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**Abstract**

In this paper we study the convergence rate of a finite volume approximation of the compressible Navier–Stokes–Fourier system. To this end we first show the local existence of a highly regular unique strong solution and analyse its global extension in time as far as the density and temperature remain bounded. We make a physically reasonable assumption that the numerical density and temperature are uniformly bounded from above and below. The relative energy provides us an elegant way to derive a priori error estimates between finite volume solutions and the strong solution.

**Keywords:** compressible Navier–Stokes–Fourier system, finite volume method, error estimates, weak–strong uniqueness, relative energy

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# 1 Introduction

Numerical simulation of gas dynamics plays an important role in a wide range of modern industrial and real life applications, such as vehicle engineering, engines design, aerospace or weather forecast. In the past decades, lots of effort was put in the development of efficient (high order, high resolution and parallel) and accurate (stable and convergent) numerical methods. Comparing to the great success in the development of reliable numerical methods, the progress in their analysis is much slower.

In this paper, we study the convergence rate of a finite volume (FV) method proposed by Feireisl et al. [6] for the compressible Navier–Stokes–Fourier system. Our main tool is the relative energy functional originally proposed by Dafermos [2] in order to measure the distance of a weak and a classical solution. The idea has been adapted to study the error between a numerical solution and a classical solution, see Gallouët et al. [12] and Feireisl et al. [4]. These results are based on a discrete version of the relative energy functional, where the target smooth solution is projected into the discrete space of numerical solutions. Recently, Feireisl et al. [8] showed that using the continuous version of the relative energy functional yields a better convergence rate compared to its discrete version. This new strategy directly employs the consistency error of the numerical scheme and avoids lengthy calculations in the discrete space. This approach has been already successfully applied for the error estimates of the compressible Euler equations, see Feireisl et al. [7] for the barotropic case and see Lukáčová et al. [19] for the full Euler system. In the present paper we use the *continuous form of the relative energy* and study *the convergence rate of a FV method* for the Navier–Stokes–Fourier system. To this end, we need a new refined version of the consistency errors as well as stability estimates. As far as we know, this is the first result in the literature, where the error estimates of a fully discrete numerical scheme for the compressible viscous heat-conducting fluids are derived. In this paper we concentrate on theoretical error analysis, numerical simulations illustrating reliability of the studied FV method were shown in [18].

The Navier–Stokes–Fourier system describes the motion of compressible, viscous, and heat conducting fluids

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}, \tag{1.1b}$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \boldsymbol{u}) - \mathrm{div}_x(\kappa \nabla_x \vartheta) = \mathbb{S} : \nabla_x \boldsymbol{u} - p \mathrm{div}_x \boldsymbol{u} \tag{1.1c}$$

in the time–space cylinder $(0,T) \times \Omega$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here, $\varrho, \boldsymbol{u}, \vartheta, p, e$ are the fluid density, velocity, absolute temperature, pressure and internal energy, respectively. Further, $\kappa > 0$ is a heat conductivity constant and $\mathbb{S}$ is the viscous stress tensor

$$\mathbb{S} = \mathbb{S}(\nabla_x \boldsymbol{u}) = 2\mu \mathbb{D}(\boldsymbol{u}) + \lambda \mathrm{div}_x \boldsymbol{u} \mathbb{I} \quad \text{with} \quad \mathbb{D}(\boldsymbol{u}) = \frac{\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u}}{2},$$

where $\mu > 0$ and $\lambda \geq 0$ are constant viscosity coefficients. We consider the standard pressure law of a perfect gas

$$p = (\gamma - 1)\varrho e, \ e = c_v \vartheta,$$

where $\gamma > 1$ is the adiabatic coefficient and $c_v = \frac{1}{\gamma - 1}$ is the specific heat per constant volume. For simplicity, we consider periodic boundary conditions and identify the fluid domain with a flat torus, i.e., $\Omega = \mathbb{T}^d = \left([0,1]_{[0,1]}\right)^d$. To close the system we impose the initial conditions

$$(\varrho(0), \boldsymbol{u}(0), \vartheta(0)) = (\varrho_0, \boldsymbol{u}_0, \vartheta_0) \text{ with } \varrho_0 > 0 \text{ and } \vartheta_0 > 0. \tag{1.2}$$

In view of the second law of thermodynamics, any solution to the system (1.1)–(1.2) satisfies the entropy balance

$$\varrho(\partial_t s + \boldsymbol{u}\cdot\nabla_x s) - \mathrm{div}_x\left(\frac{\kappa\nabla_x\vartheta}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x\boldsymbol{u}) : \nabla_x\boldsymbol{u} + \kappa\frac{|\nabla_x\vartheta|^2}{\vartheta}\right) \geq 0, \tag{1.3}$$

where $s$ is the physical entropy

$$s(\varrho,\vartheta) = \log\left(\frac{\vartheta^{c_v}}{\varrho}\right) \qquad \text{for } \varrho > 0,\ \vartheta > 0.$$

The rest of the paper is structured as follows. In Section 2 we show the existence of a strong solution. Next, we introduce a FV method in Section. 3. Then, we show the stability and consistency of the FV approximation in Section 4. Finally, we present the convergence rate of the FV method in Section 5.

## 2 Strong solution

In this section, we focus on the existence results to the Navier–Stokes–Fourier system (1.1) in the class of strong solutions. In particular, we emphasize that the classical solution inherits the $W^{k,2}$-regularity from the initial data for any $k \geq 4$, as a consequence of the fact that (1.1) can be seen as a symmetric hyperbolic-parabolic system, cf. Theorem A.1. Moreover, due to the blow-up criterion established by Huang, Li [14] and by Feireisl, Wen, Zhu [11], the solution exists globally in time, as long as it is bounded.

**Proposition 2.1.** *Let $k \geq 6$ and let the initial data (1.2) be such that*

$$\varrho_0 > 0, \quad (\varrho_0, \boldsymbol{u}_0, \vartheta_0) \in W^{k,2}(\mathbb{T}^d;\mathbb{R}^{1+d+1}).$$

*Let $(\varrho,\boldsymbol{u},\vartheta)$ be a weak solution to problem (1.1), (1.2) such that*

$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho}, \quad 0 < \underline{\vartheta} \leq \vartheta \leq \overline{\vartheta} \quad \text{a.e. in } (0,T)\times\mathbb{T}^d. \tag{2.1}$$

*Then $(\varrho,\boldsymbol{u},\vartheta)$ is a strong solution of (1.1), (1.2) in $[0,T]\times\mathbb{T}^d$ satisfying*

$$(\varrho,\boldsymbol{u},\vartheta) \in C([0,T];W^{k,2}(\mathbb{T}^d;\mathbb{R}^{1+d+1})).$$

*Moreover, the following estimate*

$$\begin{aligned}&\|(\varrho,\boldsymbol{u},\vartheta)\|_{C([0,T];W^{k,2}(\mathbb{T}^d))} + \|(\varrho,\boldsymbol{u},\vartheta)\|_{C^1([0,T]\times\mathbb{T}^d)}\\ &+\|(\partial_t\varrho,\partial_t\boldsymbol{u},\partial_t\vartheta)\|_{C([0,T]\times\mathbb{T}^d)} + \|(\partial_t^2\varrho,\partial_t^2\boldsymbol{u},\partial_t^2\vartheta)\|_{C([0,T]\times\mathbb{T}^d)} \leq C\|(\varrho_0,\boldsymbol{u}_0,\vartheta_0)\|_{W^{k,2}}\end{aligned} \tag{2.2}$$

*holds, where the positive constant $C$ depends solely on $T,\underline{\varrho},\overline{\varrho},\underline{\vartheta},\overline{\vartheta}$ and $\|(\varrho_0,\boldsymbol{u}_0,\vartheta_0)\|_{W^{k,2}}$.*

*Proof.* Applying the local-existence result developed in Theorem A.1, there exists $T_1$, $0 < T_1 \leq T$, such that a unique strong solution $(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})$ to problem (1.1), (1.2) exists on $[0,T_1]$ and

$$(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta}) \in C([0,T_1];W^{k,2}(\mathbb{T}^d;\mathbb{R}^{1+d+1})).$$

Moreover, from the Sobolev embedding $W^{k-4,2}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ whenever $k \geq 6$, from (A.2) we can deduce that there exists a positive constant $C = C(T_1)$ such that

$$\begin{aligned} &\|(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|_{C([0,T_1];W^{k,2}(\mathbb{T}^d))} + \|(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|_{C^1([0,T_1]\times\mathbb{T}^d)} \\ &+\|(\partial_t\widetilde{\varrho}, \partial_t\widetilde{\boldsymbol{u}}, \partial_t\widetilde{\vartheta})\|_{C([0,T_1]\times\mathbb{T}^d)} + \|(\partial_t^2\widetilde{\varrho}, \partial_t^2\widetilde{\boldsymbol{u}}, \partial_t^2\widetilde{\vartheta})\|_{C([0,T_1]\times\mathbb{T}^d)} \leq C\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}. \end{aligned} \tag{2.3}$$

Now, from the weak-strong uniqueness principle proven in [9, Theorem 2.1], the weak solution $(\varrho, \boldsymbol{u}, \vartheta)$ must coincide with the strong solution $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$ in $[0, T_1] \times \mathbb{T}^d$. According to the blow-up criterion developed by Feireisl, Wen and Zhu [11, Theorem 1.4], if $T_1 < T$ were the maximal time of existence, then

$$\limsup_{t\to T_1^-} (\|\varrho(t)\|_{L^\infty} + \|\vartheta(t)\|_{L^\infty}) = \infty, \tag{2.4}$$

which would lead to a contradiction due to hypothesis (2.1). Consequently $T_1 = T$ which concludes the proof. □

**Remark 2.2.** *We point out that the compatibility condition (1.10) required for the initial data in [11] is not necessary in this context as we are not allowing the presence of vacuum zones.*

**Remark 2.3.** *Proposition 2.1 still holds if we replace the boundedness of temperature with the one of velocity; specifically, instead of having $\vartheta \leq \overline{\vartheta}$ in (2.1), we could require $|\boldsymbol{u}| \leq \overline{\boldsymbol{u}}$. Then, instead of (2.4), we could use the Serrin-type blow-up criterion developed by Huang and Li [14, Theorem 1.2]: if $T_1$ is the maximal time of existence, then*

$$\limsup_{t\to T_1^-} (\|\varrho(t)\|_{L^\infty} + \|\boldsymbol{u}(t)\|_{L^\infty}) = \infty.$$

# 3 The finite volume method

In this section we introduce the finite volume method originally proposed in [6].

## 3.1 Notations

**Mesh.** Let $\mathcal{T}_h$ be a uniform structured mesh with the mesh size $h$ formed by squares in 2D, and cubes in 3D such that $\mathbb{T}^d = \bigcup_{K\in\mathcal{T}_h} K$.

We denote by $\mathcal{E}$ the set of all faces of $\mathcal{T}_h$, and by $\mathcal{E}_i$ the set of all faces that are orthogonal to the $i^{\text{th}}$ basis vector $\boldsymbol{e}_i$ of the canonical system. The set of all faces of an element $K$ is denoted by $\mathcal{E}(K)$. Moreover, we write $\sigma = K|L$ if $\sigma \in \mathcal{E}$ is the common face of elements $K$ and $L$. Further, we denote by $|K|$ and $|\sigma|$ the Lebesgue measure of an element $K \in \mathcal{T}_h$ and a face $\sigma \in \mathcal{E}$, respectively. Moreover, $x_K$ and $x_\sigma$ stand for the barycenter of $K$ and $\sigma$, respectively.

In what follows we shall use an abbreviated notation for the integrals,

$$\int_{\mathbb{T}^d} \cdot \,\mathrm{d}x := \sum_{K\in\mathcal{T}_h} \int_K \cdot \,\mathrm{d}x \quad \text{and} \quad \int_{\mathcal{E}} \cdot \,\mathrm{d}S_x := \sum_{\sigma\in\mathcal{E}} \int_\sigma \cdot \,\mathrm{d}S_x.$$

**Dual grid.** For any $\sigma = K|L \in \mathcal{E}_i$, we define a dual cell $D_\sigma := D_{\sigma,K} \cup D_{\sigma,L}$, where the polygon $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is a half of $K$ (resp. $L$), i.e.,

$$D_{\sigma,K} = \{x \in K \mid x_i \in \mathrm{co}\{(x_K)_i, (x_\sigma)_i\}\},$$
$$\mathrm{co}\{A, B\} \equiv [\min\{A, B\}, \max\{A, B\}],$$

see Figure 1 for a two-dimensional example of dual cells. Further, we denote the $i^{\text{th}}$ dual grid $\mathcal{D}_i$ as

$$\mathcal{D}_i = \bigcup_{\sigma \in \mathcal{E}_i} D_\sigma.$$

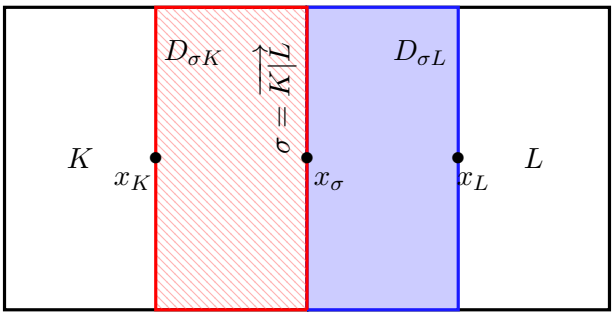


Figure 1: Dual cell $D_\sigma = D_{\sigma K} \cup D_{\sigma L}$ for 2D structured mesh.

