(0, T; H^{1,2}(\Omega))$ and we can find $\rho \in L^2(0, T; H^{1,2}(\Omega))$ such that for a chosen subsequence

$$\overline{\rho}_{\Delta t} \rightharpoonup \rho \quad \text{in } L^2\left(0, T; H^{1,2}(\Omega)\right). \tag{97}$$

Further, from the continuity equation we derive the dual estimates

$$\left\| \frac{\overline{\rho}_{\Delta t}(\cdot) - \overline{\rho}_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^2\left(0,T;(H^{1,2}(\Omega))^*\right)} \leqslant c, \quad \left\| \frac{\overline{\rho}'_{\Delta t}(\cdot) - \overline{\rho}'_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^2\left(\Delta t,T;(H^{1,2}(\Omega))^*\right)} \leqslant c \tag{98}$$

by the same arguments as the bounds (91) and (93) for the discrete time derivatives of $\overline{B}_{\Delta t}$ and $\overline{B}'_{\Delta t}$. In particular, we can again apply the discrete Aubin-Lions Lemma [11, Theorem 1] to infer

$$\overline{\rho}_{\Delta t},\ \overline{\rho}'_{\Delta t} \to \rho \quad \text{in } L^q\left(0, T; L^q\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant q < \infty, \qquad \underline{\rho} \leqslant \rho \leqslant \overline{\rho} \quad \text{a.e. in } [0, T] \times \mathbb{R}^3, \tag{99}$$

where the limit function ρ has been extended by $\underline{\rho}$ outside of Ω. Now, we sum the discrete equation (32) over all $k = 1, ..., \frac{T}{\Delta t}$ and pass to the limit by means of (97) and (99). This yields

$$-\int_0^T \int_\Omega \rho \partial_t \psi \; dxdt - \int_\Omega \rho_0 \psi(0, x) \; dx = \int_0^T \int_\Omega (\rho u) \cdot \nabla \psi + \epsilon \rho \Delta \psi \; dxdt \quad \forall \psi \in \mathcal{D}([0, T) \times \Omega). \tag{100}$$

Our next goal is to show strong convergence of $\nabla \overline{\rho}_{\Delta t}$, which is required for the limit passage in the momentum equation. The first bound in (98) further implies that for a subsequence

$$\partial_t \rho_{\Delta t} \overset{*}{\rightharpoonup} \partial_t \rho \quad \text{in } L^2\left(0, T; \left(H^{1,2}(\Omega)\right)^*\right). \tag{101}$$

Consequently, the limit of the discrete continuity equation can also be expressed in the form

$$\int_0^\tau \int_\Omega \partial_t \rho \psi - (\rho u) \cdot \nabla \psi + \epsilon \nabla \rho \cdot \nabla \psi \; dxdt = 0 \quad \forall \psi \in L^2(0, T; H^{1,2}(\Omega)), \; \tau \in [0, T]. \tag{102}$$

We now test (102) by ρ and compare it to the corresponding relation (96) on the Δt–level, which will yield convergence of $\|\nabla \overline{\rho}_{\Delta t}\|_{L^2((0,\tau);L^2(\Omega))}$ and thus the desired strong convergence of $\nabla \overline{\rho}_{\Delta t}$. Indeed, testing (102) by ρ we obtain

$$\|\rho(\tau)\|^2_{L^2(\Omega)} + 2\epsilon \int_0^\tau \int_\Omega |\nabla \rho|^2 \; dxdt = \|\rho(0)\|^2_{L^2(\Omega)}. \tag{103}$$

Further, the inequality (96) can be rewritten in the form

$$\|\overline{\rho}_{\Delta t}(l\Delta t-\gamma)\|^2_{L^2(\Omega)} + 2\epsilon\int_0^{l\Delta t-\gamma}\int_\Omega |\nabla\overline{\rho}_{\Delta t}|^2\ dxdt \leqslant \|\rho_0\|^2_{L^2(\Omega)} \quad \forall\gamma\in[0,\Delta t),\ l\in\left\{1,...,\frac{T}{\Delta t}\right\}. \tag{104}$$

Any $\tau\in(0,T]$ can be expressed as $\tau = l\Delta t-\gamma$ for some $l\in\left\{1,...,\frac{T}{\Delta t}\right\}$ and $\gamma\in[0,\Delta t)$. Thus, subtracting (103) from (104) and making use of the strong convergence (99), we infer that for a further subsequence

$$\lim_{\Delta t\to 0}\int_0^\tau\int_\Omega |\nabla\overline{\rho}_{\Delta t}|^2\ dxdt \leqslant \int_0^\tau\int_\Omega |\nabla\rho|^2\ dxdt \quad \text{for a.a. } \tau\in(0,T]. \tag{105}$$

On the other hand, for each such τ the weak lower semicontinuity of norms guarantees us the existence of $z_3 = z_3(\tau)\in\mathbb{R}$ such that for another subsequence

$$\|\nabla\overline{\rho}_{\Delta t}\|^2_{L^2(0,\tau;L^2(\Omega))} \to z_3^2 \geqslant \|\nabla\rho\|^2_{L^2(0,\tau;L^2(\Omega))}. \tag{106}$$

Combining (105) and (106), we infer that for almost all $\tau\in[0,T]$ there exists a subsequence for which

$$\|\nabla\overline{\rho}_{\Delta t}\|_{L^2(0,\tau;L^2(\Omega))} \to \|\nabla\rho\|_{L^2(0,\tau;L^2(\Omega))}.$$

In combination with the weak convergence (97) and a diagonal argument, this implies the desired relation

$$\nabla\overline{\rho}_{\Delta t}\to\nabla\rho \quad \text{in } L^2\left(0,\tau;L^2(\Omega)\right) \quad \text{for a.a. } \tau\in[0,T]. \tag{107}$$

Next, we show that the limit density satisfies a regularized and integrated version of the renormalized continuity equation (29), which will be significant in the limit passage with respect to $\epsilon\to 0$. To this end we take an arbitrary smooth and convex function β on $[\underline{\rho},\overline{\rho}]$ and test (32) by $\beta'(\rho^k_{\Delta t})$ for any $k=1,...,\frac{T}{\Delta t}$. By the convexity of β and the fact that $\operatorname{div}\overline{u}'_{\Delta t}=0$ this yields

$$\begin{aligned}\int_0^\tau\int_\Omega \partial_t\rho_{\Delta t}\beta'(\overline{\rho}_{\Delta t})\ dx =& \Delta t\left(\sum_{k=1}^{l-1}\int_\Omega \frac{\rho^k_{\Delta t}-\rho^{k-1}_{\Delta t}}{\Delta t}\beta'(\rho^k_{\Delta t})\ dx\right) + \gamma\int_\Omega\frac{\rho^l_{\Delta t}-\rho^{l-1}_{\Delta t}}{\Delta t}\beta'(\rho^l_{\Delta t})\ dx\\ =& -\int_0^\tau\int_\Omega \overline{u}'_{\Delta t}\cdot\nabla\overline{\rho}_{\Delta t}\beta'(\overline{\rho}_{\Delta t})\ dxdt - \int_0^\tau\int_\Omega \epsilon\nabla\overline{\rho}_{\Delta t}\nabla\beta'(\overline{\rho}_{\Delta t})\ dxdt\\ =& -\int_0^\tau\int_\Omega \epsilon\,|\nabla\overline{\rho}_{\Delta t}|^2\,\beta''(\overline{\rho}_{\Delta t})\ dxdt \leqslant 0\end{aligned} \tag{108}$$

for any $\tau\in(0,T]$, and $l\in\left\{1,...,\frac{T}{\Delta t}\right\}$, $\gamma\in[0,\Delta t)$ chosen such that $\tau=l\Delta t-\gamma$. Since the derivatives of β are bounded, the strong $L^2(H^{1,2})$-convergence of $\overline{\rho}_{\Delta t}$ (c.f. (99), (107)) implies

$$\begin{aligned}\beta''(\overline{\rho}_{\Delta t}) &\overset{*}{\rightharpoonup} \beta''(\rho) \quad \text{in } L^\infty((0,T)\times\Omega),\\ \beta'(\overline{\rho}_{\Delta t}) &\to \beta'(\rho) \quad \text{in } L^2\left(0,\tau;H^{1,2}(\Omega)\right) \quad \text{for a.a. } \tau\in[0,T].\end{aligned}$$

Using this in combination with (101), we can pass to the limit in (108) and obtain the desired relation

$$\int_\Omega\beta(\rho(\tau))\ dx - \int_\Omega\beta(\rho_0)\ dx = \int_0^\tau\int_\Omega\partial_t\beta(\rho)\ dxdt = -\int_0^\tau\int_\Omega\epsilon\beta''(\rho)|\nabla\rho|^2\ dxdt\leqslant 0 \quad \text{for a.a. } \tau\in[0,T]. \tag{109}$$

4.4 Momentum equation

We test the discrete momentum equation (33) by $\phi(t)$ for $\phi\in L^4(0,T;V_0^2(\Omega))$ and sum the result over all k. Using the Hölder inequality and the Gagliardo–Nirenberg interpolation inequality we estimate

$$\begin{aligned}&\int_0^T\int_\Omega\epsilon\left(\nabla\overline{u}_{\Delta t}\nabla\overline{\rho}_{\Delta t}\right)\cdot\phi\ dxdt\\ \leqslant&\epsilon\|\nabla\overline{u}_{\Delta t}\|_{L^4(0,T;L^2(\Omega))}\|\nabla\overline{\rho}_{\Delta t}\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{L^4(0,T;L^\infty(\Omega))}\\ \leqslant& c\epsilon\,\|\overline{u}_{\Delta t}\|^{\frac12}_{L^2(0,T;H^{2,2}(\mathbb{R}^3))}\|\overline{u}_{\Delta t}\|^{\frac12}_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|\nabla\overline{\rho}_{\Delta t}\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{L^4(0,T;L^\infty(\Omega))}\leqslant c.\end{aligned}$$

This allows us to infer the dual estimate

$$\left\| \frac{\overline{(\rho u)}_{\Delta t}(\cdot) - \overline{(\rho u)}_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^{\frac{4}{3}}\left(0,T;\left(V_0^2(\Omega)\right)^*\right)} = \|\partial_t (\rho u)_{\Delta t}\|_{L^{\frac{4}{3}}\left(0,T;\left(V_0^2(\Omega)\right)^*\right)} \leqslant c. \tag{110}$$

We can now mimick the compactness results for the time-dependent incompressible Navier-Stokes equations, c.f. [31, Theorem 2.4]. The estimate (110) gives us the conditions for the discrete Aubin-Lions Lemma [11, Theorem 1], which allows us to deduce

$$P\left(\overline{(\rho u)}_{\Delta t}\right) \to P(\rho u) \quad \text{in } L^2\left(0,T;\left(V_0^2(\Omega)\right)^*\right),$$

where P denotes the orthogonal projection of $L^2(\Omega)$ onto the space $V^0(\Omega)$ of weakly divergence-free L^2–functions. This, in combination with the $L^4((0,T)\times\Omega)$–bound (84) of $\overline{u}_{\Delta t}$, leads to

$$\overline{u}_{\Delta t} \to u \quad \text{in } L^q((0,T)\times\Omega) \quad \forall 1 \leqslant q < 4. \tag{111}$$