**Function space.** The symbol $Q_h$ stands for the set of piecewise constant functions on the grid $\mathcal{T}_h$. Note that hereinafter $\boldsymbol{v}_h \in Q_h$ means that every component of a vector–valued function $\boldsymbol{v}_h$ belongs to the set $Q_h$. We define the following projection operators

$$\Pi_Q \phi(x) = \sum_{K \in \mathcal{T}_h} \frac{1_K(x)}{|K|} \int_K \phi \,\mathrm{d}x, \quad \Pi_W^{(i)} \phi(x) = \sum_{\sigma \in \mathcal{E}_i} \frac{1_{D_\sigma}(x)}{|\sigma|} \int_\sigma \phi \,\mathrm{d}S_x,$$

where $1_D(x) = \begin{cases} 1 & x \in D, \\ 0 & \text{otherwise,} \end{cases}$ for $D = K, D_\sigma$ being an element of $\mathcal{T}_h$ or $\mathcal{D}_i$, respectively. Note that for any $\phi \in W^{1,\infty}(\mathbb{T}^d)$

$$\left\| \phi - \Pi_W^{(i)} \phi \right\|_{L^\infty(\mathbb{T}^d)} \leq h \, \|\phi\|_{W^{1,\infty}(\mathbb{T}^d)}. \tag{3.1}$$

The average and jump operators are denoted as

$$\{\!\{v\}\!\}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \quad [\![v]\!] = v^{\text{out}}(x) - v^{\text{in}}(x),$$

where

$$v^{\text{out}}(x) = \lim_{\delta \to 0+} v(x + \delta \boldsymbol{n}), \quad v^{\text{in}}(x) = \lim_{\delta \to 0+} v(x - \delta \boldsymbol{n}),$$

whenever $x \in \sigma \in \mathcal{E}$ and $\boldsymbol{n}$ is the outer normal vector to $\sigma$.

**Discrete operators.** The discrete gradient $\nabla_h$ and divergence $\mathrm{div}_h$ operators acting on the mesh $\mathcal{T}_h$ are defined as follows,

$$(\nabla_h r)_K = \frac{|\sigma|}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \{\!\{r\}\!\}\, \boldsymbol{n}, \qquad (\mathrm{div}_h \boldsymbol{v})_K = \frac{|\sigma|}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \{\!\{\boldsymbol{v}\}\!\} \cdot \boldsymbol{n},$$
$$\nabla_h r(x) = \sum_{K\in\mathcal{T}_h} (\nabla_h r)_K \, 1_K(x), \quad \mathrm{div}_h \boldsymbol{v}(x) = \sum_{K\in\mathcal{T}_h} (\mathrm{div}_h \boldsymbol{v})_K \, 1_K(x).$$

We point out that $\nabla_h$ and $\mathrm{div}_h$ are discrete variants of $\nabla_x$ and $\mathrm{div}_x$ on the cell $K$, respectively. Moreover, we define

$$\nabla_h \boldsymbol{v} = (\nabla_h v_1, \dots, \nabla_h v_d)^T, \quad \mathbb{D}_h(\boldsymbol{v}) = \frac{\nabla_h \boldsymbol{v} + \nabla_h^T \boldsymbol{v}}{2} \quad \text{yielding} \quad \mathrm{div}_h \boldsymbol{v} = \mathrm{tr}(\nabla_h \boldsymbol{v}) = \mathrm{tr}(\mathbb{D}_h(\boldsymbol{v}))$$

and the following difference operator for any $r_h \in Q_h$

$$\begin{aligned} \nabla_{\mathcal{E}} r_h(x) &= \sum_{\sigma\in\mathcal{E}} (\nabla_{\mathcal{E}} r_h)_\sigma \, 1_{D_\sigma}(x), \quad (\nabla_{\mathcal{E}} r_h)_\sigma = \frac{[\![ r_h ]\!]}{h} \boldsymbol{n}. \\ \Delta_h r_h(x) &= \sum_{K\in\mathcal{T}_h} (\Delta_h r_h)_K \, 1_K(x), \quad (\Delta_h r_h)_K = (\mathrm{div}_h \nabla_{\mathcal{E}} r_h)_K = \frac{|\sigma|}{|K|} \sum_{\sigma\in\mathcal{E}(K)} \frac{[\![ r_h ]\!]}{h}. \end{aligned} \tag{3.2}$$

It is crucial that the following duality relationships between discrete operators hold

$$\int_{\mathbb{T}^d} r_h \mathrm{div}_h \boldsymbol{v}_h \, \mathrm{d}x = -\int_{\mathbb{T}^d} \nabla_h r_h \cdot \boldsymbol{v}_h \, \mathrm{d}x, \tag{3.3a}$$
$$\int_{\mathbb{T}^d} \Delta_h r_h \cdot f_h \, \mathrm{d}x = -\int_{\mathbb{T}^d} \nabla_{\mathcal{E}} r_h \cdot \nabla_{\mathcal{E}} f_h \, \mathrm{d}x = \int_{\mathbb{T}^d} r_h \cdot \Delta_h f_h \, \mathrm{d}x, \tag{3.3b}$$
$$\int_{\mathbb{T}^d} r_h \mathrm{div}_x \boldsymbol{\phi} \, \mathrm{d}x = -\int_{\mathbb{T}^d} \nabla_{\mathcal{E}} r_h \cdot \Pi_W \boldsymbol{\phi} \, \mathrm{d}x \quad \text{for } \boldsymbol{\phi} \in W^{1,1}(\mathbb{T}^d; \mathbb{R}^d), \tag{3.3c}$$

where $\boldsymbol{v}_h, r_h, f_h \in Q_h$ and $\Pi_W \boldsymbol{\phi} = (\Pi_W^{(1)} \phi_1, \dots, \Pi_W^{(d)} \phi_d)$. We refer to [7, Lemma 5 & Lemma 7] for proofs.

**Time discretization.** Given a time step $\Delta t > 0$ we divide the time interval $[0, T]$ into $N_T = T/\Delta t$ uniform parts and denote $t^k = k\Delta t$. Then $v_h^k \approx v_h(t^k)$ is the approximation of a function $v_h$ at time $t^k$, $k = 0, 1, \dots, N_T$. By $v_h \in L_{\Delta t}(0, T; Q_h)$ we denote a piecewise constant in time function

$$v_h(t, \cdot) = v_h^0 \ \text{ for } \ t < \Delta t, \quad v_h(t, \cdot) = v_h^k \ \text{ for } \ t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, \cdots, N_T.$$

Further, we define the discrete time derivative by

$$D_t v_h(t) = \frac{v_h(t) - v_h^{\triangleleft}}{\Delta t} \quad \text{with} \quad v_h^{\triangleleft} = v_h(t - \Delta t).$$

### 3.2 Numerical scheme

We are now ready to propose a FV method for the compressible Navier–Stokes-Fourier system (1.1) in a weak form, see also [6].

**Definition 3.1** (FV method). *Given the initial data (1.2) we set* $(\varrho_h^0, \boldsymbol{u}_h^0, \vartheta_h^0) = (\Pi_Q \varrho_0, \Pi_Q \boldsymbol{u}_0, \Pi_Q \vartheta_0)$. *We say that the triple* $(\varrho_h, \boldsymbol{u}_h, \vartheta_h) \in L_{\Delta t}(0,T; Q_h \times Q_h \times Q_h)$ *is a finite volume approximation of the Navier–Stokes–Fourier system (1.1), (1.2), if the following system of algebraic equations holds*

$$\int_{\mathbb{T}^d} D_t \varrho_h \phi_h \,\mathrm{d}x - \int_{\mathcal{E}} F_h^{\varepsilon}(\varrho_h, \boldsymbol{u}_h) \,[\![\phi_h]\!] \,\mathrm{d}S_x = 0 \quad \text{for all } \phi_h \in Q_h, \tag{3.4a}$$

$$\int_{\mathbb{T}^d} D_t(\varrho_h \boldsymbol{u}_h) \cdot \boldsymbol{\phi}_h \,\mathrm{d}x - \int_{\mathcal{E}} F_h^{\varepsilon}(\varrho_h \boldsymbol{u}_h, \boldsymbol{u}_h) \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S_x + \int_{\mathbb{T}^d} (\mathbb{S}_h - p_h \mathbb{I}) : \nabla_h \boldsymbol{\phi}_h \,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\phi}_h \in Q_h, \tag{3.4b}$$

$$c_v \int_{\mathbb{T}^d} D_t(\varrho_h \vartheta_h)\phi_h \,\mathrm{d}x - c_v \int_{\mathcal{E}} F_h^{\varepsilon}(\varrho_h \vartheta_h, \boldsymbol{u}_h) \,[\![\phi_h]\!] \,\mathrm{d}S_x + \int_{\mathcal{E}} \frac{\kappa}{h} \,[\![\vartheta_h]\!] \,[\![\phi_h]\!] \,\mathrm{d}S_x$$
$$= \int_{\mathbb{T}^d} (\mathbb{S}_h - p_h \mathbb{I}) : \nabla_h \boldsymbol{u}_h \phi_h \,\mathrm{d}x, \quad \text{for all } \phi_h \in Q_h, \tag{3.4c}$$

*where* $\mathbb{S}_h = 2\mu \mathbb{D}_h(\boldsymbol{u}_h) + \lambda \mathrm{div}_h \boldsymbol{u}_h \mathbb{I}$, *the discrete pressure* $p_h$ *is given by*

$$p_h(t) = \varrho_h(t)\vartheta_h(t) \quad \text{for } \vartheta_h(t) > 0 \quad \text{and} \quad p_h(t) = 0 \quad \text{for } \vartheta_h(t) \le 0,$$

*and* $F_h^{\varepsilon}$, $\varepsilon \in (-1,1)$, *is the so-called diffusive upwind flux*

$$F_h^{\varepsilon}(r_h, \boldsymbol{u}_h) = \mathrm{Up}[r_h, \boldsymbol{u}_h] - h^{\varepsilon} \,[\![r_h]\!]\,, \ \mathrm{Up}[r_h, \boldsymbol{u}_h] = \{\!\{r_h\}\!\} \ \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n} - \frac{1}{2} |\,\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}|\,[\![r_h]\!]\,.$$

**Remark 3.2.** *Note that the existence of a solution* $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ *to the nonlinear algebraic system (3.4) can be done using the fixed point theorem [7, Theorem 15] in the same way as in [7, Lemma 11.3]. In our recent paper [18] we have presented several numerical experiments obtained by the FV method (3.4) that illustrate its reliability and robustness.*

## 4 Stability and consistency of the FV method

In this section we discuss the stability and consistency of the FV method (3.4). To begin, we recall from [6] the following structure-preserving properties of the method.

**Lemma 4.1.** *Let* $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ *be a numerical solution of the FV method (3.4) with positive initial data* $\varrho_h^0 > 0$ *and* $\vartheta_h^0 > 0$. *Then we have*

- ***Positivity of the density.***
$$\varrho_h(t) > 0 \quad \text{for all} \quad t \in (0,T).$$
- ***Positivity of the temperature.***
$$\vartheta_h(t) > 0 \quad \text{for all} \quad t \in (0,T).$$

- ***Mass conservation.***
$$\int_{\mathbb{T}^d} \varrho_h(t)\,\mathrm{d}x = \int_{\mathbb{T}^d} \varrho_h(0)\,\mathrm{d}x = \int_{\mathbb{T}^d} \varrho_0\,\mathrm{d}x > 0.$$

- ***Energy balance.***
$$D_t \int_{\mathbb{T}^d} \left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2 + c_v\varrho_h\vartheta_h\right)\mathrm{d}x + h^\varepsilon \int_{\mathcal{E}} \{\!\{\varrho_h\}\!\}\,|\,[\![\boldsymbol{u}_h]\!]\,|^2\,\mathrm{d}S_x$$
$$+ \frac{\Delta t}{2}\int_{\mathbb{T}^d} \varrho_h^{\triangleleft}|D_t\boldsymbol{u}_h|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{E}} \varrho_h^{\mathrm{up}}|\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}||\,[\![\boldsymbol{u}_h]\!]\,|^2\,\mathrm{d}S_x = 0. \tag{4.1}$$

- ***Entropy balance.***
$$\int_{\mathbb{T}^d} D_t\left(\varrho_h s_h\right)\phi_h\,\mathrm{d}x - \int_{\mathcal{E}} \mathrm{Up}(\varrho_h s_h, \boldsymbol{u}_h)\,[\![\phi_h]\!]\,\mathrm{d}S_x + \int_{\mathcal{E}} \frac{\kappa}{h}\,[\![\vartheta_h]\!]\left[\!\!\left[\frac{\phi_h}{\vartheta_h}\right]\!\!\right]\mathrm{d}S_x$$
$$- \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_h\boldsymbol{u}_h\frac{\phi_h}{\vartheta_h}\,\mathrm{d}x = D_s(\phi_h) + R_s(\phi_h), \tag{4.2}$$

*where* $\phi_h \in Q_h$, $D_s(\phi_h) = D_1(\phi_h) + D_2(\phi_h) + D_3(\phi_h)$ *and*

$$\begin{aligned}
D_1(\phi_h) &:= \Delta t\int_{\mathbb{T}^d}\left(\frac{|D_t\varrho_h|^2}{2\xi_{\varrho,h}} + \frac{c_v\varrho_h^{\triangleleft}|D_t\vartheta_h|^2}{2|\xi_{\vartheta,h}|^2}\right)\phi_h\,\mathrm{d}x,\\
D_2(\phi_h) &:= \frac{1}{2}\int_{\mathcal{E}} \phi_h^{\mathrm{down}}|\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,[\![(\varrho_h, p_h)]\!]\cdot\nabla^2_{(\varrho,p)}(-\varrho s)|_{\boldsymbol{v}_1^*}\cdot[\![(\varrho_h,p_h)]\!]\,\mathrm{d}S_x,\\
D_3(\phi_h) &:= h^\varepsilon\int_{\mathcal{E}} \{\!\{\phi_h\}\!\}\,[\![(\varrho_h,p_h)]\!]\cdot\nabla^2_{(\varrho,p)}(-\varrho s)|_{\boldsymbol{v}_2^*}\cdot[\![(\varrho_h,p_h)]\!]\,\mathrm{d}S_x,\\
R_s(\phi_h) &:= h^\varepsilon\int_{\mathcal{E}} [\![\phi_h]\!]\cdot\left(\{\!\{\nabla_\varrho(-\varrho_h s_h)\}\!\}\,[\![\varrho_h]\!] + \{\!\{\nabla_p(-\varrho_h s_h)\}\!\}\,[\![p_h]\!]\right)\mathrm{d}S_x,\\
&\xi_{\varrho,h}\in\mathrm{co}\{\varrho_h^{\triangleleft},\varrho_h\},\ \xi_{\vartheta,h}\in\mathrm{co}\{\vartheta_h^{\triangleleft},\vartheta_h\},\ \boldsymbol{v}_1^*,\boldsymbol{v}_2^*\in\mathrm{co}\{(\varrho_h^{\mathrm{in}},p_h^{\mathrm{in}}),(\varrho_h^{\mathrm{out}},p_h^{\mathrm{out}})\}.
\end{aligned} \tag{4.3}$$

*Moreover, it holds that* $D_s(\phi_h)\geq 0$ *for* $\phi_h\geq 0$.