With this strong convergence at hand, we derive the following limit version of the momentum equation

$$\begin{aligned} &-\int_0^T \int_\Omega \rho u \cdot \partial_t \phi \; dxdt - \int_\Omega \rho_0 u_0 \cdot \phi(0,x) \; dx \\ &= \int_0^T \int_\Omega \rho(u \otimes u) : \nabla\phi - 2\nu \mathbb{D}(u) : \nabla\phi - \frac{1}{\eta}\rho\chi\,(u - \Pi)\cdot\phi \\ &\quad + \rho g \cdot \phi + \frac{1}{\mu}\left(\text{curl}B \times B\right)\cdot\phi - \epsilon\left(\nabla u \nabla\rho\right)\cdot\phi - \epsilon\Delta u \cdot \Delta\phi \; dxdt \end{aligned} \tag{112}$$

for any $\phi \in \mathcal{D}([0,T)\times\Omega)$ with $\text{div}\phi = 0$, where Π was defined in (74). Here we further used the strong convergence (107) of $\nabla\overline{\rho}_{\Delta t}$ and the relations (87), (90), (95) which identify the magnetic term in the limit equation. Now it only remains to identify Π. We start by remarking that

$$\int_{\mathbb{R}^3} \rho(t)\chi(t) dx \geqslant \underline{\rho}|S| > 0 \quad \text{for a.a. } t \in [0,T]. \tag{113}$$

We pick an arbitrary ball $B_R \subset \mathbb{R}^3$ with radius $R > 0$, centered at 0. The weak-$*$ convergence (73) of $\overline{u}'_{\Delta t}$, the uniform convergence (79) of the characteristic function and the strong convergence (99) of the density yield that

$$\begin{aligned} \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\; \overline{u}'_{\Delta t} \; dx &\overset{*}{\rightharpoonup} \int_{\mathbb{R}^3} \rho\chi \; u \; dx \quad \text{in } L^\infty\left((0,T)\times B_R\right), \\ \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; x \; dx &\to \int_{\mathbb{R}^3} \rho\chi \; x \; dx \quad \text{in } L^p\left((0,T)\times B_R\right) \quad \forall 1 \leqslant p < \infty, \\ \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; dx &\to \int_{\mathbb{R}^3} \rho\chi \; dx \quad \text{in } L^p\left((0,T)\times B_R\right) \quad \forall 1 \leqslant p < \infty. \end{aligned}$$

Combining the latter convergence with the bounds (52), (113) away from 0 we further see

$$\begin{aligned} \frac{1}{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; dx} &\to \frac{1}{\int_{\mathbb{R}^3} \rho\chi \; dx} \quad \text{in } L^p\left((0,T)\times B_R\right) \quad \forall 1 \leqslant p < \infty, \\ \frac{1}{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; dx} &\overset{*}{\rightharpoonup} \frac{1}{\int_{\mathbb{R}^3} \rho\chi \; dx} \quad \text{in } L^\infty\left((0,T)\times B_R\right) \end{aligned}$$

and altogether

$$\frac{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\overline{u}'_{\Delta t} \; dx}{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; dx} = \overline{(u_G)}'_{\Delta t} \overset{*}{\rightharpoonup} (u_G)_{[\chi,\rho,u]} := \frac{\int_{\mathbb{R}^3} \rho\chi u \; dx}{\int_{\mathbb{R}^3} \rho\chi \; dx} \quad \text{in } L^\infty\left((0,T)\times B_R\right) \tag{114}$$

as well as

$$\frac{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} x \; dx}{\int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \; dx} = \overline{a}'_{\Delta t} \to a_{[\chi,\rho]} := \frac{\int_{\mathbb{R}^3} \rho\chi x \; dx}{\int_{\mathbb{R}^3} \rho\chi \; dx} \quad \text{in } L^p\left((0,T)\times B_R\right) \quad \forall 1 \leqslant p < \infty. \tag{115}$$

Next, we consider the matrix

$$\overline{I}'_{\Delta t} = \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \left(|x - \overline{a}'_{\Delta t}|^2 \mathrm{id} - \left(x - \overline{a}'_{\Delta t}\right) \otimes \left(x - \overline{a}'_{\Delta t}\right)\right) \, dx,$$

for which similar arguments together with the already proven convergence (115) lead to

$$\overline{I}'_{\Delta t} \to I_{[\chi,\rho]} := \int_{\mathbb{R}^3} \rho\chi \left(|x - a_{[\chi,\rho]}|^2 \mathrm{id} - \left(x - a_{[\chi,\rho]}\right) \otimes \left(x - a_{[\chi,\rho]}\right)\right) \, dx \quad \text{in } L^p\left((0,T) \times B_R\right) \tag{116}$$

for any $1 \leqslant p < \infty$. From this and the bound of the eigenvalues of $\overline{I}'_{\Delta t}(t)$ away from 0, c.f. (54), it is possible to derive

$$(\overline{I}'_{\Delta t})^{-1} \to \left(I_{[\chi,\rho]}\right)^{-1} \quad \text{in } L^p((0,T) \times B_R) \quad \forall 1 \leqslant p < \infty. \tag{117}$$

This, together with arguments similar to the ones used for (114), yields

$$\begin{aligned} \overline{\omega}'_{\Delta t} &= \left(\overline{I}'_{\Delta t}\right)^{-1} \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t} \left(x - \overline{a}'_{\Delta t}\right) \times \overline{u}'_{\Delta t} \, dx \\ \rightharpoonup \omega_{[\chi,\rho,u]} &:= I^{-1}_{[\chi,\rho]} \int_{\mathbb{R}^3} \rho\chi \left(\left(x - a_{[\chi,\rho]}\right) \times u\right) \, dx \quad \text{in } L^p((0,T) \times B_R) \quad \forall 1 \leqslant p < \infty. \end{aligned} \tag{118}$$

Now (114), (115) and (118) imply

$$\Pi = (u_G)_{[\chi,\rho,u]} + \omega_{[\chi,\rho,u]} \times (x - a_{[\chi,\rho]}) =: \Pi_{[\chi,\rho,u]}. \tag{119}$$

4.5 Energy inequality

In order to derive an energy inequality for the limit system, we first derive a slightly modified version of the discrete energy inequality (50). More precisely, we again add the estimates (48) and (49) and sum over all $l = 1, ..., k$, $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$. Since each $\tau \in (0, T]$ can be written in the form $\tau = k\Delta t - \gamma$ for some $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$ and $\gamma \in [0, \Delta t)$, this leads to

$$\begin{aligned} &\frac{1}{2} \left\|\sqrt{\overline{\rho}_{\Delta t}}(\tau)\overline{u}_{\Delta t}(\tau)\right\|^2_{L^2(\Omega)} + \int_0^\tau \int_\Omega 2\nu \left|\nabla\overline{u}_{\Delta t}(t,x)\right|^2 \, dxdt + \int_0^\tau \int_\Omega \epsilon \left|\Delta\overline{u}_{\Delta t}(t,x)\right|^2 \, dxdt \\ &+ \frac{1}{2\mu} \left\|\overline{B}_{\Delta t}(\tau)\right\|^2_{L^2(\Omega)} + \int_0^\tau \int_\Omega \frac{\epsilon}{\mu^3} \left|\mathrm{curl}\overline{B}_{\Delta t}(t,x)\right|^4 \, dxdt + \int_0^\tau \int_\Omega \frac{\epsilon}{\mu} \left|\Delta\overline{B}_{\Delta t}(t,x)\right|^2 \, dxdt \\ &+ \int_0^\tau \int_\Omega \frac{1}{\sigma\mu^2} \left|\mathrm{curl}\overline{B}_{\Delta t}(t,x)\right|^2 \, dxdt \\ \leqslant &\frac{1}{2} \left\|\sqrt{\rho_0}u_0\right\|^2_{L^2(\Omega)} + \frac{1}{2\mu} \left\|B_0\right\|^2_{L^2(\Omega)} + \int_0^\tau \int_\Omega -\frac{1}{\eta}\overline{\rho}'_{\Delta t}(t,x)\overline{\chi}_{\Delta t}(t,x) \left(\overline{u}'_{\Delta t}(t,x) - \overline{\Pi}'_{\Delta t}(t,x)\right) \cdot \overline{u}_{\Delta t}(t,x) \\ &+ \overline{\rho}'_{\Delta t}(t,x)\overline{g}_{\Delta t}(t,x) \cdot \overline{u}_{\Delta t}(t,x) + \frac{1}{\mu} \left(\mathrm{curl}\overline{B}'_{\Delta t}(t,x) \times \overline{B}'_{\Delta t}(t,x)\right) \cdot \overline{u}_{\Delta t}(t,x) \\ &+ \frac{1}{\mu} \left(\overline{u}_{\Delta t}(t,x) \times \overline{B}'_{\Delta t}(t,x)\right) \cdot \mathrm{curl}\overline{B}_{\Delta t}(t,x) + \frac{1}{\sigma}\overline{J}_{\Delta t}(t,x) \cdot \mathrm{curl}\overline{B}_{\Delta t}(t,x) \, dxdt + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right]. \end{aligned}$$

On the right-hand side of this inequality we can pass to the limit by using in particular the strong convergence (111) of $\overline{u}_{\Delta t}$ and the relations (87), (90), (95) which identify the limits of the mixed terms. Using the weak lower semicontinuity of norms on the left-hand side, we end up with

$$\begin{aligned} &\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2 + \frac{1}{2\mu}|B(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu \left|\nabla u(t,x)\right|^2 + \epsilon|\Delta u(t,x)|^2 + \frac{\epsilon}{\mu^3} \left|z(t,x)\right|^{\frac{4}{3}} \\ &+ \frac{\epsilon}{\mu} \left|\Delta B(t,x)\right|^2 + \frac{1}{\sigma\mu^2} \left|\mathrm{curl}B(t,x)\right|^2 + \frac{1}{\eta}\rho(t,x)\chi(t,x) \left|u(t,x) - \Pi_{[\chi,\rho,u]}(t,x)\right|^2 \, dxdt \\ \leqslant &\int_\Omega \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|B_0|^2 \, dx + \int_0^\tau \int_\Omega \rho(t,x)g(t,x) \cdot u(t,x) + \frac{1}{\sigma}J(t,x) \cdot \mathrm{curl}B(t,x) \, dxdt \end{aligned} \tag{120}$$

for almost all $\tau \in [0,T]$. Here, the mixed terms canceled each other by the identity

$$(\operatorname{curl} B \times B) \cdot u = -(u \times B) \cdot \operatorname{curl} B,$$

and the term involving $u - \Pi_{[\chi,\rho,u]}$ was rewritten by means of the relation

$$\int_0^\tau \int_\Omega \rho(t,x)\chi(t,x) \left(u(t,x) - \Pi_{[\chi,\rho,u]}(t,x)\right) \cdot \Pi_{[\chi,\rho,u]}(t,x)\ dxdt = 0, \quad \tau \in [0,T],$$

c.f. [4, Lemma 3.1]. In summary, we have shown

Proposition 4.1. *Let all the assumptions of Theorem 1.1 be satisfied and let $\epsilon > 0$. Assume in addition that*

$$\rho_0 \in H^{1,2}(\Omega), \qquad u_0,\ B_0 \in H^{2,2}(\Omega).$$

Then, there exist

$$\rho_\epsilon \in \left\{\psi \in L^2\left(0,T;H^{1,2}(\Omega)\right):\ \underline{\rho} \leqslant \psi \leqslant \overline{\rho}\ \text{a.e. in } Q\right\}, \tag{121}$$

$$\chi_\epsilon \in C\left([0,T];L^p\left(\mathbb{R}^3\right)\right),\ 1 \leqslant p < \infty, \qquad z_\epsilon \in L^{\frac{4}{3}}((0,T)\times\Omega), \tag{122}$$

$$B_\epsilon \in \Big\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^{2,2}(\Omega)\right):\ \operatorname{div} b = 0 \text{ in } Q,$$
$$\operatorname{curl} b = 0 \text{ in } Q^S(\chi_\epsilon,T) \bigcap Q,\ b \cdot n|_{\partial\Omega} = 0\Big\}, \tag{123}$$

$$u_\epsilon \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;V_0^2(\Omega)\right) \tag{124}$$

such that

$$-\int_0^T \int_{\mathbb{R}^3} \chi_\epsilon \partial_t \Theta\ dxdt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} \left(\chi_\epsilon \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right) \cdot \nabla\Theta\ dxdt, \tag{125}$$

$$-\int_0^T \int_\Omega \rho_\epsilon \partial_t \psi\ dxdt - \int_\Omega \rho_0 \psi(0,x)\ dx = \int_0^T \int_\Omega (\rho_\epsilon u_\epsilon) \cdot \nabla\psi + \epsilon\rho_\epsilon \Delta\psi\ dxdt, \tag{126}$$

$$\begin{aligned}-\int_0^T \int_\Omega \rho_\epsilon u_\epsilon \cdot \partial_t \phi\ dxdt - \int_\Omega \rho_0 u_0 \cdot \phi(0,x)\ dx = &\int_0^T \int_\Omega \rho_\epsilon (u_\epsilon \otimes u_\epsilon) : \nabla\phi - 2\nu\mathbb{D}(u_\epsilon) : \nabla\phi \\ &- \frac{1}{\eta}\rho_\epsilon\chi_\epsilon\left(u_\epsilon - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right) \cdot \phi + \rho_\epsilon g \cdot \phi \\ &+ \frac{1}{\mu}\left(\operatorname{curl} B_\epsilon \times B_\epsilon\right) \cdot \phi - \epsilon\left(\nabla u_\epsilon \nabla\rho_\epsilon\right) \cdot \phi \\ &- \epsilon\Delta u_\epsilon \cdot \Delta\phi\ dxdt,\end{aligned} \tag{127}$$

$$\begin{aligned}-\int_0^T \int_\Omega B_\epsilon \cdot \partial_t b\ dxdt - \int_\Omega B_0 \cdot b(0,x)\ dx = &\int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu}\operatorname{curl} B_\epsilon + u_\epsilon \times B_\epsilon + \frac{1}{\sigma}J - \frac{\epsilon}{\mu^2}z_\epsilon\right] \cdot \operatorname{curl} b \\ &- \epsilon \operatorname{curl}\left(\operatorname{curl} B_\epsilon\right) : \operatorname{curl}\left(\operatorname{curl} b\right)\ dxdt\end{aligned} \tag{128}$$

for all $\Theta \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\psi, \phi \in \mathcal{D}([0,T)\times\Omega)$ and all $b \in Y(\chi,T)$. Moreover, these functions satisfy the energy inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2}\rho_\epsilon(\tau)|u_\epsilon(\tau)|^2 + \frac{1}{2\mu}|B_\epsilon(\tau)|^2\ dx + \int_0^\tau \int_\Omega 2\nu\left|\nabla u_\epsilon(t,x)\right|^2 + \epsilon|\Delta u_\epsilon(t,x)|^2 + \frac{\epsilon}{\mu^3}\left|z_\epsilon(t,x)\right|^{\frac{4}{3}} \\ &+ \frac{\epsilon}{\mu}\left|\Delta B_\epsilon(t,x)\right|^2 + \frac{1}{\sigma\mu^2}\left|\operatorname{curl} B_\epsilon(t,x)\right|^2 + \frac{1}{\eta}\rho_\epsilon(t,x)\chi_\epsilon(t,x)\left|u_\epsilon(t,x) - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}(t,x)\right|^2\ dxdt \\ \leqslant &\int_\Omega \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|B_0|^2\ dx + \int_0^\tau \int_\Omega \rho_\epsilon(t,x)g(t,x) \cdot u_\epsilon(t,x) + \frac{1}{\sigma}J(t,x) \cdot \operatorname{curl} B_\epsilon(t,x)\ dxdt\end{aligned} \tag{129}$$

for almost all $\tau \in [0,T]$ *and the characteristic function* χ_ϵ *is connected to the solution* $X^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}$ *of the initial value problem*

$$\frac{dX^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;t,x)}{dt} = \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\left(t, X^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;t,x)\right), \qquad X^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;s,x) = x \tag{130}$$

by

$$\chi(t,x) = \chi_0\left(X^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(t;0,x)\right). \tag{131}$$

5 Limit passage with respect to $\epsilon \to 0$

From the energy inequality (129) we infer the existence of a constant $c > 0$, independent of ϵ, such that

$$\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|B_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\epsilon\|_{L^2(0,T;H^{1,2}(\Omega))} + \|B_\epsilon\|_{L^2(0,T;H^{1,2}(\Omega))} \leqslant c, \tag{132}$$

$$\epsilon^{\frac{1}{2}}\|\Delta u_\epsilon\|_{L^2((0,T)\times\Omega)} + \epsilon^{\frac{3}{4}}\|z_\epsilon\|_{L^{\frac{4}{3}}((0,T)\times\Omega)} + \epsilon^{\frac{1}{2}}\|\Delta B_\epsilon\|_{L^2((0,T)\times\Omega)} \leqslant c. \tag{133}$$

The continuity equation on the ϵ-level tested by ρ_ϵ, c.f. (103), yields

$$\epsilon\|\nabla\rho_\epsilon\|^2_{L^2((0,T)\times\Omega)} \leqslant c. \tag{134}$$

Further, from the lower bound (113) for the total mass of the solid we deduce, similarly to (53) and (54), the estimates

$$\left|a_{[\chi_\epsilon,\rho_\epsilon]}(t)\right| \leqslant c, \quad \left|(u_G)_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}(t)\right| \leqslant c\|u_\epsilon(t)\|_{L^2(\Omega)}, \quad \left|\omega_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}(t)\right| \leqslant c\|u_\epsilon(t)\|_{L^2(\Omega)}, \tag{135}$$

$$v\cdot\left(I_{[\chi_\epsilon,\rho_\epsilon]}(t)v\right) \geqslant c|v|^2 \quad \forall v \in \mathbb{R}^3, \tag{136}$$

for the quantities $a_{[\chi_\epsilon,\rho_\epsilon]}$, $(u_G)_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}$, $\omega_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}$ and $I_{[\chi_\epsilon,\rho_\epsilon]}$, introduced in (114)–(118), with c independent of t and ϵ and therefore

$$\|\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}(t)\|_{L^\infty(\Omega)} \leqslant c\|u_\epsilon(t)\|_{L^2(\Omega)} \quad \text{for a.a. } t \in [0,T]. \tag{137}$$

By this, the bounds in (121) for the density and the uniform bounds (132)–(134) we find functions

$$\rho \in L^\infty((0,T)\times\Omega), \quad \Pi \in L^\infty\left(0,T;W^{1,\infty}_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{138}$$

$$B \in \left\{b \in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;V^1(\Omega)\right):\ b\cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \tag{139}$$

$$u \in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;V^1_0(\Omega)\right) \tag{140}$$

such that for chosen subsequences

$$u_\epsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad u_\epsilon \rightharpoonup u \quad \text{in } L^2\left(0,T;H^{1,2}(\Omega)\right), \tag{141}$$

$$B_\epsilon \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad B_\epsilon \rightharpoonup B \quad \text{in } L^2\left(0,T;H^{1,2}(\Omega)\right), \tag{142}$$

$$\rho_\epsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T;L^\infty(\Omega)), \qquad \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]} \overset{*}{\rightharpoonup} \Pi \quad \text{in } L^\infty\left(0,T;W^{1,\infty}_{\text{loc}}\left(\mathbb{R}^3\right)\right) \tag{143}$$

and

$$\epsilon\nabla\rho_\epsilon,\ \epsilon\Delta u_\epsilon,\ \epsilon\Delta B_\epsilon \to 0 \quad \text{in } L^2((0,T)\times\Omega), \qquad \epsilon z_\epsilon \to 0 \quad \text{in } L^{\frac{4}{3}}((0,T)\times\Omega). \tag{144}$$

5.1 Characteristic function

The transport equation (125), the equation (130) for the associated characteristics and the estimates (135) correspond directly to the conditions for Lemma 7.2 and Remark 7.1, which therefore yield

$$X_\epsilon^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}} \to X^\Pi \quad \text{in } C([0,T]\times[0,T];C_{\text{loc}}\left(\mathbb{R}^3\right)), \tag{145}$$

$$\chi_\epsilon \to \chi \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \qquad \chi(t,x) = \chi_0\left(X^\Pi(t;0,x)\right), \tag{146}$$

where χ and X^{Π} are the unique solutions to

$$-\int_0^T \int_{\mathbb{R}^3} \chi \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} (\chi \Pi) \cdot \nabla \Theta\ dx dt \quad \forall \Theta \in \mathcal{D}\left([0,T) \times \mathbb{R}^3\right), \tag{147}$$

$$\frac{dX^{\Pi}(s;t,x)}{dt} = \Pi\left(t, X^{\Pi}(s;t,x)\right), \qquad X^{\Pi}(s;s,x) = x. \tag{148}$$

5.2 Induction equation

Interpolating the bounds for B_ϵ in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;L^6(\Omega))$ we see that B_ϵ is also bounded in $L^3((0,T)\times\Omega)$. Hence, using the Hölder inequality, we find $z_4, z_5 \in L^{\frac{6}{5}}((0,T)\times\Omega)$ such that for selected subsequences

$$u_\epsilon \times B_\epsilon \rightharpoonup z_4 \quad \text{in } L^{\frac{6}{5}}\left((0,T)\times\Omega\right), \qquad \text{curl} B_\epsilon \times B_\epsilon \rightharpoonup z_5 \quad \text{in } L^{\frac{6}{5}}\left((0,T)\times\Omega\right). \tag{149}$$

Further, for any $\gamma > 0$ we again find, by (145) and (146), some $\delta(\gamma) > 0$ such that

$$S_\gamma(\chi(t)) \subset S(\chi_\epsilon(t)) \subset S^{\frac{\gamma}{2}}(\chi(t)) \subset S^\gamma(\chi(t)) \quad \forall t \in [0,T],\ \epsilon < \delta(\gamma). \tag{150}$$

We fix arbitrary $b \in Y(\chi,T)$, so b is curl-free in $S^\gamma(\chi(t))$ for some $\gamma > 0$ and all $t \in [0,T]$. Now (150) implies that b is also curl-free in a $\frac{\gamma}{2}$-neighbourhood of the solid region on the ϵ-level for all sufficiently small $\epsilon > 0$. In particular it holds $b \in Y(\chi_\epsilon, T)$ for all such ϵ. Thus, letting $\epsilon \to 0$ in (128), we obtain

$$-\int_0^T \int_\Omega B \cdot \partial_t b\ dx dt - \int_\Omega B_0 \cdot b(0)\ dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu}\text{curl} B + z_4 + \frac{1}{\sigma} J\right] \cdot \text{curl} b\ dx dt \tag{151}$$

for any $b \in Y(\chi,T)$, where the regularization terms vanished as stated in (144). It remains to identify z_4 and z_5. On the solid domain, we can argue as in Section 4.2 and see from the fact that B_ϵ is curl-free in $Q^S(\chi_\epsilon,T)\bigcap Q$ and (150) that

$$z_4 \cdot \text{curl} b = 0 = (u \times B) \cdot \text{curl} b, \quad \text{curl} B \times B = \text{curl} B = 0 = z_5 \qquad \text{a.e. in } Q^S(\chi,T)\bigcap Q \tag{152}$$

for $b \in Y(\chi,T)$. In the fluid region we again consider an arbitrary set of the form $\overline{I \times U} \subset Q^F(\chi,T)$, where $I \subset (0,T)$ is an interval and $U \subset \Omega$ is a ball. For any sufficiently small $\epsilon > 0$ the first inclusion in (150) implies that, for all functions $\psi \in \mathcal{D}(I)$ and $b \in \mathcal{D}(U)$ extended by 0 outside of I and U, the product ψb is an admissible test function in the induction equation (128) on the ϵ-level. This, together with the uniform estimates (132), (133) leads to the dual estimate

$$\left\| \partial_t \int_U B_\epsilon \cdot b dx \right\|_{L^{\frac{4}{3}}(I)} \leqslant c. \tag{153}$$