**Remark 4.2.** *We point out that the entropy Hessian matrix* $\nabla^2_{(\varrho,p)}(-\varrho s)$ *is symmetric positive definite for* $(\varrho,\vartheta)\in(0,\infty)^2$, *i.e.*
$$\nabla^2_{(\varrho,p)}(-\varrho s) = \begin{pmatrix} \frac{1+c_v}{\varrho} & -\frac{c_v}{\varrho\vartheta} \\ -\frac{c_v}{\varrho\vartheta} & \frac{c_v}{\varrho\vartheta^2}\end{pmatrix} > 0. \tag{4.4}$$
*Assuming that* $\varrho_h$ *and* $\vartheta_h$ *are uniformly bounded from above and below by positive constants, see* (4.5) *below, we obtain that the entropy Hessian matrix* $\nabla^2_{(\varrho,p)}(-\varrho s)(\varrho_h,\vartheta_h)$ *is uniformly bounded from above and below by some positive constants, see Appendix B for details.*

With the boundedness of the entropy Hessian matrix $\nabla^2_{(\varrho,p)}(-\varrho s)(\varrho_h,\vartheta_h)$, we obtain the following uniform bounds from the entropy balance (4.2) with test function $\phi_h = 1$.

**Corollary 4.3** (Uniform bounds). *Let $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ be a numerical solution of the FV method (3.4) with positive initial data $\varrho_h^0 > 0$ and $\vartheta_h^0 > 0$. Under the assumption*

$$0 < \underline{\vartheta} \le \vartheta_h \le \overline{\vartheta}, \ 0 < \underline{\varrho} \le \varrho_h \le \overline{\varrho} \ \text{ uniformly for } \ h \to 0, \tag{4.5}$$

*there exist some constants $\underline{s} \le \overline{s}$, $0 < \underline{p} \le \overline{p}$ such that the entropy and pressure are bounded,*

$$\underline{s} \le s_h \le \overline{s} \ \text{ and } \underline{p} \le p_h \le \overline{p} \ \text{ uniformly for } \ h \to 0. \tag{4.6}$$

*Moreover, we have the following estimates*

$$\|\boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \|\nabla_{\mathcal{E}}\vartheta_h\|_{L^2((0,T)\times\mathbb{T}^d)} + \|\mathbb{S}_h\|_{L^2((0,T)\times\mathbb{T}^d)} \lesssim 1, \tag{4.7a}$$

$$\int_0^\tau \int_{\mathcal{E}} (h^\varepsilon + |\{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}|) \left( [\![\varrho_h]\!]^2 + [\![p_h]\!]^2 + |[\![\boldsymbol{u}_h]\!]|^2 \right) \mathrm{d}S_x \,\mathrm{d}t \lesssim 1. \tag{4.7b}$$

**Remark 4.4.** *It is easy to verify that*

$$h^{\varepsilon+1} \|\nabla_{\mathcal{E}}\boldsymbol{u}_h\|_{L^2L^2}^2 = h^\varepsilon \int_{\mathcal{E}} |[\![\boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S_x \lesssim 1 \tag{4.8}$$

*and* $\int_0^\tau \int_{\mathbb{T}^d} \nabla_h \boldsymbol{u}_h : \nabla_h^T \boldsymbol{u}_h \,\mathrm{d}x\mathrm{d}t = \|\mathrm{div}_h \boldsymbol{u}_h\|_{L^2L^2}^2 \ge 0$, *which further implies*

$$\|\nabla_h \boldsymbol{u}_h\|_{L^2L^2} \approx \|\mathbb{D}_h(\boldsymbol{u}_h)\|_{L^2L^2} \approx \|\mathbb{S}_h\|_{L^2L^2} \lesssim 1. \tag{4.9}$$

*Note that the notation "$a \approx b$" means $a \lesssim b$ and $b \lesssim a$.*

Now we are ready to analyse the consistency of the FV method (3.4).

**Lemma 4.5** (Consistency formulation). *Let $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ be a numerical solution obtained by the FV scheme (3.4) with $(\Delta t, h) \in (0,1)^2$, $-1 < \varepsilon < 1$. Let the assumption (4.5) hold. Then,*

$$\left[\int_{\mathbb{T}^d} \varrho_h \phi \,\mathrm{d}x\right]_{t=0}^\tau = \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \phi) \,\mathrm{d}x\mathrm{d}t + e_\varrho(\phi, \Delta t, h, \tau) \tag{4.10a}$$

*for any $\phi \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d))$, $\partial_t\phi, \partial_t^2\phi \in L^\infty((0,T)\times\mathbb{T}^d)$ ;*

$$\begin{aligned}\left[\int_{\mathbb{T}^d} \varrho_h \boldsymbol{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}x\right]_{t=0}^\tau &= \int_0^\tau \int_{\mathbb{T}^d} (\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\phi}) \,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\mathbb{T}^d} (p_h \mathbb{I} - \mathbb{S}_h) : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x\mathrm{d}t + e_{\boldsymbol{m}}(\boldsymbol{\phi}, \Delta t, h, \tau)\end{aligned} \tag{4.10b}$$

*for any $\boldsymbol{\phi} \in L^\infty(0,T; W^{2,\infty}(\mathbb{T}^d; \mathbb{R}^d))$, $\partial_t\boldsymbol{\phi}, \partial_t^2\boldsymbol{\phi} \in L^\infty((0,T)\times\mathbb{T}^d; \mathbb{R}^d)$;*

$$\begin{aligned}\left[\int_{\mathbb{T}^d} \varrho_h s_h \phi \,\mathrm{d}x\right]_{t=0}^\tau &\ge \int_0^\tau \int_{\mathbb{T}^d} \varrho_h s_h (\partial_t \phi + \boldsymbol{u}_h \cdot \nabla_x \phi) \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_h \boldsymbol{u}_h \frac{\phi}{\vartheta_h} \,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa\phi}{\vartheta_h^{\mathrm{out}}\vartheta_h} |\nabla_{\mathcal{E}}\vartheta_h|^2 \,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa}{\vartheta_h} \nabla_{\mathcal{E}}\vartheta_h \cdot \nabla_x \phi \,\mathrm{d}x\mathrm{d}t + e_s(\phi, \Delta t, h, \tau)\end{aligned} \tag{4.10c}$$

*for any* $\phi \in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))$, $\partial_t\phi, \partial_t^2\phi \in L^\infty((0,T)\times\mathbb{T}^d)$, $\phi \geq 0$. *Here*

$$|e_\varrho(\phi,\Delta t,h,\tau)| \leq C_\varrho\left(\Delta t + h + h^{1-\varepsilon} + h^{1+\varepsilon}\right), \tag{4.11a}$$

$$|e_{\boldsymbol{m}}(\boldsymbol{\phi},\Delta t,h,\tau)| \leq C_{\boldsymbol{m}}\left(\Delta t + h + h^{(1-\varepsilon)/2} + h^{1+\varepsilon}\right), \tag{4.11b}$$

$$|e_s(\phi,\Delta t,h,\tau)| \leq C_s\left(\Delta t + h + h^{1-\varepsilon} + h^{(1+\varepsilon)/2}\right) \tag{4.11c}$$

*and the constants* $C_\varrho$, $C_{\boldsymbol{m}}$, $C_s$ *are independent of the discretization parameters* $h$, $\Delta t$.

The proof of the above lemma has been done in [6] with more regular test functions but without a clear order of the consistency error. Here we present an optimal consistency errors with minimum regularity assumptions on the test functions. We note that in the next section the exact strong solution will be used as a test function for the error estimates. For completeness we provide the details of the proof in Appendix C.

# 5 Error estimates

In this section we show the main result, that is the error estimate of the FV approximation (3.4). To begin, we recall the relative energy functional which measures the distance between the numerical solution $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ and the strong solution $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$

$$R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h) \mid (\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right) = \int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|^2 + \mathbb{E}_{\mathcal{H}}\big((\varrho_h,\vartheta_h)|(\widetilde{\varrho},\widetilde{\vartheta})\big)\right)\mathrm{d}x$$

with

$$\mathbb{E}_{\mathcal{H}}\big((\varrho,\vartheta)|(\widetilde{\varrho},\widetilde{\vartheta})\big) = \mathcal{H}_{\widetilde{\vartheta}}(\varrho,\vartheta) - \frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho}(\varrho-\widetilde{\varrho}) - \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta}) \quad \text{and} \quad \mathcal{H}_{\widetilde{\vartheta}}(\varrho,\vartheta) = \varrho(c_v\vartheta - \widetilde{\vartheta}s(\varrho,\vartheta)).$$

Then it is straightforward to check

$$\begin{aligned}
&\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\varrho,\vartheta)}{\partial\varrho} = c_v\vartheta - \widetilde{\vartheta}\big(s(\varrho,\vartheta)-1\big), \quad \widetilde{\varrho}\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho} - \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta}) = \widetilde{p},\\
&\partial_y\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho} + \widetilde{s}\partial_y\widetilde{\vartheta} = \frac{1}{\widetilde{\varrho}}\partial_y\widetilde{p}, \quad y = t,x,
\end{aligned} \tag{5.1}$$

where we have denoted $\widetilde{p} = p(\widetilde{\varrho},\widetilde{\vartheta})$, $\widetilde{s} = s(\widetilde{\varrho},\widetilde{\vartheta})$.

**Remark 5.1.** *Let* $\eta = \varrho s$. *Taking* $(\varrho,\eta)$ *as independent variables, we reformulate* $\mathbb{E}_{\mathcal{H}}\big((\varrho,\vartheta)|(\widetilde{\varrho},\widetilde{\vartheta})\big)$ *as*

$$\mathbb{E}_{\mathcal{H}}\big((\varrho,\vartheta)|(\widetilde{\varrho},\widetilde{\vartheta})\big) = \varrho e - \frac{\partial(\varrho e)}{\partial\varrho}\Big|_{(\widetilde{\varrho},\widetilde{\eta})}(\varrho-\widetilde{\varrho}) - \frac{\partial(\varrho e)}{\partial\eta}\Big|_{(\widetilde{\varrho},\widetilde{\eta})}(\eta-\widetilde{\eta}) - \widetilde{\varrho}\widetilde{e}.$$

*Due to the convexity of* $\varrho e$ *with respect to* $(\varrho,\eta)$, *we know that* $\mathbb{E}_{\mathcal{H}}\big((\varrho,\vartheta)|(\widetilde{\varrho},\widetilde{\vartheta})\big) \geq 0$. *Moreover, it holds under the assumption* (4.5) *that*

$$\begin{aligned}
&R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h) \mid (\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right) \approx \|\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d)} + \|\eta_h-\widetilde{\eta}\|^2_{L^2(\mathbb{T}^d)} + \|\varrho_h-\widetilde{\varrho}\|^2_{L^2(\mathbb{T}^d)}\\
&\approx \|\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{T}^d)} + \left\|\vartheta_h-\widetilde{\vartheta}\right\|^2_{L^2(\mathbb{T}^d)} + \|\varrho_h-\widetilde{\varrho}\|^2_{L^2(\mathbb{T}^d)} + \|s_h-\widetilde{s}\|^2_{L^2(\mathbb{T}^d)}.
\end{aligned} \tag{5.2}$$

*For more details, we refer to [19] and [10, Chapter 3.5].*

**Theorem 5.2** (Main result: Error estimates)**.** *Let $\gamma > 1$ and the initial data $(\varrho_0, \boldsymbol{u}_0, \vartheta_0)$ satisfy*

$$\varrho_0 > 0, \quad (\varrho_0, \boldsymbol{u}_0, \vartheta_0) \in W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1}), \quad k \geq 6.$$

*Further, let $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ be a numerical solution obtained by the FV scheme (3.4) emanating from the same initial data with $(\Delta t, h) \in (0,1)^2$ and $-1 < \varepsilon < 1$. In addition, let the numerical density $\varrho_h$ and temperature $\vartheta_h$ be uniformly bounded, i.e. there exist $\underline{\vartheta}, \overline{\vartheta}, \underline{\varrho}, \overline{\varrho}$ such that*

$$0 < \underline{\vartheta} \leq \vartheta_h \leq \overline{\vartheta}, \ 0 < \underline{\varrho} \leq \varrho_h \leq \overline{\varrho} \ \text{ uniformly for } \ h \to 0. \tag{5.3}$$

*Then there exists a positive number*

$$c = c\left(T, \underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}, \|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}\right)$$

*such that*

$$\begin{aligned} &\sup_{0\leq t\leq \tau} R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right) + \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa\widetilde{\vartheta}}{\vartheta_h \vartheta_h^{\text{out}}} |\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta}|^2 \,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\mathbb{T}^d} \left( \frac{\widetilde{\vartheta}}{\vartheta_h} \left( 2\mu \left| \mathbb{D}_h(\boldsymbol{u}_h) - \frac{\vartheta_h}{\widetilde{\vartheta}} \mathbb{D}(\widetilde{\boldsymbol{u}}) \right|^2 + \lambda \left| \mathrm{div}_h \boldsymbol{u}_h - \frac{\vartheta_h}{\widetilde{\vartheta}} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right|^2 \right) \right) \mathrm{d}x \\ &\quad \leq c(\Delta t + h^A) \qquad \text{for all } \ \tau \in [0, T], \end{aligned} \tag{5.4}$$

*where $A = \min\{(1-\varepsilon)/2, (1+\varepsilon)/2\}$ and $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$ is a strong solution satisfying (2.2). In particular, we have*

$$\begin{aligned} &\|\varrho_h - \widetilde{\varrho}\|_{L^\infty L^2} + \|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}\|_{L^\infty L^2} + \left\|\vartheta_h - \widetilde{\vartheta}\right\|_{L^\infty L^2} \\ &\qquad + \|\nabla_h \boldsymbol{u}_h - \nabla_x \widetilde{\boldsymbol{u}}\|_{L^2 L^2} + \left\|\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta}\right\|_{L^2 L^2} \leq c(\Delta t^{1/2} + h^{A/2}). \end{aligned} \tag{5.5}$$

**Remark 5.3.**

- *Recalling Proposition 2.1, the choice of initial data with $k \geq 6$ and the assumption (4.5) used in Theorem 5.2 ensure the global existence of strong solution satisfying (2.2).*
- *By choosing $\varepsilon = 0$ we obtain $A = 1/2$, i.e. the optimal convergence rate in space.*

*Proof of Theorem 5.2.* First, by choosing suitable test functions in the consistency formulation we get

$$\int_0^\tau (4.1)\,\mathrm{d}t - (4.10\text{c})|_{\phi=\widetilde{\vartheta}} + (4.10\text{a})|_{\phi=\frac{|\widetilde{\boldsymbol{u}}|^2}{2} - \frac{\partial \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta})}{\partial \varrho}}} - (4.10\text{b})|_{\phi=\widetilde{\boldsymbol{u}}} + \int_0^\tau \int_{\mathbb{T}^d} \partial_t \widetilde{p} \,\mathrm{d}x\mathrm{d}t$$

and derive the relative energy inequality (see Appendix D for the detailed proof)

$$\begin{aligned} &\left[ R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right) \right]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \frac{\widetilde{\vartheta}}{\vartheta_h} \left( 2\mu \left| \mathbb{D}_h(\boldsymbol{u}_h) - \frac{\vartheta_h}{\widetilde{\vartheta}} \mathbb{D}(\widetilde{\boldsymbol{u}}) \right|^2 + \lambda \left| \mathrm{div}_h \boldsymbol{u}_h - \frac{\vartheta_h}{\widetilde{\vartheta}} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right|^2 \right) \mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa\widetilde{\vartheta}}{\vartheta_h \vartheta_h^{\text{out}}} |\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta}|^2 \,\mathrm{d}x\mathrm{d}t \leq R_C + \sum_{i=1}^5 R_i - R_{\mathbb{S}} - R_\vartheta, \end{aligned} \tag{5.6}$$

where

$$R_C = e_\varrho\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - \frac{\partial \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial \varrho}, \Delta t, h, \tau\right) + e_{\boldsymbol{u}}(\widetilde{\boldsymbol{u}}, \Delta t, h, \tau) - e_s(\widetilde{\vartheta}, \Delta t, h, \tau),$$

$$R_1 = -\int_0^\tau \int_{\mathbb{T}^d} \varrho_h(s_h - \widetilde{s})(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \cdot \nabla_x \widetilde{\vartheta}\,\mathrm{d}x\mathrm{d}t, \quad R_2 = -\int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \otimes (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) : \nabla_x \widetilde{\boldsymbol{u}}\,\mathrm{d}x,$$

$$R_3 = \int_0^\tau \int_{\mathbb{T}^d} \frac{\varrho_h - \widetilde{\varrho}}{\widetilde{\varrho}}(\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot \mathrm{div}_x \widetilde{\mathbb{S}}\,\mathrm{d}x, \quad R_4 = \int_0^\tau \int_{\mathbb{T}^d} (\widetilde{p} - p_h - \partial_\varrho \widetilde{p}(\widetilde{\varrho} - \varrho_h) - \partial_\vartheta \widetilde{p}(\widetilde{\vartheta} - \vartheta_h))\mathrm{div}_x \widetilde{\boldsymbol{u}}\,\mathrm{d}x\mathrm{d}t,$$

$$R_5 = -\int_0^\tau \int_{\mathbb{T}^d} \Big((\varrho_h - \widetilde{\varrho})(s_h - \widetilde{s}) + \widetilde{\varrho}\big(s_h - \widetilde{s} - \partial_\varrho \widetilde{s}(\varrho_h - \widetilde{\varrho}) - \partial_\vartheta \widetilde{s}(\vartheta_h - \widetilde{\vartheta})\big)\Big)(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta})\,\mathrm{d}x\mathrm{d}t,$$

$$R_{\mathbb{S}} = \int_0^\tau \int_{\mathbb{T}^d} \Big(\boldsymbol{u}_h \cdot \mathrm{div}_x \widetilde{\mathbb{S}} + \nabla_h \boldsymbol{u}_h : \widetilde{\mathbb{S}}\Big)\,\mathrm{d}x\mathrm{d}t,$$

$$R_\vartheta = \kappa \int_0^\tau \int_{\mathbb{T}^d} \left(\frac{\vartheta_h^2 \vartheta_h^{\mathrm{out}} - \widetilde{\vartheta}^3}{\widetilde{\vartheta}^2 \vartheta_h \vartheta_h^{\mathrm{out}}}|\nabla_x \widetilde{\vartheta}|^2 + \vartheta_h \mathrm{div}_x \frac{\nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} + \frac{2\widetilde{\vartheta} - \vartheta_h^{\mathrm{out}}}{\vartheta_h \vartheta_h^{\mathrm{out}}} \nabla_x \widetilde{\vartheta} \cdot \nabla_{\mathcal{E}} \vartheta_h\right) \mathrm{d}x\mathrm{d}t.$$

Note that $\widetilde{\mathbb{S}} = \mathbb{S}(\nabla_x \widetilde{\boldsymbol{u}})$ and $\mathbb{S}_h = \mathbb{S}(\nabla_h \boldsymbol{u}_h) = 2\mu \mathbb{D}_h(\boldsymbol{u}_h) + \lambda \mathrm{div}_h \boldsymbol{u}_h \mathbb{I}$.

Next, we analyse the right hand side of (5.6). Recalling the consistency error (4.11) and thanks to the triangular inequality we estimate $R_C$ by

$$\begin{aligned} |R_C| &\leq \left| e_\varrho\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - \frac{\partial \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial \varrho}, \Delta t, h, \tau\right)\right| + |e_{\boldsymbol{u}}(\widetilde{\boldsymbol{u}}, \Delta t, h, \tau)| + |e_s(\widetilde{\vartheta}, \Delta t, h, \tau)| \\ &\lesssim \Delta t + h^{(1-\varepsilon)/2} + h^{(1+\varepsilon)/2} \qquad \text{for any } \ \varepsilon \in (-1, 1). \end{aligned}$$

Using Young's inequality, the uniform bounds (4.7) and the relation (5.2) we estimate the rest terms

$$\begin{aligned} \left|\sum_{i=1}^5 R_i\right| &\lesssim \|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}\|_{L^2L^2}^2 + \|s_h - \widetilde{s}\|_{L^2L^2}^2 + \|\varrho_h - \widetilde{\varrho}\|_{L^2L^2}^2 + \left\|\vartheta_h - \widetilde{\vartheta}\right\|_{L^2L^2}^2 \\ &\lesssim \int_0^\tau R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right)(t)\,\mathrm{d}t. \end{aligned} \tag{5.7}$$

Via Hölder's inequality, Young's inequality, the uniform bounds (4.7) and the relation (5.2), we can control $R_{\mathbb{S}}, R_\vartheta$ as follows (see Appendix E for details):

$$|R_{\mathbb{S}}| \lesssim h\,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2L^2}\,, \ |R_\vartheta| \lesssim h + \int_0^\tau R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right)(t)\,\mathrm{d}t + \delta \left\|\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta}\right\|_{L^2L^2}^2, \tag{5.8}$$

where $h\,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2L^2} \lesssim h^{(1-\varepsilon)/2}$, cf. (4.8). Choosing a small $\delta \in (0, \kappa \underline{\vartheta}/\overline{\vartheta}^2)$, we obtain the desired version of the energy inequality

$$\begin{aligned} &\Big[R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right)\Big]_0^\tau + \int_0^\tau \int_{\mathbb{T}^d} \frac{\widetilde{\vartheta}}{\vartheta_h}\left(2\mu\left|\mathbb{D}_h(\boldsymbol{u}_h) - \frac{\vartheta_h}{\widetilde{\vartheta}}\mathbb{D}(\widetilde{\boldsymbol{u}})\right|^2 + \lambda\left|\mathrm{div}_h \boldsymbol{u}_h - \frac{\vartheta_h}{\widetilde{\vartheta}}\mathrm{div}_x \widetilde{\boldsymbol{u}}\right|^2\right)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa \widetilde{\vartheta}}{\vartheta_h \vartheta_h^{\mathrm{out}}}|\nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta}|^2\,\mathrm{d}x\mathrm{d}t \lesssim \Delta t + h^{(1-\varepsilon)/2} + h^{(1+\varepsilon)/2} + \int_0^\tau R_E\left((\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\right)(t)\,\mathrm{d}t. \end{aligned}$$

Further, applying Gronwall's lemma, together with the error estimate caused by regular initial data

$$R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right)(0)\lesssim\int_{\mathbb{T}^d}\left(|\boldsymbol{u}_h^0-\widetilde{\boldsymbol{u}}(0)|^2+|\vartheta_h^0-\widetilde{\vartheta}(0)|^2+|\varrho_h^0-\widetilde{\varrho}(0)|^2\right)\mathrm{d}x\lesssim h^2$$

we obtain

$$R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right)(\tau)+\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\widetilde{\vartheta}}{\vartheta_h}\left(2\mu\left|\mathbb{D}_h(\boldsymbol{u}_h)-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathbb{D}(\widetilde{\boldsymbol{u}})\right|^2+\lambda\left|\mathrm{div}_h\boldsymbol{u}_h-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathrm{div}_x\widetilde{\boldsymbol{u}}\right|^2\right)\right)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa\widetilde{\vartheta}}{\vartheta_h\vartheta_h^{\mathrm{out}}}|\nabla_{\mathcal{E}}\vartheta_h-\nabla_x\widetilde{\vartheta}|^2\,\mathrm{d}x\mathrm{d}t\lesssim\Delta t+h^{(1-\varepsilon)/2}+h^{(1+\varepsilon)/2}\tag{5.9}$$

for any $\varepsilon\in(-1,1)$, which proves (5.4). Finally, (5.5) is a consequence of the uniform bounds of density and temperature, (5.4) and (5.2). □

**Remark 5.4.** *From the consistency proof and the error estimates proof above, we know that the error of order $h^{(1-\varepsilon)/2}$ in (5.9) is indeed $h\,\|\nabla_{\mathcal{E}}\boldsymbol{u}_h\|_{L^2L^2}$.*

*Inspired by [5], let us add some additional artificial viscosity in the momentum equation (3.4b), i.e.*

$$\int_{\mathbb{T}^d}D_t(\varrho_h\boldsymbol{u}_h)\cdot\boldsymbol{\phi}_h\,\mathrm{d}x-\int_{\mathcal{E}}F_h^{\varepsilon}(\varrho_h\boldsymbol{u}_h,\boldsymbol{u}_h)\cdot[\![\boldsymbol{\phi}_h]\!]\,\mathrm{d}S_x+\int_{\mathbb{T}^d}(\mathbb{S}_h-p\mathbb{I}):\nabla_h\boldsymbol{\phi}_h\,\mathrm{d}x$$
$$=-h^\alpha\int_{\mathbb{T}^d}\nabla_{\mathcal{E}}\boldsymbol{u}_h:\nabla_{\mathcal{E}}(\boldsymbol{\phi}_h)\,\mathrm{d}x,\quad\alpha>0.$$

*Then we obtain one more dissipation in the energy balance, i.e.*

$$D_t\int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2+c_v\varrho_h\vartheta_h\right)\mathrm{d}x+h^\varepsilon\int_{\mathcal{E}}\{\!\{\varrho_h\}\!\}\,|\,[\![\boldsymbol{u}_h]\!]\,|^2\,\mathrm{d}S_x+h^\alpha\int_{\mathbb{T}^d}|\nabla_{\mathcal{E}}\boldsymbol{u}_h|^2\,\mathrm{d}x$$
$$+\frac{\Delta t}{2}\int_{\mathbb{T}^d}\varrho_h^{\triangleleft}|D_t\boldsymbol{u}_h|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathcal{E}}\varrho_h^{\mathrm{up}}|\,\{\!\{\boldsymbol{u}_h\}\!\}\cdot\boldsymbol{n}|\,[\![\boldsymbol{u}_h]\!]\,|^2\,\mathrm{d}S_x=0,$$