This allows us to apply the Arzelà-Ascoli theorem and deduce

$$B_\epsilon \to B \quad \text{in } C_{\text{weak}}\left(\overline{I}; L^2(U)\right) \text{ and thus in } L^p\left(I; H^{-1,2}(U)\right) \quad \forall 1 \leqslant p < \infty. \tag{154}$$

Hence, writing

$$\int_I \int_U (\text{curl} B_\epsilon \times B_\epsilon) \cdot b\ dx dt = \int_I \int_U \text{div}\left(B_\epsilon \otimes B_\epsilon\right) \cdot b - \nabla\left(\frac{1}{2}|B_\epsilon|^2\right) \cdot b\ dx dt, \quad b \in \mathcal{D}(I \times U), \tag{155}$$

which allows us, after integration by parts, to shift the derivatives to the test function b, we conclude the desired identities

$$z_4 = u \times B \quad \text{a.e. in } Q^F(\chi,T), \qquad z_5 = \text{curl} B \times B \quad \text{a.e. in } Q^F(\chi,T). \tag{156}$$

5.3 Continuity equation

We test the continuity equation (126) by $\psi\Phi$, where $\psi \in \mathcal{D}(0,T)$ and $\Phi \in \mathcal{D}(\Omega)$, to find that

$$\left\| \partial_t \int_\Omega \rho_\epsilon \Phi dx \right\|_{L^2(0,T)} \leqslant c.$$

This again gives us the conditions for the Arzelà-Ascoli theorem, from which we obtain

$$\rho_\epsilon \to \rho \quad \text{in } C_{\text{weak}}\left([0,T]; L^2(\Omega)\right) \text{ and thus in } L^p\left(0,T; \left(H^{1,2}(\Omega)\right)^*\right) \quad \forall 1 \leqslant p < \infty.$$

Combining this with the weak convergence (141) of u_ϵ and the fact that $\epsilon\nabla\rho_\epsilon$ converges to 0 in $L^2((0,T)\times\Omega)$ according to (144), we may pass to the limit in (126) and obtain

$$-\int_0^T \int_\Omega \rho \partial_t \psi dx dt - \int_\Omega \rho_0 \psi(0,x)\, dx = \int_0^T \int_\Omega (\rho u)\cdot\nabla\psi\, dx dt \quad \forall \psi \in \mathcal{D}([0,T)\times\Omega). \tag{157}$$

Since $\rho \in L^2((0,T)\times\Omega)$, the transport theorem by DiPerna and Lions [10] implies that ρ also satisfies, in the sense of distributions, the renormalized continuity equation (29) for any bounded $\beta \in C^1(\mathbb{R})$ vanishing near 0 and such that also $(\beta'(1+|\cdot|))^{-1}$ is bounded. As ρ is bounded from above and away from 0, we can actually choose $\beta(z) = z\ln(z)$. Using the same choice in the corresponding relation (109) on the ϵ-level, letting $\epsilon \to 0$ and comparing the results, we conclude

$$\lim_{\epsilon\to 0} \int_\Omega \rho_\epsilon \ln(\rho_\epsilon(\tau))\, dx \leqslant \int_\Omega \rho \ln(\rho(\tau))\, dx \quad \text{for a.a. } \tau \in [0,T].$$

Following e.g. [15, Theorem 10.20], this implies, by the strict convexity of $z \mapsto z\ln(z)$, that

$$\rho_\epsilon \to \rho \quad \text{a.e. in } (0,T)\times\Omega.$$

In particular it follows

$$\rho_\epsilon \to \rho \quad \text{in } L^p\left((0,T)\times\mathbb{R}^3\right) \quad \forall 1 \leqslant p < \infty, \qquad \underline{\rho} \leqslant \rho \leqslant \overline{\rho} \quad \text{a.e. in } [0,T]\times\mathbb{R}^3, \tag{158}$$

where ρ has again been extended by $\underline{\rho}$ outside of Ω.

5.4 Momentum equation

In order to pass to the limit in the momentum equation we further need strong convergence of the velocity field. We test the momentum equation (127) on the ϵ-level by $\psi\Phi$, where $\psi \in \mathcal{D}(0,T)$ and $\Phi \in \mathcal{D}(\Omega)$ with $\mathrm{div}\Phi = 0$. This yields

$$\left\| \partial_t \int_\Omega P(\rho_\epsilon u_\epsilon)\cdot\Phi dx \right\|_{L^{\frac{4}{3}}(0,T)} \leqslant c, \tag{159}$$

where P again denotes the orthogonal projection of $L^2(\Omega)$ onto $V^0(\Omega)$. The estimate (159) leads, under exploitation of the Arzelà-Ascoli theorem, to

$$P(\rho_\epsilon u_\epsilon) \to P(\rho u) \quad \text{in } C_{\text{weak}}([0,T]; L^2(\Omega)) \text{ and thus in } L^2\left(0,T; \left(V_0^1(\Omega)\right)^*\right).$$

By the same arguments as in the proof of the classical compactness results for the incompressible Navier-Stokes equations, c.f. [31, Theorem 2.4], this yields strong convergence of u_ϵ in $L^2((0,T)\times\Omega)$ and in particular

$$\rho_\epsilon u_\epsilon \otimes u_\epsilon \rightharpoonup \rho u \otimes u \quad \text{in } L^2\left(0,T; L^{\frac{3}{2}}(\Omega)\right). \tag{160}$$

Moreover, we can use the strong convergence (146) of the characteristic function and the strong convergence (158) of the density to identify the limit function Π from (143) as $\Pi = \Pi_{[\chi,\rho,u]}$ just as in (119). Combining this with (149), (152), (156) for the identification of the magnetic term and (160), we can pass to the limit in (127). The regularization terms again vanish as stated in (144) and so we end up with

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\ dxdt-\int_\Omega \rho_0 u_0\cdot\phi(0,x)\ dx\\
&=\int_0^T\int_\Omega \rho(u\otimes u):\nabla\phi-2\nu\mathbb{D}(u):\nabla\phi-\frac{1}{\eta}\rho\chi\left(u-\Pi_{[\chi,\rho,u]}\right)\cdot\phi+\rho g\cdot\phi+\frac{1}{\mu}\left(\mathrm{curl}B\times B\right)\cdot\phi\ dxdt
\end{aligned}
\tag{161}
$$

for any $\phi\in\mathcal{D}([0,T)\times\Omega)$ with $\mathrm{div}\phi=0$.

5.5 Energy inequality

We drop the (nonnegative) regularization terms from the left-hand side of the energy inequality (129). Using weak lower semicontinuity of norms, we then let ϵ tend to 0 and obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2+\frac{1}{2}|B(\tau)|^2\ dx+\int_0^\tau\int_\Omega 2\nu\left|\nabla u(t,x)\right|^2+\frac{1}{\sigma\mu^2}\left|\mathrm{curl}B(t,x)\right|^2\\
&+\frac{1}{\eta}\rho(t,x)\chi(t,x)\left|\left(u(t,x)-\Pi_{[\chi,\rho,u]}(t,x)\right)\right|^2\ dxdt\\
&\leqslant\int_\Omega \frac{1}{2}\rho(0)|u(0)|^2+\frac{1}{2}|B(0)|^2\ dx+\int_0^\tau\int_\Omega \rho(t,x)g(t,x)\cdot u(t,x)+\frac{1}{\sigma}J(t,x)\cdot\mathrm{curl}B(t,x)\ dxdt
\end{aligned}
\tag{162}
$$

for almost all $\tau\in[0,T]$. Altogether we have shown

Proposition 5.1. *Let all the assumptions of Theorem 1.1 be satisfied and let $\eta>0$. Assume in addition that*

$$
\rho_{0,\eta}\in H^{1,2}(\Omega),\qquad u_{0,\eta},\ B_{0,\eta}\in H^{2,2}(\Omega).
$$

Then there exist

$$
\rho_\eta\in\left\{\psi\in L^\infty\left((0,T)\times\Omega\right):\ \underline{\rho}\leqslant\psi\leqslant\overline{\rho}\ a.e.\ in\ Q\right\},\tag{163}
$$

$$
\chi_\eta\in C\left([0,T];L^p\left(\mathbb{R}^3\right)\right),\ 1\leqslant p<\infty,\tag{164}
$$

$$
\begin{aligned}
B_\eta\in\Big\{&b\in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^{1,2}(\Omega)\right):\ \mathrm{div}\,b=0\ in\ Q,\\
&\mathrm{curl}\,b=0\ in\ Q^S(\chi_\eta,T)\bigcap Q,\ b\cdot n|_{\partial\Omega}=0\Big\},
\end{aligned}\tag{165}
$$

$$
u_\eta\in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;V_0^1(\Omega)\right)\tag{166}
$$

such that

$$
-\int_0^T\int_{\mathbb{R}^3}\chi_\eta\partial_t\Theta dxdt-\int_{\mathbb{R}^3}\chi_0\Theta(0,x)\ dx=\int_0^T\int_{\mathbb{R}^3}\left(\chi_\eta\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\right)\cdot\nabla\Theta\ dxdt,\tag{167}
$$

$$
-\int_0^T\int_\Omega\rho_\eta\partial_t\psi dxdt-\int_\Omega\rho_{0,\eta}\psi(0,x)\ dx=\int_0^T\int_\Omega\left(\rho_\eta u_\eta\right)\cdot\nabla\psi\ dxdt,\tag{168}
$$

$$
\begin{aligned}
-\int_0^T\int_\Omega\rho_\eta u_\eta\cdot\partial_t\phi\ dxdt-\int_\Omega\rho_{0,\eta}u_{0,\eta}\cdot\phi(0,x)\ dx=&\int_0^T\int_\Omega\rho_\eta(u_\eta\otimes u_\eta):\nabla\phi-2\nu\mathbb{D}(u_\eta):\nabla\phi\\
&-\frac{1}{\eta}\rho_\eta\chi_\eta\left(u_\eta-\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\right)\cdot\phi+\rho_\eta g\cdot\phi\\
&+\frac{1}{\mu}\left(\mathrm{curl}\,B_\eta\times B_\eta\right)\cdot\phi\ dxdt,
\end{aligned}\tag{169}
$$

$$
-\int_0^T\int_\Omega B_\eta\cdot\partial_t b\ dxdt-\int_\Omega B_{0,\eta}\cdot b(0,x)\ dx=\int_0^T\int_\Omega\left[-\frac{1}{\sigma\mu}\mathrm{curl}\,B_\eta+u_\eta\times B_\eta+\frac{1}{\sigma}J\right]\cdot\mathrm{curl}\,b\ dxdt\tag{170}
$$

for all $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$, $\psi, \phi \in \mathcal{D}([0,T) \times \Omega)$ *and all* $b \in Y(\chi, T)$. *Moreover, these functions satisfy the energy inequality*