*yielding a priori bound on the velocity gradient*

$$\|\nabla_{\mathcal{E}}\boldsymbol{u}_h\|_{L^2L^2}^2\lesssim h^{-\alpha}$$

*and a new consistency error in momentum equality, i.e. $|e_{\boldsymbol{m}}^{new}|\lesssim|e_{\boldsymbol{m}}|+h^\alpha$.*

*Letting $\alpha<\varepsilon+1$ we obtain a better estimate of $\|\nabla_{\mathcal{E}}\boldsymbol{u}_h\|_{L^2L^2}$ than (4.8). Hence, we obtain a new formula of $A$, i.e. $A=\min\{1-\alpha/2,\alpha,(1+\varepsilon)/2\}$. This yields the optimal rate $A=2/3$ by choosing $\alpha=2/3,\varepsilon=1/3$.*

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# A Local existence of strong solution

The goal of this section is to prove the existence of local-in-time highly-regular strong solutions for the Navier–Stokes–Fourier system (1.1). We pursue the same ideas developed by Kawashima [16] and by Breit, Feireisl, Hofmanová [1]: equations (1.1) can be seen as a symmetric hyperbolic-parabolic system and therefore, all the necessary bounds can be deduced from the Moser-type calculus inequalities. We can consider more general pressure $p = p(\varrho, \vartheta)$ and internal energy $e = e(\varrho, \vartheta)$ to be smooth functions of $\varrho$ and $\vartheta$ satisfying

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} \geq \underline{p}_{\varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} \geq \underline{e}_{\vartheta} > 0. \tag{A.1}$$

Note that condition (A.1) implies that the thermodynamic stability condition holds. In particular, for the perfect gas we have $\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} = \vartheta \geq \underline{\vartheta} > 0$ and $\frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} = c_v = 1/(\gamma - 1) > 0$. We can now state our main result.

**Theorem A.1.** *Let $k \geq 4$ and let the initial data (1.2) be such that*

$$\varrho_0 > 0, \quad (\varrho_0, \boldsymbol{u}_0, \vartheta_0) \in W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1}).$$

*Then there exists $T^* > 0$ and a unique strong solution $(\varrho, \boldsymbol{u}, \vartheta)$ to problem (1.1), (1.2) such that*

$$(\varrho, \boldsymbol{u}, \vartheta) \in C([0, T^*]; W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1})).$$

*Moreover, the following estimate*

$$\begin{aligned} \sup_{t\in[0,T^*]} \Big( \|(\varrho, \boldsymbol{u}, \vartheta)(t)\|_{W^{k,2}} + \|\partial_t \varrho(t)\|_{W^{k-1,2}} + \|(\partial_t \boldsymbol{u}, \partial_t \vartheta)(t)\|_{W^{k-2,2}} \Big) \\ + \sup_{t\in[0,T^*]} \Big( \|\partial_t^2 \varrho(t)\|_{W^{k-3,2}} + \|(\partial_t^2 \boldsymbol{u}, \partial_t^2 \vartheta)(t)\|_{W^{k-4,2}} \Big) \leq C \|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}} \end{aligned} \tag{A.2}$$

*holds, where the positive constant $C$ depends solely on $T^*$, $\inf \varrho_0$, $\underline{e}_{\vartheta}$, and $\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}$.*

Before proving Theorem A.1, we briefly recall some estimates which are a straightforward consequence of the Moser-type calculus, see e.g. [16, Lemmas 2.3, 2.4 and 2.5].

**Lemma A.2.**

*(i) Let $k \geq 2$ and $0 \leq m \leq k$. If $u \in W^{k,2}(\mathbb{T}^d)$ and $v \in W^{m,2}(\mathbb{T}^d)$, then $uv \in W^{m,2}(\mathbb{T}^d)$ and*

$$\|uv\|_{W^{m,2}} \leq c \|u\|_{W^{k,2}} \|v\|_{W^{m,2}}. \tag{A.3}$$

*(ii) Let $k \geq 1$ and let $F = F(u)$ be a $C^\infty$- function of $u$. If $u \in (W^{k,2} \cap L^\infty)(\mathbb{T}^d)$ then $\nabla_x F(u) \in W^{k-1,2}(\mathbb{T}^d)$ and*

$$\|\nabla_x F(u)\|_{W^{k-1,2}} \leq cM(1 + \|u\|_{L^\infty})^{k-1} \|\nabla_x u\|_{W^{k-1,2}}, \tag{A.4}$$

*with $M = \sum_{|\alpha|=1}^{k} \sup_u |\partial_u^\alpha F(u)|$ ($\sup_u$ is taken over all $u$ such that $|u| \leq \|u\|_{L^\infty}$).*

*(iii) Let $k \geq 4$ and $1 \leq m \leq k$. If $u \in (W^{k,2} \cap L^\infty)(\mathbb{T}^d)$ and $v \in W^{m-1,2}(\mathbb{T}^d)$, then*

$$[\partial_x^\alpha, u]v := \partial_x^\alpha(uv) - u\partial_x^\alpha v \in L^2(\mathbb{T}^d), \tag{A.5}$$

*and*

$$\sum_{|\alpha|=0}^{m} \|[\partial_x^\alpha, u]v\|_{L^2} \leq c\|\nabla_x u\|_{W^{k-1,2}}\|v\|_{W^{m-1,2}}. \tag{A.6}$$

*Proof of Theorem A.1.* As shown by Kawashima and Shizuta [17, Section 4], the Navier–Stokes–Fourier system (1.1) can be seen as a symmetric hyperbolic-parabolic system, i.e. as a coupled system of a hyperbolic equation for $\varrho$ and a symmetric strongly parabolic system for $(\boldsymbol{u}, \vartheta)$. More precisely, system (1.1) can be written as

$$\partial_t \varrho + \boldsymbol{u} \cdot \nabla_x \varrho = f_1(\varrho, \nabla_x \boldsymbol{u}), \tag{A.7}$$

$$\varrho \partial_t \boldsymbol{u} - \left[\mu \Delta_x \boldsymbol{u} + (\mu + \lambda)\mathrm{div}_x \nabla_x^\top \boldsymbol{u}\right] = \boldsymbol{f}_2(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \varrho, \nabla_x \boldsymbol{u}, \nabla_x \vartheta) \tag{A.8}$$

$$\varrho \partial_\vartheta e \ \partial_t \vartheta - \kappa \Delta_x \vartheta = f_3(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \boldsymbol{u}, \nabla_x \vartheta), \tag{A.9}$$

with

$$\begin{aligned} f_1(\varrho, \nabla_x \boldsymbol{u}) &:= -\varrho \mathrm{div}_x \boldsymbol{u}, \\ \boldsymbol{f}_2(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \varrho, \nabla_x \boldsymbol{u}, \nabla_x \vartheta) &:= -\big(\partial_\varrho p \nabla_x \varrho + \partial_\vartheta p \nabla_x \vartheta + \varrho \boldsymbol{u} \cdot \nabla_x \boldsymbol{u}\big), \\ f_3(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \boldsymbol{u}, \nabla_x \vartheta) &:= \frac{\mu}{2}|\nabla_x \boldsymbol{u} + \nabla_x^\top \boldsymbol{u}|^2 + \lambda(\mathrm{div}_x \boldsymbol{u})^2 - \big(\vartheta \partial_\vartheta p \ \mathrm{div}_x \boldsymbol{u} + \varrho \partial_\vartheta e \ \boldsymbol{u} \cdot \nabla_x \vartheta\big). \end{aligned}$$

For any $T > 0$, we define the Banach space $X(T) := X_\varrho(T) \times X_{\boldsymbol{u},\vartheta}(T)$ with

$$\begin{aligned} X_\varrho(T) &:= C([0,T]; W^{k,2}(\mathbb{T}^d)) \cap C^1([0,T]; W^{k-1,2}(\mathbb{T}^d)), \\ X_{\boldsymbol{u},\vartheta}(T) &:= C([0,T]; W^{k,2}(\mathbb{T}^d; \mathbb{R}^{d+1})) \cap C^1([0,T]; W^{k-2,2}(\mathbb{T}^d; \mathbb{R}^{d+1})) \cap \\ &\quad L^2(0,T; W^{k+1,2}(\mathbb{T}^d; \mathbb{R}^{d+1})) \cap W^{1,2}(0,T; W^{k-1,2}(\mathbb{T}^d; \mathbb{R}^{d+1})), \end{aligned}$$

and the set

$$\mathcal{R}(T, M, M_t) := \left\{ (\varrho, \boldsymbol{u}, \vartheta) \in X(T) \;\middle|\; \begin{array}{r} \sup\limits_{t\in[0,T]} \|(\varrho, \boldsymbol{u}, \vartheta)(t)\|_{W^{k,2}} + \|(\boldsymbol{u}, \vartheta)\|_{L^2 W^{k+1,2}} \leq M, \\ \|(\partial_t \varrho, \partial_t \boldsymbol{u}, \partial_t \vartheta)(t)\|_{L^2 W^{k-1,2}} \leq M_t, \end{array} \right\}.$$

Let us now consider the map

$$(\varrho, \boldsymbol{u}, \vartheta) \mapsto G(\varrho, \boldsymbol{u}, \vartheta) = (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta}), \tag{A.10}$$

where the trio $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$ satisfies the linearized system associated to (A.7)–(A.9),

$$\partial_t \widetilde{\varrho} + \boldsymbol{u} \cdot \nabla_x \widetilde{\varrho} = f_1(\varrho, \nabla_x \boldsymbol{u}), \tag{A.11}$$

$$\varrho \partial_t \widetilde{\boldsymbol{u}} - \left[\mu \Delta_x \widetilde{\boldsymbol{u}} + (\mu + \lambda)\mathrm{div}_x \nabla_x^\top \widetilde{\boldsymbol{u}}\right] = \boldsymbol{f}_2(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \varrho, \nabla_x \boldsymbol{u}, \nabla_x \vartheta) \tag{A.12}$$

$$\varrho \partial_\vartheta e \ \partial_t \widetilde{\vartheta} - \kappa \Delta_x \widetilde{\vartheta} = f_3(\varrho, \boldsymbol{u}, \vartheta, \nabla_x \boldsymbol{u}, \nabla_x \vartheta). \tag{A.13}$$

**Step 1.** Our first goal is to determine $T = T_0, M, M_t > 0$ in such a way that $G : \mathcal{R}(T_0, M, M_t) \to \mathcal{R}(T_0, M, M_t)$ is well-defined. Let $(\varrho, \boldsymbol{u}, \vartheta) \in \mathcal{R}(T_0, M, M_t)$ be fixed. We can deduce in particular that

$$(f_1, \boldsymbol{f}_2, f_3) \in L^\infty(0, T_0; W^{k-1,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1})), \quad f_1 \in L^2(0, T_0; W^{k,2}(\mathbb{T}^d)), \tag{A.14}$$

and that there exists a constant $c_1 = c_1(M) > 0$ such that

$$\sup_{t \in [0,T_0]} \|(f_1, \boldsymbol{f}_2, f_3)(t)\|_{W^{k-1,2}} \leq c_1 M. \tag{A.15}$$

The existence of a solution $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta}) \in X(T_0)$ for any $T_0 > 0$ of the linearized problem follows from already existing results that can be found in literature, see e.g. Kato [15, Theorem I]. We point out that the strict positivity of the initial density $\varrho_0$ is sufficient to guarantee that

$$\varrho \geq \underline{\varrho} > 0.$$

In order to prove that $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta}) \in \mathcal{R}(T_0, M, M_t)$, we need to deduce some *energy estimates*. Applying $\partial_x^\alpha$ with $|\alpha| \leq k$ to equations (A.11)–(A.13), we obtain

$$\partial_t \partial_x^\alpha \widetilde{\varrho} + \boldsymbol{u} \cdot \nabla_x \partial_x^\alpha \widetilde{\varrho} = F_1^\alpha, \tag{A.16}$$

$$\varrho\, \partial_t \partial_x^\alpha \widetilde{\boldsymbol{u}} - \left[\mu \Delta_x \partial_x^\alpha \widetilde{\boldsymbol{u}} + (\mu + \lambda) \mathrm{div}_x \nabla_x^\top \partial_x^\alpha \widetilde{\boldsymbol{u}}\right] = \boldsymbol{F}_2^\alpha, \tag{A.17}$$

$$\varrho \partial_\vartheta e\, \partial_t \partial_x^\alpha \widetilde{\vartheta} - \kappa \Delta_x \partial_x^\alpha \widetilde{\vartheta} = F_3^\alpha, \tag{A.18}$$

with

$$\begin{aligned}
F_1^\alpha &:= \partial_x^\alpha f_1 - \left[\partial_x^\alpha, \boldsymbol{u}\right] \nabla_x \widetilde{\varrho}, \\
\boldsymbol{F}_2^\alpha &:= \varrho\, \partial_x^\alpha \left(\frac{\boldsymbol{f}_2}{\varrho}\right) + \varrho \left(\left[\partial_x^\alpha, \frac{\mu}{\varrho}\right] \Delta_x \widetilde{\boldsymbol{u}} + \left[\partial_x^\alpha, \frac{\mu + \lambda}{\varrho}\right] \mathrm{div}_x \nabla_x^\top \widetilde{\boldsymbol{u}}\right), \\
F_3^\alpha &:= \varrho \partial_\vartheta e\, \partial_x^\alpha \left(\frac{f_3}{\varrho \partial_\vartheta e}\right) + \varrho \partial_\vartheta e \left[\partial_x^\alpha, \frac{\kappa}{\varrho \partial_\vartheta e}\right] \Delta_x \widetilde{\vartheta};
\end{aligned}$$

the commutator $[\partial_x^\alpha,\ \cdot\ ]\ \cdot$ is defined as in (A.5). We now multiply equations (A.16), (A.17) and (A.18) by $\partial_x^\alpha \widetilde{\varrho}$, $\partial_x^\alpha \widetilde{\boldsymbol{u}}$ and $\partial_x^\alpha \widetilde{\vartheta}$, respectively, integrate over $\mathbb{T}^d$ and sum over $0 \leq |\alpha| \leq k$; from hypothesis (A.1), Gårding's inequality for strongly elliptic operators, Hölder and Young's inequalities, the Sobolev embedding $W^{k-1,2}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, and from (A.3), (A.4), (A.6), (A.14), we find constants $c_2 > 0$, $c_3 = c_3(\underline{\varrho}, \underline{e}_\vartheta, M) > 0$ and $c_4 = c_4(\underline{\varrho}, \underline{e}_\vartheta, M) > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\widetilde{\varrho}\|_{W^{k,2}} \leq c_2 \left(\|f_1\|_{W^{k,2}} + M \|\widetilde{\varrho}\|_{W^{k,2}}\right), \tag{A.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \|(\widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|_{W^{k,2}}^2 + c_3 \|(\widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|_{W^{k+1,2}}^2 \leq c_4 \left[\left(1 + \|(\partial_t \varrho, \partial_t \vartheta)\|_{W^{k-1,2}}\right) \|(\widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|_{W^{k,2}}^2 + \|(\boldsymbol{f}_2, f_3)\|_{W^{k-1,2}}^2\right]. \tag{A.20}$$

Integrating (A.19), (A.20) over $[0, \tau]$, from Hölder's inequality and the Gronwall argument, there exist $c_5 = c_5(\underline{\varrho}, \underline{e}_\vartheta) > 0$ and $c_6 = c_6(\underline{\varrho}, \underline{e}_\vartheta, M) > 0$ such that

$$\begin{aligned}
&\|(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})(\tau)\|_{W^{k,2}}^2 + \int_0^\tau \|(\widetilde{\boldsymbol{u}}, \widetilde{\vartheta})(t)\|_{W^{k+1,2}}^2 \ \mathrm{d}t \\
&\leq c_5 e^{c_6\left(\tau + \tau^{\frac{1}{2}} M_t\right)} \left[\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}^2 + c_6 \int_0^\tau \left(\tau \|f_1(t)\|_{W^{k,2}}^2 + \|(\boldsymbol{f}_2, f_3)(t)\|_{W^{k-1,2}}^2\right) \ \mathrm{d}t\right].
\end{aligned} \tag{A.21}$$

From (A.3), (A.4) and (A.15), we can finally conclude that there exist $C_1 = C_1(\underline{\varrho}, \underline{e}_\vartheta) > 1$ and $C_2 = C_2(\underline{\varrho}, \underline{e}_\vartheta, M) > 0$ such that

$$\|(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})(\tau)\|^2_{W^{k,2}} + \|(\widetilde{\boldsymbol{u}}, \widetilde{\vartheta})\|^2_{L^2W^{k+1,2}} \leq C_1^2 e^{C_2\left(\tau+\tau^{\frac{1}{2}}M_t\right)} \left[\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|^2_{W^{k,2}} + C_2M^2(1+\tau)\tau\right].$$

Moreover, directly from equations (A.7)–(A.9), we deduce that there exists a constant $C_3 = C_3(M) > 0$ such that

$$\int_0^\tau \|(\partial_t\widetilde{\varrho}, \partial_t\widetilde{\boldsymbol{u}}, \partial_t\widetilde{\vartheta})(t)\|^2_{W^{k-1,2}}\ \mathrm{d}t \leq C_3^2M^2(1+\tau).$$

To get that $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta}) \in \mathcal{R}(T_0, M, M_t)$, it is now enough to choose $M$, $M_t$ and $T_0$ such that

$$\begin{aligned} M &:= 2C_1\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}, \\ M_t &:= 4C_1C_3\|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}, \\ e^{C_2\left(T_0+T_0^{\frac{1}{2}}M_t\right)} \leq 2, \quad C_2M^2(1+T_0)T_0 &\leq \|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|^2_{W^{k,2}}, \quad T_0 \leq 3. \end{aligned}$$

**Step 2.** Let us now consider the approximation sequence $\{(\varrho^n, \boldsymbol{u}^n, \vartheta^n)\}_{n\in\mathbb{N}}$ such that

$$(\varrho^0, \boldsymbol{u}^0, \vartheta^0) = (\varrho_0, \boldsymbol{u}_0, \vartheta_0), \tag{A.22}$$
$$(\varrho^n, \boldsymbol{u}^n, \vartheta^n) = G(\varrho^{n-1}, \boldsymbol{u}^{n-1}, \vartheta^{n-1}), \quad n \geq 1, \tag{A.23}$$

and take the difference of two subsequent solutions:

$$\begin{aligned} \partial_t(\varrho^{n+1} - \varrho^n) + \boldsymbol{u}^n \cdot \nabla_x(\varrho^{n+1} - \varrho^n) &= f_1^n, \\ \varrho^n\partial_t(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) - \mu\Delta_x(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) - (\mu+\lambda)\mathrm{div}_x\nabla_x^\top(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) &= \boldsymbol{f}_2^n, \\ \varrho^n\partial_\vartheta e^n\ \partial_t(\vartheta^{n+1} - \vartheta^n) - \kappa\Delta_x(\vartheta^{n+1} - \vartheta^n) &= f_3^n, \end{aligned}$$

where, from the fact that $(\varrho^n, \boldsymbol{u}^n, \vartheta^n) \in \mathcal{R}(T_0, M, M_t)$ for any $n \in \mathbb{N}$, it is not difficult to show that there exists a constant $C = C(\underline{\varrho}, \underline{e}_\vartheta, \|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}) > 0$ such that

$$\begin{aligned} \|f_1^n\|_{W^{k-1,2}} &\leq C\left(\|\varrho^n - \varrho^{n-1}\|_{W^{k-1,2}} + \|(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \vartheta^n - \vartheta^{n-1})\|_{W^{k,2}}\right), \\ \|(\boldsymbol{f}_2^n, f_3^n)\|_{W^{k-2,2}} &\leq C\|(\varrho^n - \varrho^{n-1}, \boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \vartheta^n - \vartheta^{n-1})\|_{W^{k-1,2}}. \end{aligned}$$

Therefore, the trio $(\varrho^{n+1} - \varrho^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \vartheta^{n+1} - \vartheta^n)$ satisfies the linearized equations (A.11)–(A.13) with $(f_1, \boldsymbol{f}_2, f_3)$ replaced by $(f_1^n, \boldsymbol{f}_2^n, f_3^n)$ and $(\varrho^{n+1} - \varrho^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \vartheta^{n+1} - \vartheta^n)(0) = \mathbf{0}$; hence an analogous of (A.21) holds with $k$ replaced by $k-1$:

$$\begin{aligned} &\sup_{t\in[0,\tau]} \|(\varrho^{n+1} - \varrho^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \vartheta^{n+1} - \vartheta^n)(t)\|^2_{W^{k-1,2}} + \int_0^\tau \|(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \vartheta^{n+1} - \vartheta^n)(t)\|_{W^{k,2}}\,\mathrm{d}t \\ &\quad\leq C_4(1+\tau)\tau e^{C_4\left(\tau+M_t\tau^{\frac{1}{2}}\right)}\Big[\sup_{t\in[0,\tau]} \|(\varrho^n - \varrho^{n-1}, \boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \vartheta^n - \vartheta^{n-1})(t)\|^2_{W^{k-1,2}} \\ &\qquad\qquad + \int_0^\tau \|(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \vartheta^n - \vartheta^{n-1})(t)\|_{W^{k,2}}\,\mathrm{d}t\Big], \end{aligned}$$

where $C_4 = C_4(\underline{\varrho}, \underline{e}_\vartheta, \|(\varrho_0, \boldsymbol{u}_0, \vartheta_0)\|_{W^{k,2}}) > 0$ is a constant independent of $n$. Taking $0 < T_1 \leq T_0$ such that

$$C_4 T_1 (1 + T_1) e^{C_4 (T_1 + M_t T_1^{\frac{1}{2}})} < 1,$$

we get that $\{(\varrho^n, \boldsymbol{u}^n, \vartheta^n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0, T_1]; W^{k-1,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1}))$. Consequently, there exists $(\varrho, \boldsymbol{u}, \vartheta)$ such that

$$(\varrho^n, \boldsymbol{u}^n, \vartheta^n) \to (\varrho, \boldsymbol{u}, \vartheta) \quad \text{in } C([0, T_1]; W^{k-1,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1})).$$

Furthermore, $\{(\varrho^n, \boldsymbol{u}^n, \vartheta^n)\}_{n\in\mathbb{N}} \subset \mathcal{R}(T_0, M, M_t) \subset \mathcal{R}(T_1, M, M_t)$ is uniformly bounded and therefore there exists a subsequence, not relabeled, such that

$$(\varrho^n, \boldsymbol{u}^n, \vartheta^n) \overset{*}{\rightharpoonup} (\varrho, \boldsymbol{u}, \vartheta) \quad \text{in } L^\infty(0, T_1; W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1})).$$

Moreover, we obtain that

$$(\partial_t \varrho, \partial_t \boldsymbol{u}, \partial_t \vartheta) \in L^\infty(0, T_1; W^{k-1,2}(\mathbb{T}^d) \times W^{k-2,2}(\mathbb{T}^d; \mathbb{R}^{d+1})).$$

**Step 3.** Finally, to show that $(\varrho, \boldsymbol{u}, \vartheta) \in C([0, T_1]; W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1}))$, it is enough to consider a family of mollifiers $\{\phi_\delta\}_{\delta>0}$ and define $(\varrho_\delta, \boldsymbol{u}_\delta, \vartheta_\delta) := (\varrho * \phi_\delta, \boldsymbol{u} * \phi_\delta, \vartheta * \phi_\delta)$. Clearly, $(\varrho_\delta, \boldsymbol{u}_\delta, \vartheta_\delta) \in C^0([0, T_1]; W^{k-1,2}(\mathbb{T}^d))$ for any $\delta > 0$. Applying $\phi_\delta *$ to system (A.11)–(A.13) and proceeding as in the previous step, the differences $(\varrho_\delta - \varrho_{\delta'}, \boldsymbol{u}_\delta - \boldsymbol{u}_{\delta'}, \vartheta_\delta - \vartheta_{\delta'})$ satisfy the linearized equations (A.11)–(A.13) with $(f_1, \boldsymbol{f}_2, f_3)$ replaced by some $(f_1^\delta, \boldsymbol{f}_2^\delta, f_3^\delta)$. Therefore, an analogous of (A.21) holds and it is easy to check that $\{(\varrho_\delta, \boldsymbol{u}_\delta, \vartheta_\delta)\}_{\delta>0}$ is a Cauchy sequence in $C([0, T_1]; W^{k,2}(\mathbb{T}^d; \mathbb{R}^{1+d+1}))$. Consequently,

$$(\partial_t \varrho, \partial_t \boldsymbol{u}, \partial_t \vartheta) \in C([0, T_1]; W^{k-1,2}(\mathbb{T}^d) \times W^{k-2,2}(\mathbb{T}^d; \mathbb{R}^{d+1})),$$

and, differentiating (A.7)–(A.9) in time, we can deduce that

$$(\partial_t^2 \varrho, \partial_t^2 \boldsymbol{u}, \partial_t^2 \vartheta) \in C([0, T_1]; W^{k-3,2}(\mathbb{T}^d) \times W^{k-4,2}(\mathbb{T}^d; \mathbb{R}^{d+1})).$$

To conclude the proof, it is sufficient to choose $T^* := T_1$. □

# B Boundedness of the entropy Hessian matrix

In this section we derive the lower and upper bounds of the eigenvalues of the entropy Hessian matrix $\nabla^2_{(\varrho,p)}(-\varrho s)$ with $(\varrho, \vartheta) \in (0, \infty)^2$. Denote

$$f(\lambda) = \left| \nabla^2_{(\varrho,p)}(-\varrho s) - \lambda \mathbb{I} \right| = \lambda^2 - \frac{c_v + (1 + c_v)\vartheta^2}{\varrho \vartheta^2} \lambda + \frac{c_v}{\varrho^2 \vartheta^2}$$

and let $\lambda_1 \leq \lambda_2$ be the roots of $f(\lambda) = 0$.