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho_\eta(\tau)|u_\eta(\tau)|^2 + \frac{1}{2\mu}|B_\eta(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu \left|\nabla u_\eta(t,x)\right|^2 \\
&+ \frac{1}{\sigma\mu^2} \left|\mathrm{curl}\, B_\eta(t,x)\right|^2 + \frac{1}{\eta}\rho_\eta(t,x)\chi_\eta(t,x) \left|u_\eta(t,x) - \Pi_{[\chi_\eta,\rho_\eta,u_\eta]}(t,x)\right|^2 \, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,\eta}|u_{0,\eta}|^2 + \frac{1}{2}|B_{0,\eta}|^2 \, dx + \int_0^\tau \int_\Omega \rho_\eta(t,x) g(t,x) \cdot u_\eta(t,x) + \frac{1}{\sigma} J(t,x) \cdot curl B_\eta(t,x) \, dxdt
\end{aligned} \tag{171}$$

for almost all $\tau \in [0,T]$ *and the characteristic function* χ_η *is connected to the solution* $X^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}}$ *of the initial value problem*

$$\frac{dX^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}}(s;t,x)}{dt} = \Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\left(t, X^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}}(s;t,x)\right), \qquad X^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}}(s;s,x) = x \tag{172}$$

by

$$\chi(t,x) = \chi_0\left(X^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}}(t;0,x)\right). \tag{173}$$

6 Limit passage with respect to $\eta \to 0$

6.1 Uniform bounds and convergent terms

In order to prove Theorem 1.1, we assume in this section further that the regularized initial data we had chosen on the Δt-level and the ϵ-level satisfy

$$\rho_{0,\eta} \to \rho_0 \quad \text{in } L^2(\Omega), \qquad u_{0,\eta} \to u_0 \quad \text{in } L^2(\Omega), \qquad B_{0,\eta} \to B_0 \quad \text{in } L^2(\Omega), \tag{174}$$

where ρ_0, u_0, B_0 denote the initial data in Theorem 1.1. The energy inequality (171) implies the existence of a constant $c > 0$, independent of η, such that

$$\|u_\eta\|_{L^\infty(0,T;L^2(\Omega))} + \|B_\eta\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\eta\|_{L^2(0,T;H^{1,2}(\Omega))} + \|B_\eta\|_{L^2(0,T;H^{1,2}(\Omega))} \leqslant c, \tag{175}$$

$$\frac{1}{\eta^{\frac{1}{2}}} \left\|\chi_\eta \left(u_\eta - \Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\right)\right\|_{L^2(0,T;L^2(\Omega))} \leqslant c, \tag{176}$$

and as in the corresponding estimates (135)–(137) on the ϵ-level, we deduce that

$$\left|a_{[\chi_\eta,\rho_\eta]}(t)\right| \leqslant c, \quad \left|(u_G)_{[\chi_\eta,\rho_\eta,u_\eta]}(t)\right| \leqslant c \|u_\eta(t)\|_{L^2(\Omega)}, \quad \left|\omega_{[\chi_\eta,\rho_\eta,u_\eta]}(t)\right| \leqslant c \|u_\eta(t)\|_{L^2(\Omega)}, \tag{177}$$

$$v \cdot \left(I_{[\chi_\eta,\rho_\eta]}(t) v\right) \geqslant c|v|^2 \quad \forall v \in \mathbb{R}^3, \tag{178}$$

$$\left\|\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}(t,\cdot)\right\|_{W^{1,\infty}(\Omega)} \leqslant c \|u_\eta(t)\|_{L^2(\Omega)} \quad \text{for a.a. } t \in [0,T] \tag{179}$$

with c independent of η and t. The above bounds, together with the uniform bounds for the density in (163), allow us to find functions

$$\rho \in L^\infty((0,T) \times \Omega), \tag{180}$$

$$B \in \left\{b \in L^\infty\left(0,T; L^2(\Omega)\right) \bigcap L^2\left(0,T; V^1(\Omega)\right) : \ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\}, \tag{181}$$

$$u \in L^\infty\left(0,T; L^2(\Omega)\right) \bigcap L^2\left(0,T; V_0^1(\Omega)\right) \tag{182}$$

such that for extracted subsequences

$$u_\eta \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T; L^2(\Omega)\right), \qquad u_\eta \rightharpoonup u \qquad \text{in } L^2\left(0,T; H^{1,2}(\Omega)\right), \tag{183}$$

$$B_\eta \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T; L^2(\Omega)\right), \qquad B_\eta \rightharpoonup B \qquad \text{in } L^2\left(0,T; H^{1,2}(\Omega)\right), \tag{184}$$

$$\rho_\eta \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L^\infty(\Omega)\right), \qquad \Pi_{[\chi_\eta,\rho_\eta,u_\eta]} \overset{*}{\rightharpoonup} \Pi_{[\chi,\rho,u]} \quad \text{in } L^\infty\left(0,T; W^{1,\infty}_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right). \tag{185}$$

The identification of the limit function $\Pi_{[\chi,\rho,u]} = (u_G)_{[\chi,\rho,u]} + \omega_{[\chi,\rho,u]} \times (x - a_{[\chi,\rho]})$ in (185) can be obtained as in the derivation of (119), under the exploitation of the strong convergence of χ_η in (187) and ρ_η in (190) below.

6.2 Characteristic function

The transport equation (167), the equation (172) for the corresponding characteristics and the bounds (177) allow us to once more apply Lemma 7.2 and Remark 7.1, which yield

$$X_\eta^{\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}} \to X^{\Pi_{[\chi,\rho,u]}} \quad \text{in } C([0,T]\times[0,T];C_{\text{loc}}(\mathbb{R}^3)), \tag{186}$$

$$\chi_\eta \to \chi \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \qquad \chi(t,x) = \chi_0\left(X^{\Pi}(t;0,x)\right), \tag{187}$$

where $X^{\Pi_{[\chi,\rho,u]}}$ and χ denote the unique solutions of

$$-\int_0^T\int_{\mathbb{R}^3}\chi\partial_t\Theta dxdt - \int_{\mathbb{R}^3}\chi_0\Theta(0,x)\,dx = \int_0^T\int_{\mathbb{R}^3}\left(\chi\Pi_{[\chi,\rho,u]}\right)\cdot\nabla\Theta\,dxdt \quad \forall\Theta\in\mathcal{D}\left([0,T)\times\mathbb{R}^3\right), \tag{188}$$

$$\frac{dX^{\Pi_{[\chi,\rho,u]}}(s;t,x)}{dt} = \Pi_{[\chi,\rho,u]}\left(t, X^{\Pi_{[\chi,\rho,u]}}(s;t,x)\right), \qquad X^{\Pi_{[\chi,\rho,u]}}(s;s,x) = x. \tag{189}$$

6.3 Continuity equation

For the strong convergence of the density we can apply classical compactness results for the incompressible Navier-Stokes equations, c.f. [31, Theorem 2.4, Remark 2.4 3)], and infer that

$$\rho_\eta \to \rho \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \tag{190}$$

with ρ once again extended by $\underline{\rho}$ outside of Ω. Passing to the limit in (168), we see that ρ is the solution to

$$-\int_0^T\int_\Omega \rho\partial_t\psi dxdt - \int_\Omega \rho_0\psi(0,x)\,dx = \int_0^T\int_\Omega(\rho u)\cdot\nabla\psi\,dxdt \quad \forall\psi\in\mathcal{D}([0,T)\times\Omega). \tag{191}$$

6.4 Induction equation

In the induction equation, all the approximation terms already vanished during the last limit passage. Thus the limit passage with respect to $\eta \to 0$ works by the same arguments as before. Indeed, we can first use the uniform convergence (186) to check that for any $\gamma > 0$ there exists $\delta(\gamma) > 0$ such that

$$S_\gamma(\chi(t)) \subset S(\chi_\eta(t)) \subset S^{\frac{\gamma}{2}}(\chi(t)) \subset S^\gamma(\chi(t)) \quad \forall t\in[0,T],\ \eta < \delta(\gamma). \tag{192}$$

Then we can argue as in Section 5.2 to conclude

$$\text{curl}B = 0 \quad \text{a.e. in } Q^S(\chi,T)\bigcap Q, \tag{193}$$

and

$$u_\epsilon \times B_\epsilon \rightharpoonup u\times B \quad \text{in } L^{\frac{6}{5}}((0,T)\times\Omega), \qquad \text{curl}\,B_\epsilon\times B_\epsilon \rightharpoonup \text{curl}\,B\times B \quad \text{in } L^{\frac{6}{5}}((0,T)\times\Omega). \tag{194}$$

Exploiting further the convergence (174) of the initial data, we can pass to the limit in (170) and obtain

$$-\int_0^T\int_\Omega B\cdot\partial_t b\,dxdt - \int_\Omega B_0\cdot b(0,x)\,dx = \int_0^T\int_\Omega\left[-\frac{1}{\sigma\mu}\text{curl}B + u\times B + \frac{1}{\sigma}J\right]\cdot\text{curl}b\,dxdt \tag{195}$$

for all $b\in Y(\chi,T)$.

6.5 Momentum equation

Let now T' be given by (28), i.e. T' denotes the first time at which the rigid body $S(\chi(\cdot))$ collides with $\partial\Omega$ or, if this never happens in $[0,T]$, then $T'=T$. Since the initial distance between the body and $\partial\Omega$ is positive by (15), the uniform convergence (186) implies $T'>0$ and, for any $T_0<T'$, there is some $\gamma>0$ such that

$$\operatorname{dist}\Big(\partial\Omega,\ S(\chi(t))\Big)>\gamma \quad \forall t\in[0,T_0]. \tag{196}$$

Our first goal in this section is to show that the limit velocity indeed coincides with a rigid velocity field in the solid region. To this end we consider an arbitrary compact set $\overline{I\times U}\subset Q^S(\chi,T')$ with an interval $I\subset(0,T')$ and some ball $U\subset\Omega$. From the first inclusion in (192) we see that for sufficiently small η it holds

$$\overline{I\times U}\subset Q^S(\chi_\eta,T')\bigcap Q \quad\Leftrightarrow\quad \chi_\eta=1 \quad \text{on } \overline{I\times U}.$$