It is easy to check that

$$f(0) = \frac{c_v}{\varrho^2 \vartheta^2} > 0$$

and

$$f\left(\frac{1}{\alpha\varrho}\right)=\frac{[1-\alpha(1+c_v)]\vartheta^2+\alpha(\alpha-1)c_v}{\alpha^2\varrho^2\vartheta^2},\quad f\left(\frac{1}{\beta\varrho\vartheta^2}\right)=\frac{[c_v\beta-(1+c_v)]\beta\vartheta^2+(1-c_v\beta)}{\beta^2\varrho^2\vartheta^4}.$$

Hence, we take $\alpha=2+c_v$, $\beta=\frac{2+c_v}{c_v}$ and obtain

$$1-\alpha(1+c_v)<0,\quad f\left(\frac{1}{\alpha\varrho}\right)\geq\frac{\Big(1-(2+c_v)\alpha+\alpha^2\Big)c_v}{\alpha^2\varrho^2\vartheta^2}>0\qquad\text{if }\vartheta^2\leq c_v,$$

$$c_v\beta-(1+c_v)>0,\quad f\left(\frac{1}{\beta\varrho\vartheta^2}\right)>\frac{1+\Big(c_v\beta-(2+c_v)\Big)c_v\beta}{\beta^2\varrho^2\vartheta^4}>0\qquad\text{if }\vartheta^2>c_v.$$

Hence, it holds

$$0<\min\left(\frac{1}{(2+c_v)\varrho},\frac{c_v}{(2+c_v)\varrho\vartheta^2}\right)<\lambda_1<\frac{c_v+(1+c_v)\vartheta^2}{2\varrho\vartheta^2}<\lambda_2<\frac{c_v+(1+c_v)\vartheta^2}{\varrho\vartheta^2}.$$

# C Consistency proof

In order to prove the consistency, we integrate (3.4a), (3.4b), and (4.2) from $t=0$ to $t=t^{n+1}$ and choose $\phi_h=\Pi_Q\phi$ as the test function for any $\phi\in L^2(0,T;W^{2,\infty}(\mathbb{T}^d))$, $\partial_t\phi,\partial_t^2\phi\in L^\infty((0,T)\times\mathbb{T}^d)$. We estimate all resulting terms in six steps.

### Step 1 – time derivative terms

Let $r_h$ stand for $\varrho_h$, $\varrho_h\boldsymbol{u}_h$ or $\varrho_h s_h$. Recalling [8, (2.17)] and the estimates (4.5), (4.6), and (4.7a) we know that

$$\begin{aligned}&\left|\left[\int_{\mathbb{T}^d}r_h\phi\,\mathrm{d}x\right]_{t=0}^{\tau}-\int_0^{t^{n+1}}\int_{\mathbb{T}^d}\left(D_t r_h(t)\Pi_Q\phi(t)+r_h(t)\partial_t\phi(t)\right)\mathrm{d}x\,\mathrm{d}t\right|\\&\lesssim\Delta t\left(\left\|\partial_t^2\phi\right\|_{L^\infty L^\infty}+\|\partial_t\phi\|_{L^\infty L^\infty}\right)\|r_h\|_{L^\infty L^1}\lesssim\Delta t\end{aligned}$$

for any $\tau\in[t_n,t_{n+1})$.

In what follows we shall also need some standard interpolation inequalities used in the numerical analysis of finite volume methods, see, e.g. [3, 12, 13]. For completeness, we list them below.

For $\phi\in W^{2,\infty}(\mathbb{T}^d)$ we have the following estimates,

$$|\,[\![\Pi_Q\phi]\!]\,|\lesssim h\|\nabla_x\phi\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)},\quad|\Pi_Q\phi-\{\!\{\Pi_Q\phi\}\!\}\,|\lesssim h\|\nabla_x\phi\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)}\ \text{ for }\sigma\in\mathcal{E}\tag{C.1}$$

and for all $1\leq p\leq\infty$,

$$\begin{aligned}&\left\|\nabla_x\phi-\nabla_{\mathcal{E}}\big(\Pi_Q\phi\big)\right\|_{L^p}\lesssim h\|\nabla_x^2\phi\|_{L^\infty(\mathbb{T}^d)},\quad\left\|\nabla_x\phi-\nabla_h\big(\Pi_Q\phi\big)\right\|_{L^p}\lesssim h\|\nabla_x^2\phi\|_{L^\infty(\mathbb{T}^d)},\\&\|\phi-\Pi_Q\phi\|_{L^p(\mathbb{T}^d)}\lesssim h\|\nabla_x\phi\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)}.\end{aligned}\tag{C.2}$$

Moreover, the discrete version of product rule holds

$$[\![f_hg_h]\!]=[\![f_h]\!]\,\{\!\{g_h\}\!\}+\{\!\{f_h\}\!\}\,[\![g_h]\!]\,,\quad f_h,g_h\in Q_h.\tag{C.3}$$

### Step 2 – convective terms

To deal with the convective terms, it is convenient to recall [6, Lemma 2.5],

$$\int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \boldsymbol{u}_h \cdot \nabla_x \phi \,\mathrm{d}x \,\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathcal{E}} F_h^{\varepsilon}[r_h, \boldsymbol{u}_h] \, [\![\Pi_Q \phi]\!] \,\mathrm{d}S_x \,\mathrm{d}t = \sum_{j=1}^{4} E_j(r_h)$$

with

$$\begin{aligned}
E_1(r_h) &= \frac{1}{2} \int_0^{t^{n+1}} \int_{\mathcal{E}} |\, \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n}| \, [\![r_h]\!] \, [\![\Pi_Q \phi]\!] \,\mathrm{d}S_x \,\mathrm{d}t, \\
E_2(r_h) &= \frac{1}{4} \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n} \, [\![r_h]\!] \, [\![\Pi_Q \phi]\!] \,\mathrm{d}S_x \,\mathrm{d}t, \\
E_3(r_h) &= \int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \boldsymbol{u}_h \cdot \Big( \nabla_x \phi - \nabla_h \big( \Pi_Q \phi \big) \Big) \,\mathrm{d}x \,\mathrm{d}t, \\
E_4(r_h) &= h^{\varepsilon} \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![r_h]\!] \, [\![\Pi_Q \phi]\!] \,\mathrm{d}S_x \,\mathrm{d}t = -h^{1+\varepsilon} \int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \Delta_h \Pi_Q \phi \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

**Error terms** $E_1(r_h)$

Directly recalling the proof of [7, Theorem 11.2] we have

$$\begin{aligned}
|E_1(r_h)| &= \left| h \sum_{j=1}^{d} \int_0^{t^{n+1}} \int_{\mathbb{T}^d} r_h \cdot \Big( \Delta_h^{(j)} \Pi_Q \phi \Pi_Q |\, \{\!\{u_{j,h}\}\!\} \,| + (\Pi_Q \eth_{\mathcal{E}}^{(j)} \Pi_Q \phi) \eth_{\mathcal{T}}^{(j)} |\, \{\!\{u_{j,h}\}\!\} \,| \Big) \,\mathrm{d}x \right| \\
&\lesssim h \, \|r_h\|_{L^2 L^2} \left( \left\| \nabla_x^2 \phi \right\|_{L^\infty L^\infty} \|\boldsymbol{u}_h\|_{L^2 L^2} + \|\nabla_x \phi\|_{L^\infty L^\infty} \|\nabla_h \boldsymbol{u}_h\|_{L^2 L^2} \right) \lesssim h
\end{aligned}$$

for $r_h = \varrho_h, \varrho_h u_{i,h}$ or $\varrho_h s_h$, where $\Delta_h^{(j)} = \eth_{\mathcal{T}}^{(j)} \eth_{\mathcal{E}}^{(j)}$ and $\eth_{\mathcal{E}}^{(j)}, \eth_{\mathcal{T}}^{(j)}$ are the $j^{\text{th}}$-component of $\nabla_{\mathcal{E}}$ and $\nabla_h$, respectively.

**Error terms** $E_2(r_h)$

To deal with $E_2(r_h)$ we first set $r_h = \varrho_h$ and obtain

$$|E_2(\varrho_h)| \lesssim h \, \|\nabla_x \phi\|_{L^\infty L^\infty} \left( \int_0^{t^{n+1}} \int_{\mathcal{E}} |\, [\![\boldsymbol{u}_h]\!] \,|^2 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \left( \int_0^{t^{n+1}} \int_{\mathcal{E}} [\![\varrho_h]\!]^2 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \lesssim h^{(3-\varepsilon)/2} \, \|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2}$$

due to (C.1), (4.7b) and Hölder's inequality.

Then, inserting $r_h = \varrho_h u_{i,h}$ into $E_2(r_h)$ and taking into account (C.1), the assumption (4.5) and the

estimate (4.7b) with (C.3) we get

$$
\begin{aligned}
|E_2(\varrho_h u_{i,h})| &\lesssim h \,\|\nabla_x \phi\|_{L^\infty L^\infty} \int_0^{t^{n+1}} \int_{\mathcal{E}} \big|\, [\![ \boldsymbol{u}_h ]\!] \cdot \boldsymbol{n} \big( [\![ \varrho_h ]\!] \{\!\{ u_{i,h} \}\!\} + \{\!\{ \varrho_h \}\!\} [\![ u_{i,h} ]\!] \big) \big| \,\mathrm{d}S_x \,\mathrm{d}t \\
&\lesssim h \int_0^{t^{n+1}} \int_{\mathcal{E}} |\, [\![ \varrho_h ]\!] \,|\, |\, [\![ \boldsymbol{u}_h ]\!] \cdot \boldsymbol{n} \,|\, |\, \{\!\{ \boldsymbol{u}_h \}\!\} \,| \,\mathrm{d}S_x \,\mathrm{d}t + h \int_0^{t^{n+1}} \int_{\mathcal{E}} \{\!\{ \varrho_h \}\!\} \,|\, [\![ \boldsymbol{u}_h ]\!] \,|^2 \,\mathrm{d}S_x \,\mathrm{d}t \\
&\lesssim h \left( \int_0^{t^{n+1}} \int_{\mathcal{E}} |\, \{\!\{ \boldsymbol{u}_h \}\!\} \,|^2 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} \left( \int_0^{t^{n+1}} \int_{\mathcal{E}} |\, [\![ \boldsymbol{u}_h ]\!] \,|^2 \,\mathrm{d}S_x \,\mathrm{d}t \right)^{1/2} + h^2 \,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2}^2 \\
&\lesssim h \,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2} + h^2 \,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2}^2 \,.
\end{aligned}
$$

From the Taylor expansion

$$
|\, [\![ \varrho_h s_h ]\!] \,| = |\partial_\varrho(-\varrho s)(\boldsymbol{v}^*) \,[\![ \varrho_h ]\!] + \partial_p(-\varrho s)(\boldsymbol{v}^*) \,[\![ p_h ]\!] \,| \lesssim |\, [\![ \varrho_h ]\!] \,| + |\, [\![ p_h ]\!] \,|
$$

with $\boldsymbol{v}^* \in \mathrm{co}\{(\varrho_h^{\mathrm{in}}, p_h^{\mathrm{in}}), (\varrho_h^{\mathrm{out}}, p_h^{\mathrm{out}})\}$. As $[\![ p_h ]\!]$ has the same bound as $[\![ \varrho_h ]\!]$, see (4.7b), we know that $E_2(\varrho_h s_h)$ has the same estimate as $E_2(\varrho_h)$, meaning

$$
|E_2(\varrho_h s_h)| \lesssim h^{(3-\varepsilon)/2} \,\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2} \,.
$$

**Error terms** $E_3(r_h)$

The estimates of the third error terms are straightforward due to (C.2), and the uniform bounds (4.5) – (4.7a). Indeed,

$$
\begin{aligned}
|E_3(\varrho_h)| &\lesssim h \left\|\nabla_x^2 \phi\right\|_{L^\infty L^\infty} \int_0^{t^{n+1}} \int_{\mathbb{T}^d} |\varrho_h \boldsymbol{u}_h| \,\mathrm{d}x \,\mathrm{d}t \lesssim h \,\|\boldsymbol{u}_h\|_{L^\infty L^2} \lesssim h, \\
|E_3(\varrho_h u_{i,h})| &\lesssim h \left\|\nabla_x^2 \phi\right\|_{L^\infty L^\infty} \int_0^{t^{n+1}} \int_{\mathbb{T}^d} |\varrho_h u_{i,h} \boldsymbol{u}_h| \,\mathrm{d}x \,\mathrm{d}t \lesssim h \,\|\boldsymbol{u}_h\|_{L^\infty L^2}^2 \lesssim h, \\
|E_3(\varrho_h s_h)| &\lesssim h \left\|\nabla_x^2 \phi\right\|_{L^\infty L^\infty} \int_0^{t^{n+1}} \int_{\mathbb{T}^d} |\varrho_h s_h \boldsymbol{u}_h| \,\mathrm{d}x \,\mathrm{d}t \lesssim h \,\|\boldsymbol{u}_h\|_{L^\infty L^2} \lesssim h.
\end{aligned}
$$

**Error terms** $E_4(r_h)$

Finally, we treat the fourth error terms. For $r_h = \varrho_h s_h$ the term is not present, i.e. $E_4(\varrho_h s_h) = 0$. For $r_h = \varrho_h, \varrho_h u_{i,h}$ we have

$$
|E_4(r_h)| \lesssim h^{1+\varepsilon} \left\|\nabla_x^2 \phi\right\|_{L^\infty L^\infty} \|r_h\|_{L^1 L^1} \lesssim h^{1+\varepsilon}.
$$