By the estimate (176) this means

$$u_\eta-\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\to 0 \quad \text{in } L^2(I\times U),$$

and as $\overline{I\times U}$ was chosen arbitrarily we get, as desired,

$$u=\Pi_{[\chi,\rho,u]} \quad \text{a.e. on } Q^S(\chi,T'). \tag{197}$$

Next, we show that the projection term vanishes in the limit of the momentum equation (169). We fix some arbitrary test function $\phi\in\mathcal{T}(\chi,T')$, i.e. $\phi\in\mathcal{D}([0,T)\times\Omega)$, $\operatorname{div}\phi=0$ and there exists $\sigma>0$ such that

$$\mathbb{D}(\phi)=0 \quad \text{in } \Big\{(t,x)\in Q(T'):\ \operatorname{dist}\Big((t,x),\overline{Q}^S(\chi,T')\Big)<\sigma\Big\}, \tag{198}$$

c.f. (16). We choose $T_0<T'$ such that

$$\operatorname{supp}\phi\subset[0,T_0]\times\Omega \tag{199}$$

and a corresponding $\gamma>0$ according to (196). By (198) $\phi\in\mathcal{T}(\chi,T')$, there is some $0<\sigma<\gamma$ such for all $t\in[0,T_0]$ the function $\phi(t,\cdot)$ coincides with a rigid velocity field $\phi^S(t,\cdot)$ on $S^\sigma(\chi(t))\subset\Omega$. As

$$\chi_\eta(t,x)=0 \quad \text{for } x\in\Omega\backslash S\left(\chi_\eta(t)\right),$$

the inclusion (192) implies that for sufficiently small $\eta>0$ it holds

$$\int_0^{T'}\int_\Omega -\frac{1}{\eta}\rho_\eta\chi_\eta\left(u_\eta-\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\right)\cdot\phi\,dxdt=\int_0^{T'}\int_\Omega -\frac{1}{\eta}\rho_\eta\chi_\eta\left(u_\eta-\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}\right)\cdot\phi^S\,dxdt=0, \tag{200}$$

where the second equality is a consequence of the fact that $\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}(t,\cdot)$ is the orthogonal projection of $u_\eta(t,\cdot)$ onto rigid velocity fields on $S(\chi_\eta(t))$, c.f. [4, Lemma 3.1].
We further note that by the uniform bounds for u_η in (175) there exists a function $z_6\in L^2(0,T;L^{\frac{3}{2}}(\Omega))$ such that for a chosen subsequence it holds

$$\rho_\eta u_\eta\otimes u_\eta \rightharpoonup z_6 \quad \text{in } L^2\left(0,T;L^{\frac{3}{2}}(\Omega)\right).$$

Combining this with the convergence (174) of the initial data, the strong convergence (190) of the density, the weak convergence (194) of the magnetic term and (200), we can pass to the limit in (169) and obtain

$$\begin{aligned}-\int_0^{T'}\int_\Omega \rho u\cdot\partial_t\phi\,dxdt-\int_\Omega \rho_0u_0\cdot\phi(0,x)\,dx=\int_0^{T'}\int_\Omega & z_6:\nabla\phi-2\nu\mathbb{D}(u):\nabla\phi\\ &+\rho g\cdot\phi+\frac{1}{\mu}\left(\operatorname{curl}B\times B\right)\cdot\phi\,dxdt\end{aligned} \tag{201}$$

for any $\phi \in \mathcal{T}(\chi, T')$. It remains to identify z_6. To this end it is sufficient to show that

$$\int_0^{T_0} \int_\Omega \rho_\eta |u_\eta|^2 \, dxdt \to \int_0^{T_0} \int_\Omega \rho |u|^2 \, dxdt \tag{202}$$

for arbitrary $0 < T_0 < T'$. Indeed, as in the proof of the classical compactness result [31, Theorem 2.4], this leads to strong convergence of u_η in $L^2((0,T_0) \times \Omega)$ and in particular to

$$z_6 = \rho u \otimes u \quad \text{a.e. on } (0,T_0) \times \Omega. \tag{203}$$

Since for any arbitrary but fixed test function $\phi \in \mathcal{T}(\chi, T')$ we can find $T_0 < T'$ such that the inclusion (199) holds true, (203) suffices to identify z_6 in the momentum equation (201). The proof of (202) is achieved by following mostly [4] and using further arguments from [14]. More precisely, for fixed $0 < T_0 < T'$, we choose $\gamma_{\text{sup}} = \gamma_{\text{sup}}(T_0) > 0$ as the supremum over all γ which satisfy (196). Then for any $0 \leqslant \gamma \leqslant \frac{\gamma_{\text{sup}}}{4}$, $t \in [0,T_0]$ and $r \in [0,1]$ we define

$$K^r_{t,\gamma}(\Omega) := \left\{ v(t) \in V^r_0(\Omega) : \ \mathbb{D}(v(t)) = 0 \text{ in } \mathcal{D}'\left(S^\gamma(\chi(t))\right) \right\} \tag{204}$$

together with the orthogonal projection

$$P^r_\gamma(t) : \ H^{r,2}(\Omega) \to K^r_{t,\gamma}(\Omega). \tag{205}$$

By the triangle inequality we estimate, for arbitrary $\psi \in \mathcal{D}(0,T_0)$, $r \in (0,1)$ and $\gamma \in (0, \frac{\gamma_{\text{sup}}}{4}]$,

$$\begin{aligned} &\left| \int_0^{T_0} \int_\Omega \psi \rho_\eta |u_\eta|^2 \, dxdt - \int_0^{T_0} \int_\Omega \psi \rho |u|^2 \, dxdt \right| \\ \leqslant &\overline{\rho} \, \|\psi\|_{L^\infty(0,T_0)} \|u_\eta\|_{L^2(0,T_0;L^2(\Omega))} \left\| P^r_\gamma u_\eta - u_\eta \right\|_{L^2(0,T_0;L^2(\Omega))} + \left| \int_0^{T_0} \int_\Omega \psi \left(\rho_\eta u_\eta \cdot P^r_\gamma u_\eta - \rho u \cdot P^r_\gamma u \right) \, dxdt \right| \\ &+ \overline{\rho} \, \|\psi\|_{L^\infty(0,T_0)} \|u\|_{L^2(0,T_0;L^2(\Omega))} \left\| P^r_\gamma u - u \right\|_{L^2(0,T_0;L^2(\Omega))}. \end{aligned} \tag{206}$$

Keeping $r \in (0,1)$ and $\gamma \in (0, \frac{\gamma_{\text{sup}}}{4}]$ fixed, we let first η tend to 0. During this procedure, the second term on the right-hand side of (206) vanishes, c.f. Lemma 7.3 in the Appendix. Subsequently, by letting γ tend to 0, also the first and the last term on the right-hand side of (206) vanish, c.f. Lemma 7.4 in the Appendix. Finally, replacing ψ by a suitable sequence of cut-off functions on $[0,T_0]$, we infer the convergence (202) and hence the identity (203).

6.6 Proof of the main result

Summarizing the results from Sections 6.1–6.5, we can now finish the proof of Theorem 1.1. The regularities of χ and ρ in (18) and (19) follow from the choice of the spaces in (187) and (190). As

$$\mathbb{D}\left(\Pi_{[\chi,\rho,u]}\right) = 0,$$

the properties of u in (20) follow from (182) and the relation (197) between u and $\Pi_{[\chi,\rho,u]}$, while the properties of B in (21) are given by (181) and (193). The transport equations (22) and (23) were shown in (188) and (191), where in (188) the function $\Pi_{[\chi,\rho,u]}$ can indeed be replaced by u due to the relation (197) between these two functions and the fact that $\chi = 0$ outside of $Q^S(\chi, T')$. The momentum equation (24) is satisfied according to (201), where z_6 was identified in (203). The induction equation (25) was shown to hold true in (195). The energy inequality (26) follows by dropping the nonnegative projection term in the energy inequality (171) on the η-level and exploiting the weak lower semicontinuity of norms. Finally, by the group property [10, (76)], which is satisfied by the solution $X^{\Pi_{[\chi,\rho,u]}}$ to the initial value problem (189), it holds that

$$\begin{aligned} S(\chi(t)) &= \left\{ x \in \mathbb{R}^3 : \ \chi(t,x) = 1 \right\} = \left\{ X^\Pi(0;t,x) : \ x \in S \right\} = X^\Pi(0;t,S) \\ &= X^\Pi\left(s;t, \left\{ X^\Pi(0;s,S) \right\}\right) = X^\Pi(s;t,S(\chi(s))) \end{aligned}$$

for all $s,t \in [0,T']$. By (186), X^Π is the (pointwise) limit of a sequence of isometries and hence an isometry itself. Thus, the identity (27) follows for the choice $X = X^\Pi$, which concludes the proof.

7 Appendix

In the limit passage with respect to $\Delta t \to 0$ the following variant of [36, Theorem 8.9] is used, which guarantees that the weak limits of different interpolants of the same discrete functions coincide.

Lemma 7.1. *Let $f_{\Delta t}, \overline{f}_{\Delta t}, \overline{f}'_{\Delta t}$ be piecewise affine and, respectively, piecewise constant interpolants of discrete functions $f^k_{\Delta t}$, $k = 0, ..., \frac{T}{\Delta t}$ defined as in (58)–(60). Assume further that*

$$f_{\Delta t} \overset{*}{\rightharpoonup} f \quad \text{in } L^\infty(0,T;L^2(\Omega)), \quad \overline{f}_{\Delta t} \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \overline{f}'_{\Delta t} \overset{*}{\rightharpoonup} \overline{f}' \quad \text{in } L^\infty(0,T;L^2(\Omega))$$

Then it holds

$$f = \overline{f} = \overline{f}'. \tag{207}$$

Proof
The proof, which is performed by comparing the limit of the functions $\overline{f}_{\Delta t}$, $\overline{f}'_{\Delta t}$ to the one of $f_{\Delta t}$ in the pairing with piecewise constant in time functions respectively, can be found in the proof of [36, Theorem 8.9]. For the convenience of the reader, we restate the argument here: Without loss of generality, we only consider the subsequences with indices $\Delta t = 2^{-l}T$, $l \in \mathbb{N}$. We pick $L \in \mathbb{N}$, $k_1 < k_2 < 2^L$ and $\psi \in L^2(\Omega)$ and consider functions of the form $\chi_{[\tau k_1, \tau k_2]}\psi$, where $\tau := 2^{-L}T > 0$ and $\chi_{[\tau k_1, \tau k_2]}$ denotes the characteristic function of the interval $[\tau k_1, \tau k_2]$. By [36, Proposition 1.36], linear combinations of such functions are dense in $L^2(0,T;L^2(\Omega))$. For $\Delta t \leqslant \tau$, i.e. $l \geqslant L$, we calculate

$$\left| \int_0^T \int_\Omega \left(f_{\Delta t} - \overline{f}_{\Delta t}\right) \cdot \chi_{[\tau k_1, \tau k_2]}\psi \, dxdt \right| = \left| \sum_{k=\frac{\tau k_1}{\Delta t}+1}^{\frac{\tau k_2}{\Delta t}} \int_{(k-1)\Delta t}^{k\Delta t} \int_\Omega \left[\left(f^k_{\Delta t} - f^{k-1}_{\Delta t} \right) \frac{t - k\Delta t}{\Delta t} \right] \cdot \psi \, dxdt \right|$$
$$= \left| -\frac{\Delta t}{2} \sum_{k=\frac{\tau k_1}{\Delta t}+1}^{\frac{\tau k_2}{\Delta t}} \int_\Omega \left(f^k_{\Delta t} - f^{k-1}_{\Delta t} \right) \cdot \psi \, dx \right| = \left| -\frac{\Delta t}{2} \int_\Omega \left(f_{\Delta t}(\tau k_2) - f_{\Delta t}(\tau k_1) \right) \cdot \psi \, dx \right| \leqslant c\Delta t, \tag{208}$$

with c independent of Δt, since $f_{\Delta t}$ is bounded uniformly in $L^\infty(0,T;L^2(\Omega))$. We conclude

$$f_{\Delta t} - \overline{f}_{\Delta t} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^2(\Omega)),$$

which implies the first identity from (207). Using the same kind of test function again, we also see

$$\begin{aligned} &\left| \int_0^T \int_\Omega \left(f_{\Delta t} - \overline{f}'_{\Delta t} \right) \cdot \chi_{[\tau k_1, \tau k_2]}\psi \, dxdt \right| \\ =&\left| \int_0^T \int_\Omega \left(f_{\Delta t} - \overline{f}_{\Delta t} \right) \cdot \chi_{[\tau k_1, \tau k_2]}\psi \, dxdt + \int_{\tau k_2 - \Delta t}^{\tau k_2} \int_\Omega f^{\frac{\tau k_2}{\Delta t}}_{\Delta t} \cdot \psi \, dxdt - \int_{\tau k_1}^{\tau k_1 + \Delta t} \int_\Omega f^{\frac{\tau k_1}{\Delta t}}_{\Delta t} \cdot \psi \, dxdt \right| \\ \leqslant& c\Delta t + 2\Delta t \|\overline{f}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \|\psi\|_{L^2(\Omega)} \leqslant c\Delta t, \end{aligned}$$

exploiting in the first inequality the estimate we already know from (208). This implies $f = \overline{f}'$ and hence the second identity in (207).

□

For the limit passage in the transport equation we use the following result, which is a variant of [37, Lemma 5.2, Corollary 5.2, Corollary 5.3]:

Lemma 7.2. *Assume that for any $n \in \mathbb{N}$, the function*

$$\Pi_n : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \Pi_n(t,x) := v_n(t) + w_n(t) \times x, \quad v_n, w_n \in L^\infty(0,T),$$

satisfies

$$\|v_n\|_{L^\infty(0,T)}, \ \|w_n\|_{L^\infty(0,T)} \leqslant c \tag{209}$$

with c independent of n. Denote further by X_n the Carathéodory solution of

$$\frac{dX_n(s;t,x)}{dt} = \Pi_n\left(t, X_n(s;t,x)\right), \qquad X_n(s;s,x) = x, \quad x \in \mathbb{R}^3, \tag{210}$$

and by $\chi_n(t,x) = \chi_0(X_n(t;0,x))$ the corresponding solution to

$$-\int_0^T \int_{\mathbb{R}^3} \chi_n \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} (\chi_n \Pi_n) \cdot \nabla\Theta\ dxdt \quad \forall \Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3). \tag{211}$$

Then, passing to subsequences if necessary, it holds that

$$X_n \to X \quad \text{in } C\left([0,T] \times [0,T]; C_{loc}\left(\mathbb{R}^3\right)\right), \tag{212}$$

$$\chi_n \to \chi \quad \text{in } C([0,T]; L^p_{loc}(\mathbb{R}^3)) \quad \forall 1 \leqslant p < \infty \tag{213}$$

with X denoting the unique solution of

$$\frac{dX(s;t,x)}{dt} = \Pi\left(t, X(s;t,x)\right), \qquad X(s;s,x) = x, \tag{214}$$

χ the one of

$$-\int_0^T \int_{\mathbb{R}^3} \chi \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} (\chi \Pi) \cdot \nabla\Theta\ dxdt \quad \forall \Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3) \tag{215}$$

and with Π given by

$$\Pi_n \overset{*}{\rightharpoonup} \Pi \quad \text{in } L^\infty(0,T; W^{1,\infty}_{loc}(\mathbb{R}^3)), \qquad \Pi(t,x) = v(t) + w(t) \times x, \qquad v, w \in L^\infty(0,T). \tag{216}$$

Moreover,

$$\chi(t,x) = \chi_0\left(X(t;0,x)\right). \tag{217}$$

Proof
First we note that the existence of the solution X_n to (210) and the fact that χ_n is the solution to (211) are guaranteed by [10, Theorem 3.2]. The relation (216) is clear by (209). The convergence (213) and (215) then immediately follow from [31, Theorem 2.5]. From the Gronwall inequality, (209) and (210) it is possible to check that for each compact $K \subset \mathbb{R}^3$

$$\{X_n(s;t,\cdot)\} \text{ is relatively compact in } C(K) \text{ for all fixed } (s,t) \in [0,T] \times [0,T]$$

and further to show equicontinuity of the mapping

$$(s,t) \mapsto X_n(s;t,\cdot)$$

from $[0,T] \times [0,T]$ to $C(K)$. This gives us the conditions for a generalized version of the Arzelà-Ascoli theorem, [44, A_1(24i)], which allows us to infer (212). The fact that $X(s;\cdot,x)$ is the Carathéodory solution to the initial value problem (214) then follows by writing (210) in a variational form and passing to the limit with the help of (212) and (216). Since this solution is unique, it follows that the solution of (215) is given by the right-hand side of (217). But since we already determined the unique solution of (215) as the function χ given by (213), the equation (217) holds true, which concludes the proof.

□

Remark 7.1. *If χ_0 has compact support in $\mathbb{R}^3$, the relation $\chi_n(t,x) = \chi_0(X_n(t;0,x))$ allows us to improve the local convergence (213) to*

$$\chi_n \to \chi \quad \text{in } C([0,T]; L^p(\mathbb{R}^3)) \quad \forall 1 \leqslant p < \infty.$$

In the limit passage with respect to $\eta \to 0$, we exploit the following two results, Lemma 7.3 and Lemma 7.4, which are versions of [14, Lemma 3.4] (c.f. [4, Lemma 3.8] for a related result) and [4, Lemma 3.6, Lemma 3.7] respectively. For the convenience of the reader we outline the proofs of these lemmata below.

Lemma 7.3. *Let $0 < T_0 < T'$ be fixed, where T' is defined by (28). Let further $\gamma_{sup} = \gamma_{sup}(T_0) > 0$ be the supremum of all γ which satisfy (196). Then, for any $\gamma \in (0, \frac{\gamma_{sup}}{4}]$ and any $0 < r < 1$, it holds*

$$\left| \int_0^{T_0} \int_\Omega \psi \left(\rho_\eta u_\eta \cdot P_\gamma^r u_\eta - \rho u \cdot P_\gamma^r u \right) \, dxdt \right| \to 0 \quad \textit{for } \eta \to 0$$

Proof
The argument of the proof is the same as in [14, Lemma 3.4]. Then we test the momentum equation (169) on the η-level by test functions $\tilde{\psi}\phi$, where $\phi \in \mathcal{D}(\Omega)$ satisfies $\mathbb{D}(\phi) = 0$ in $S^{\frac{\gamma}{4}}(\chi(\tau))$ and $\tilde{\psi} \in \mathcal{D}(J(\tau))$ for an open neighbourhood $J(\tau)$ of τ in $[0, T_0]$, which, by the inclusion (192) and the fact that $X^{\Pi_{[\chi,\rho,u]}} \in C([0,T] \times [0,T]; C_{\text{loc}}(\mathbb{R}^3))$ (c.f. (186)), can be chosen sufficiently small such that

$$S^{\frac{\gamma}{8}}(\chi_\eta(t)) \subset S^{\frac{\gamma}{4}}(\chi(\tau)) \quad \text{and thus} \quad \mathbb{D}(\phi) = 0 \text{ in } S^{\frac{\gamma}{8}}(\chi_\eta(t))$$

for all $t \in J(\tau)$ and all sufficiently small $\eta > 0$. By the same arguments as in (200) it then follows that, for all such η, the term $\frac{1}{\eta}\rho_\eta \chi_\eta (u_\eta - \Pi_{[\chi_\eta, \rho_\eta, u_\eta]})$ vanishes from the momentum equation (169) tested by any $\tilde{\psi}\phi$ as chosen above. This leads to the dual estimate

$$\left\| \partial_t \int_\Omega \rho_\eta u_\eta \cdot \phi \, dx \right\|_{L^{\frac{4}{3}}(J(\tau))} \leqslant c,$$

which implies, since $K^0_{\tau,\frac{\gamma}{4}}(\Omega) \bigcap \mathcal{D}(\Omega)$ is dense in $K^0_{\tau,\frac{\gamma}{2}}(\Omega)$, that

$$\rho_\eta u_\eta \to \rho u \quad \text{in } C_{\text{weak}}\left(J(\tau); \left(K^0_{\tau,\frac{\gamma}{2}}(\Omega)\right)^*\right) \quad \text{and thus in} \quad L^2\left(J(\tau); \left(K^r_{\tau,\frac{\gamma}{2}}(\Omega)\right)^*\right). \tag{218}$$

From this it is easy to see that the assertion holds on $J(\tau)$ and, by the compactness of $[0, T_0]$, also on $[0, T_0]$.

□

Lemma 7.4. *Let T_0 be as in Lemma 7.3. For any fixed $r \in (0,1)$ it holds*

$$(i) \quad \lim_{\gamma \to 0} \lim_{\eta \to 0} \left\| P_\gamma^r u_\eta - u_\eta \right\|_{L^2(0,T_0;L^2(\Omega))} = 0,$$
$$(ii) \quad \lim_{\gamma \to 0} \left\| P_\gamma^r u - u \right\|_{L^2(0,T_0;L^2(\Omega))} = 0.$$

For the proof of Lemma 7.4 we introduce some additional notation: To this end we remark that by the second inclusion in (192), we find some $\eta_{\text{sup}} = \eta_{\text{sup}}(\gamma_{\text{sup}}) > 0$ such that

$$\text{dist}\Big(\partial\Omega, \; S(\chi_\eta(t))\Big) > \frac{\gamma_{\text{sup}}}{2} \quad \forall \eta \in (0, \eta_{\text{sup}}], \; t \in [0, T_0],$$

where $\gamma_{\text{sup}} = \gamma_{\text{sup}}(T_0)$ is as in Lemma 7.3. For any $0 \leqslant \gamma \leqslant \frac{\gamma_{\text{sup}}}{4}$, $0 < \eta \leqslant \eta_{\text{sup}}$, $t \in [0, T_0]$ and $r \in [0,1]$ we define, corresponding to (204) and (205), the space

$$K^r_{t,\gamma,\eta}(\Omega) := \left\{ v(t) \in V^r_0(\Omega) : \; \mathbb{D}(v(t)) = 0 \text{ in } \mathcal{D}'\left(S^\gamma(\chi_\eta(t))\right) \right\}$$

and the associated orthogonal projection

$$P^r_{\gamma,\eta}(t) : \; H^{r,2}(\Omega) \to K^r_{t,\gamma,\eta}(\Omega).$$

We further need the following two auxiliary results:

Lemma 7.5. *There are constants* $\frac{\gamma_{\sup}}{4} > \gamma_0 > 0$ *and* $c > 0$ *such that for all* $t \in [0, T_0]$, $\gamma \in [0, \gamma_0]$ *and* $\eta \in (0, \eta_{sup}]$ *the trace inequality*

$$\|f(t,\cdot)\|^2_{L^2(\partial S^\gamma(\chi_\eta(t)))} \leqslant c \|f(t,\cdot)\|^{\frac{1}{2}}_{L^2(S^\gamma(\chi_\eta(t)))} \|f(t,\cdot)\|^{\frac{3}{2}}_{H^{1,2}(S^\gamma(\chi_\eta(t)))} \tag{219}$$

holds true for functions $f(t,\cdot) \in H^{1,2}(S^\gamma(\chi_\eta(t)))$ *and the Poincaré-type estimate*

$$\|f(t,\cdot)\|^2_{L^2(S^\gamma(\chi_\eta(t))\setminus\overline{S}(\chi_\eta(t)))} \leqslant c\left(\gamma \|f(t,\cdot)\|^2_{L^2(\partial S(\chi_\eta(t)))} + \gamma^2 \|\nabla f(t,\cdot)\|^2_{L^2(S^\gamma(\chi_\eta(t))\setminus\overline{S}(\chi_\eta(t)))}\right) \tag{220}$$

holds true for functions $f(t,\cdot) \in H^{1,2}(S^\gamma(\chi_\eta(t))\setminus\overline{S}(\chi_\eta(t)))$.

Proof
We first sketch the proof of (219). The idea is to consider, for $\gamma_0 > 0$ sufficiently small, a mapping $\Phi_{t,\eta}$ on $\partial S(\chi_\eta(t)) \times [-\gamma_0, \gamma_0]$ such that $\Phi_{t,\eta}(\cdot, 0) = \text{id}$ and

$$\Phi_{t,\eta}(\partial S(\chi_\eta(t)), \gamma) = \begin{cases} \{x \in S(\chi_\eta(t)) : \ \text{dist}(x, \partial S(\chi_\eta(t))) = -\gamma\} & \text{for } \gamma < 0, \\ \{x \in \Omega\setminus S(\chi_\eta(t)) : \ \text{dist}(x, \partial S(\chi_\eta(t))) = \gamma\} & \text{for } \gamma > 0. \end{cases}$$

We further choose $\Phi_{t,\eta}$ to be bi-Lipschitz continuous uniformly with respect to t and η, i.e. both $\Phi_{t,\eta}$ and its inverse are Lipschitz-continuous with Lipschitz-constants independent of t and η. Such a mapping exists, since $S(\chi_\eta(t))$ is a Lipschitz domain by the assumptions of Theorem 1.1. For $a, b \in [-\gamma_0, \gamma_0]$ we denote by $S_{t,\eta,[a,b]}$ the set $\Phi_{t,\eta}(\partial S(\chi_\eta(t)), [a, b])$. By means of some integral transformations, we can now transfer the problem to $S(\chi_\eta(t))$, where we can make use of the trace inequality

$$\|\cdot\|_{L^2(\partial S(\chi_\eta(t)))} \leqslant \|\cdot\|_{L^2\left(\partial S_{t,\eta,[-\gamma_0,0]}\right)} \leqslant c \|\cdot\|_{H^{\frac{3}{4},2}\left(S_{t,\eta,[-\gamma_0,0]}\right)}, \tag{221}$$

c.f. [35, Theorem 2.3], with a constant c independent of t, γ and η. The estimate (221) leads to

$$\|f(t,\cdot)\|^2_{L^2(\partial S^\gamma(\chi_\eta(t)))} \leqslant c \|f(t, \Phi_{t,\eta}(\cdot,\gamma))\|^2_{L^2(\partial S(\chi_\eta(t)))} \leqslant c \|f(t,\cdot)\|^2_{H^{\frac{3}{4},2}(S^\gamma(\chi_\eta(t)))} \quad \forall \gamma \in [0, \gamma_0],$$

where the constants c are independent of t, γ and η due to the uniform bi-Lipschitz continuity of $\Phi_{t,\eta}$. The inequality (219) then follows by an interpolation between L^2, $H^{\frac{3}{4},2}$ and $H^{1,2}$. For the proof of (220) we also exploit the uniform bi-Lipschitz continuity of $\Phi_{t,\eta}$, which implies that

$$\int_{\partial S(\chi_\eta(t))} \int_0^\gamma |f(t, \Phi_{t,\eta}(\cdot, 0))|^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS \leqslant c\gamma \|f(t,\cdot)\|^2_{L^2(\partial S(\chi_\eta(t)))} \tag{222}$$

with a constant c uniform in t, γ and η. Using Young's inequality we can therefore estimate

$$\begin{aligned} &\|f(t,\cdot)\|^2_{L^2(S_{t,\eta,[0,\gamma]})} - c\gamma \|f(t,\cdot)\|^2_{L^2(\partial S(\chi_\eta(t)))} \\ \leqslant &\int_{\partial S(\chi_\eta(t))} \int_0^\gamma |f(t, \Phi_{t,\eta}(\cdot, s))|^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS - \int_{\partial S(\chi_\eta(t))} \int_0^\gamma |f(t, \Phi_{t,\eta}(\cdot, 0))|^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS \\ \leqslant 2 &\int_{\partial S(\chi_\eta(t))} \int_0^\gamma (|f(t, \Phi_{t,\eta}(\cdot, s))| - |f(t, \Phi_{t,\eta}(\cdot, 0))|)^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS + c\gamma \|f(t,\cdot)\|^2_{L^2(\partial S(\chi_\eta(t)))}. \end{aligned} \tag{223}$$

Making use of the uniform bi-Lipschitz continuity of $\Phi_{t,\eta}$ once more and applying Jensen's inequality, we may further estimate

$$\begin{aligned} &\int_{\partial S(\chi_\eta(t))} \int_0^\gamma (|f(t, \Phi_{t,\eta}(\cdot, s))| - |f(t, \Phi_{t,\eta}(\cdot, 0))|)^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS \\ \leqslant c &\int_{\partial S(\chi_\eta(t))} \int_0^\gamma \left(\int_0^s |\nabla f(t, \Phi_{t,\eta}(\cdot, \tilde{s}))| d\tilde{s}\right)^2 |\text{det} D\Phi_{t,\eta}(\cdot, s)| ds dS \leqslant c\gamma^2 \int_{S_{t,\eta,[0,\gamma]}} |\nabla f(t, y)|^2 dy. \end{aligned}$$

Applying this to the first term on the right-hand side of (223), we infer (220).

□

The second auxiliary result we require for the proof of Lemma 7.4 is the following variant of [4, Lemma 3.3]:

Lemma 7.6. *Let γ_0 denote the constant from Lemma 7.5. Let further, for $t \in [0, T_0]$, $\gamma \in [0, \gamma_0]$ and $\eta \in (0, \eta_{\sup}]$, the functions $w(t) \in H^{1,2}(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$, $p(t) \in L^2(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$ denote the solution to the Stokes problem*

$$\begin{aligned} -\Delta w(t,\cdot) + \nabla p(t,\cdot) &= 0 \quad \text{on } \Omega \backslash \overline{S}^\gamma(\chi_\eta(t)), \\ \operatorname{div} w(t,\cdot) &= 0 \quad \text{on } \Omega \backslash \overline{S}^\gamma(\chi_\eta(t)), \\ w(t,\cdot) &= \begin{cases} v(t,\cdot) & \text{on } \partial S^\gamma(\chi_\eta(t)), \\ 0 & \text{on } \partial\Omega, \end{cases} \end{aligned} \tag{224}$$

for $v(t) \in H^{1,2}(S^\gamma(\chi_\eta(t)))$. Then there exists a constant $c > 0$, independent of t, γ and η, such that

$$\|w(t)\|_{L^2(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))} \leqslant c \, \|v(t)\|_{L^2(S^\gamma(\chi_\eta(t)))}^{\frac{1}{4}} \|v(t)\|_{H^{1,2}(S^\gamma(\chi_\eta(t)))}^{\frac{3}{4}}.$$

The same estimate also holds true for the corresponding solution to the Stokes problem in the limit $\eta \to 0$, i.e. with $\chi_\eta(t)$ replaced by $\chi(t)$.

Proof
The proof essentially follows [4, Lemma 3.3]. The idea is to consider the Stokes problem on $\Omega \backslash \overline{S}^\gamma(\chi_\eta(t))$ with no-slip boundary condition and arbitrary right-hand side $\phi(t) \in L^2(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$. The unique solution $\tilde{w}(t) \in H^{2,2}(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$, $\tilde{p}(t) \in H^{1,2}(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$ to this problem can be seen to satisfy

$$\int_{\Omega \backslash \overline{S}^\gamma(\chi_\eta(t))} w(t,x) \cdot \phi(t,x) \, dx = -\int_{\partial(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))} (\nabla \tilde{w}(t) w(t)) \cdot \mathrm{n} \, dx + \int_{\partial(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))} \tilde{p}(t) w(t) \cdot \mathrm{n} \, dx, \tag{225}$$

where n denotes the outer unit normal vector on $\partial(\Omega \backslash \overline{S}^\gamma(\chi_\eta(t)))$. The arbitrary choice of $\phi(t)$ then yields a dual estimate for $w(t,\cdot)$ from which, together with the trace inequality (219) and the standard estimates for the Stokes problem (c.f. [42, Proposition 2.2]), the assertion follows.

□

Proof of Lemma 7.4
First we sketch the idea of the proof of (i), which follows [4, Lemma 3.7]. For almost all $t \in [0, T_0]$ we define $v_{\gamma\eta}(t,\cdot)$, $p_{\gamma\eta}(t,\cdot)$ as the solution to the Stokes problem

$$\begin{aligned} -\Delta v_{\gamma\eta}(t,\cdot) + \nabla p_{\gamma\eta}(t,\cdot) &= -\Delta u_\eta(t,\cdot) \quad \text{on } \Omega \backslash \overline{S}^\gamma(\chi_\eta(t)) \\ \operatorname{div} v_{\gamma\eta}(t,\cdot) &= 0 \quad \text{on } \Omega \backslash \overline{S}^\gamma(\chi_\eta(t)) \\ v_{\gamma\eta}(t,\cdot) &= \begin{cases} \Pi_{[\chi_\eta,\rho_\eta,u_\eta]}(t,\cdot) & \text{on } \partial S^\gamma(\chi_\eta(t)), \\ 0 & \text{on } \partial\Omega. \end{cases} \end{aligned}$$

We extend $v_{\gamma\eta}(t,\cdot)$ by $\Pi_{[\chi_\eta,\rho_\eta,u_\eta]}(t,\cdot)$ in $S^\gamma(\chi_\eta(t))$ and note that $e_{\gamma\eta}(t,\cdot) := v_{\gamma\eta}(t,\cdot) - u_\eta(t,\cdot)$ solves a Stokes problem on $\Omega \backslash \overline{S}^\gamma(\chi_\eta(t))$ with 0-right-hand side to which we can apply Lemma 7.6. This, together with the estimate (176) for $u_\eta - \Pi_{[\chi_\eta,\rho_\eta,u_\eta]}$, the trace inequality (219) and the Poincaré-type estimate (220) leads to

$$\lim_{\gamma\to0}\lim_{\eta\to0} \|e_{\gamma\eta}\|_{L^2(0,T_0;H^{r,2}(\Omega))} = 0. \tag{226}$$

Since $v_{\gamma\eta}(t,\cdot)$ coincides with a rigid velocity field in $S^\gamma(\chi_\eta(t))$ for almost all $t \in [0, T_0]$, i.e. $v_{\gamma\eta}(t,\cdot) \in K^r_{t,\gamma,\eta}(\Omega)$, the equation (226) implies

$$\lim_{\gamma\to0}\lim_{\eta\to0} \|P^r_{\gamma,\eta} u_\eta - u_\eta\|_{L^2(0,T_0;H^{r,2}(\Omega))} \leqslant \lim_{\gamma\to0}\lim_{\eta\to0} \|v_{\gamma\eta} - u_\eta\|_{L^2(0,T_0;H^{r,2}(\Omega))} = 0.$$

Moreover, by the first inclusion in (192) we have $S(\chi(t)) \subset S^\gamma(\chi_\eta(t))$ for all sufficiently small $\eta > 0$. Hence, for such η, we obtain $P^r_{2\gamma,\eta}(t) u_\eta(t,\cdot) \in K^r_{t,\gamma}(\Omega)$, which then yields

$$\begin{aligned} &\lim_{\gamma\to0}\lim_{\eta\to0} \|P^r_\gamma u_\eta - u_\eta\|_{L^2(0,T_0;L^2(\Omega))} \\ &\leqslant \lim_{\gamma\to0}\lim_{\eta\to0} \|P^r_\gamma u_\eta - u_\eta\|_{L^2(0,T_0;H^{r,2}(\Omega))} \leqslant \lim_{\gamma\to0}\lim_{\eta\to0} \|P^r_{2\gamma,\eta} u_\eta - u_\eta\|_{L^2(0,T_0;H^{r,2}(\Omega))} = 0, \end{aligned}$$

i.e. (i). The assertion (ii) follows by similar arguments, c.f. also [4, Lemma 3.6].

□

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