Similarly as in the first step, recalling (4.5) and (4.7a) we realize that

$$
\left| \int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} r_h \boldsymbol{u}_h \cdot \nabla_x \phi \,\mathrm{d}x \,\mathrm{d}t \right| \lesssim \Delta t \|\nabla_x \phi\|_{L^\infty L^\infty} \|\boldsymbol{u}_h\|_{L^\infty L^2} \|r_h\|_{L^\infty L^2} \lesssim \Delta t \tag{C.4}
$$

for $r_h \in \{\varrho_h, \varrho_h u_{i,h}, \varrho_h s_h\}$. Collecting the above estimates of $E_j(r_h), j = 1, \dots, 4$, and (C.4) we get

$$\left|\int_0^\tau \int_{\mathbb{T}^d} r_h \boldsymbol{u}_h \cdot \nabla_x \phi \,\mathrm{d}x\mathrm{d}t\,\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathcal{E}} F_h^\varepsilon[r_h, \boldsymbol{u}_h] \left[\!\left[\Pi_Q \phi\right]\!\right] \,\mathrm{d}S_x \,\mathrm{d}t\right| \lesssim \Delta t + h + h^{1+\varepsilon} + c(r_h),$$
$$c(r_h) \lesssim \begin{cases} h^{(3-\varepsilon)/2} \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\right\|_{L^2L^2} \lesssim h^{1-\varepsilon} & \text{for } r_h = \varrho_h, \varrho_h s_h, \\ h \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\right\|_{L^2L^2} + h^2 \left\|\nabla_{\mathcal{E}} \boldsymbol{u}_h\right\|_{L^2L^2}^2 \lesssim h^{(1-\varepsilon)/2} + h^{1-\varepsilon} & \text{for } r_h = \varrho_h u_{i,h}. \end{cases} \tag{C.5}$$

### Step 3 – viscosity terms in the momentum equation

The interpolation error estimate (C.2) and the *a priori* bound (4.7a) are enough to control the consistency error of the viscosity terms in the momentum equation. Indeed, we have

$$\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_h(\Pi_Q \boldsymbol{\phi}) \,\mathrm{d}x \,\mathrm{d}t\right| \\
&\leq \left|\int_0^{t^{n+1}} \int_{\mathbb{T}^d} \mathbb{S}_h : \left(\nabla_x \boldsymbol{\phi} - \nabla_h(\Pi_Q \boldsymbol{\phi})\right) \mathrm{d}x \,\mathrm{d}t\right| + \left|-\int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x\right| \\
&\lesssim h \left\|\mathbb{S}_h\right\|_{L^2L^2} \left\|\nabla_x^2 \boldsymbol{\phi}\right\|_{L^\infty L^\infty} + \left|\int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} \nabla_h \boldsymbol{u}_h : \Pi_Q \mathbb{S}(\nabla_x \boldsymbol{\phi}) \,\mathrm{d}x\right| \\
&\lesssim h + \left|\int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot \mathrm{div}_h \Pi_Q \mathbb{S}(\nabla_x \boldsymbol{\phi}) \,\mathrm{d}x\right| \\
&\lesssim h + (t^{n+1} - \tau) \left\|\boldsymbol{u}_h\right\|_{L^\infty L^2} \left\|\boldsymbol{\phi}\right\|_{L^\infty W^{2,\infty}} \lesssim h + \Delta t
\end{aligned}$$

where we have used the fact that $\mathbb{S}_h$ is piecewise constant and the identity

$$\mathbb{S}_h : \nabla_x \boldsymbol{\phi} = \nabla_h \boldsymbol{u}_h : \mathbb{S}(\nabla_x \boldsymbol{\phi}) \tag{C.6}$$

due to the symmetry of $\mathbb{S}_h$ and $\mathbb{S}(\nabla_x \boldsymbol{\phi})$.

### Step 4 – pressure term in the momentum equation

The consistency error of the pressure term in the momentum equation is controlled, thanks to the interpolation estimate (C.2) and the uniform bound on pressure (4.6), and the triangular inequality.

$$\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{T}^d} p_h \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t - \int_0^{t^{n+1}} \int_{\mathbb{T}^d} p_h \mathrm{div}_h(\Pi_Q \boldsymbol{\phi}) \,\mathrm{d}x \,\mathrm{d}t\right| \\
&\leq \left|\int_0^{t^{n+1}} \int_{\mathbb{T}^d} p_h \left(\mathrm{div}_x \boldsymbol{\phi} - \mathrm{div}_h(\Pi_Q \boldsymbol{\phi})\right) \mathrm{d}x\right| + \left|\int_\tau^{t^{n+1}} \int_{\mathbb{T}^d} p_h \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t\right| \\
&\leq \left\|p_h\right\|_{L^\infty L^1} h \left\|\nabla_x^2 \boldsymbol{\phi}\right\|_{L^\infty L^\infty} + (t^{n+1} - \tau) \left\|p_h\right\|_{L^\infty L^1} \left\|\mathrm{div}_x \boldsymbol{\phi}\right\|_{L^\infty L^\infty} \lesssim \Delta t + h.
\end{aligned}$$

### Step 5 – $\kappa$-term in the entropy inequality

Using the product rule (C.3), i.e. $\left[\!\left[\frac{\Pi_Q\phi}{\vartheta_h}\right]\!\right] = [\![\Pi_Q\phi]\!]\left\{\!\left\{\frac{1}{\vartheta_h}\right\}\!\right\} + \{\!\{\Pi_Q\phi\}\!\}\left[\!\left[\frac{1}{\vartheta_h}\right]\!\right] = [\![\Pi_Q\phi]\!]\left\{\!\left\{\frac{1}{\vartheta_h}\right\}\!\right\} - \{\!\{\Pi_Q\phi\}\!\}\frac{[\![\vartheta_h]\!]}{\vartheta_h^{\text{out}}\vartheta_h}$, we can write the consistency error of the $\kappa$-term in the following way

$$
\begin{aligned}
&-\int_0^{t^{n+1}}\int_{\mathcal{E}}\frac{\kappa}{h}[\![\vartheta_h]\!]\left[\!\left[\frac{\Pi_Q\phi}{\vartheta_h}\right]\!\right]\mathrm{d}S_x\,\mathrm{d}t - \left(\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa\phi}{\vartheta_h^{\text{out}}\vartheta_h}|\nabla_{\mathcal{E}}\vartheta_h|^2\,\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa}{\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_x\phi\,\mathrm{d}x\mathrm{d}t\right)\\
&=\kappa\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\frac{\{\!\{\Pi_Q\phi\}\!\}}{\vartheta_h\vartheta_h^{\text{out}}}|\nabla_{\mathcal{E}}\vartheta_h|^2\,\mathrm{d}x\,\mathrm{d}t + \kappa\int_0^\tau\int_{\mathbb{T}^d}\frac{\{\!\{\Pi_Q\phi\}\!\}-\phi}{\vartheta_h\vartheta_h^{\text{out}}}|\nabla_{\mathcal{E}}\vartheta_h|^2\,\mathrm{d}x\mathrm{d}t\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa}{\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot(\nabla_x\phi-\nabla_{\mathcal{E}}\Pi_Q\phi)\,\mathrm{d}x\mathrm{d}t + \kappa\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{1}{\vartheta_h}-\left\{\!\left\{\frac{1}{\vartheta_h}\right\}\!\right\}\right)\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_{\mathcal{E}}\Pi_Q\phi\,\mathrm{d}x\mathrm{d}t\\
&\quad-\kappa\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\left\{\!\left\{\frac{1}{\vartheta_h}\right\}\!\right\}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_{\mathcal{E}}\Pi_Q\phi\,\mathrm{d}x\,\mathrm{d}t\\
&=:\kappa\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\frac{\{\!\{\Pi_Q\phi\}\!\}}{\vartheta_h\vartheta_h^{\text{out}}}|\nabla_{\mathcal{E}}\vartheta_h|^2\,\mathrm{d}x\,\mathrm{d}t+\sum_{i=1}^4 I_i.
\end{aligned}
$$

Note that $\kappa\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\frac{\{\!\{\Pi_Q\phi\}\!\}}{\vartheta_h\vartheta_h^{\text{out}}}|\nabla_{\mathcal{E}}\vartheta_h|^2\,\mathrm{d}x\,\mathrm{d}t\geq 0$ contributes to the " $\geq$ " sign of (4.10c). The $I_i$, $i=1,\dots,4$, terms contribute to the consistency error, and can be controlled by using Hölder's inequality, the uniform bounds (4.5) and (4.7), and the integration by part formula (3.3b), i.e.

$$
\begin{aligned}
&|I_1|\lesssim h\,\|\phi\|_{L^\infty W^{1,\infty}}\,\|\nabla_{\mathcal{E}}\vartheta_h\|_{L^2L^2}\lesssim h,\quad |I_2|\lesssim h\,\|\phi\|_{L^2W^{2,\infty}}\,\|\nabla_{\mathcal{E}}\vartheta_h\|_{L^2L^2}\lesssim h,\\
&|I_3|=\frac{h\kappa}{2}\left|\int_0^\tau\int_{\mathbb{T}^d}\frac{\nabla_{\mathcal{E}}\vartheta_h}{\vartheta_h^{\text{out}}\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_{\mathcal{E}}\Pi_Q\phi\,\mathrm{d}x\mathrm{d}t\right|\lesssim h\,\|\phi\|_{L^\infty W^{1,\infty}}\,\|\nabla_{\mathcal{E}}\vartheta_h\|_{L^2L^2}^2\lesssim h,\\
&|I_4|\lesssim\left|\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_{\mathcal{E}}\Pi_Q\phi\,\mathrm{d}x\,\mathrm{d}t\right|=\left|\int_\tau^{t^{n+1}}\int_{\mathbb{T}^d}\vartheta_h\cdot\Delta_h\Pi_Q\phi\,\mathrm{d}x\,\mathrm{d}t\right|\lesssim\Delta t\,\|\phi\|_{L^\infty W^{2,\infty}}\lesssim\Delta t.
\end{aligned}
$$

### Step 6 – entropy production terms in the entropy inequality

Thanks to $\vartheta_h>0$ and $\phi\geq 0$ the entropy production terms are non-negative, i.e.

$$
\begin{aligned}
&\int_0^{t^{n+1}}\int_{\mathbb{T}^d}D_1(\Pi_Q\phi)+D_2(\Pi_Q\phi)+D_3(\Pi_Q\phi)\,\mathrm{d}x\,\mathrm{d}t\geq 0,\\
&\int_0^{t^{n+1}}\int_{\mathbb{T}^d}\mathbb{S}_h:\nabla_h\boldsymbol{u}_h\frac{\Pi_Q\phi}{\vartheta_h}\,\mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_{\mathbb{T}^d}\mathbb{S}_h:\nabla_h\boldsymbol{u}_h\frac{\phi}{\vartheta_h}\,\mathrm{d}x\mathrm{d}t\,\mathrm{d}t\\
&=\int_\tau^{t_{n+1}}\int_{\mathbb{T}^d}\mathbb{S}_h:\nabla_h\boldsymbol{u}_h\frac{\phi}{\vartheta_h}\,\mathrm{d}x\,\mathrm{d}t=\int_\tau^{t_{n+1}}\int_{\mathbb{T}^d}\left(2\mu|\mathbb{D}_h(\boldsymbol{u}_h)|^2+\lambda|\mathrm{div}_h\boldsymbol{u}_h|^2\right)\frac{\phi}{\vartheta_h}\,\mathrm{d}x\,\mathrm{d}t\geq 0,
\end{aligned}
$$

which again contribute to the " $\geq$ " sign of (4.10c).

In addition, by (C.1), (4.7b) and Hölder's inequality, we estimate the residual term in the entropy balance (4.2)

$$
\left|\int_0^{t^{n+1}}\int_{\mathbb{T}^d}R_s(\Pi_Q\phi)\,\mathrm{d}x\right|\lesssim h^{1+\varepsilon}\,\|\nabla_x\phi\|_{L^\infty L^\infty}\int_0^\tau\int_{\mathcal{E}}([\![\varrho_h]\!]+[\![p_h]\!])\,\mathrm{d}S_x\,\mathrm{d}t\lesssim h^{(\varepsilon+1)/2}+h^{\varepsilon+1}.
$$

Finally, collecting the estimates of the six steps above, we obtain the consistency formulation (4.10a), (4.10b), and (4.10c) from (3.4a), (3.4b), and (4.2), respectively. This proves Lemma 4.5.

# D Relative energy inequality

In this section we derive the relative energy inequality (5.6). More precisely, we want to prove

$$\Big[R_E\Big((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\Big)\Big]_0^\tau+\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\widetilde{\vartheta}}{\vartheta_h}\left(2\mu\left|\mathbb{D}_h(\boldsymbol{u}_h)-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathbb{D}(\widetilde{\boldsymbol{u}})\right|^2+\lambda\left|\mathrm{div}_h\boldsymbol{u}_h-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathrm{div}_x\widetilde{\boldsymbol{u}}\right|^2\right)\right)\,\mathrm{d}x$$
$$+\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa\widetilde{\vartheta}}{\vartheta_h\vartheta_h^{\mathrm{out}}}|\nabla_{\mathcal{E}}\vartheta_h-\nabla_x\widetilde{\vartheta}|^2\,\mathrm{d}x\mathrm{d}t\le R_C+\sum_{i=1}^5R_i-R_{\mathbb{S}}-R_\vartheta,\tag{D.1}$$

where

$$R_C=e_\varrho\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2-\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho},\Delta t,h,\tau\right)+e_{\boldsymbol{u}}(\widetilde{\boldsymbol{u}},\Delta t,h,\tau)-e_s(\widetilde{\vartheta},\Delta t,h,\tau),$$
$$R_1=-\int_0^\tau\int_{\mathbb{T}^d}\varrho_h(s_h-\widetilde{s})(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})\cdot\nabla_x\widetilde{\vartheta}\,\mathrm{d}x\mathrm{d}t,\quad R_2=-\int_0^\tau\int_{\mathbb{T}^d}\varrho_h(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})\otimes(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}):\nabla_x\widetilde{\boldsymbol{u}}\,\mathrm{d}x,$$
$$R_3=\int_0^\tau\int_{\mathbb{T}^d}\frac{\varrho_h-\widetilde{\varrho}}{\widetilde{\varrho}}(\widetilde{\boldsymbol{u}}-\boldsymbol{u}_h)\cdot\mathrm{div}_x\widetilde{\mathbb{S}}\,\mathrm{d}x,\quad R_4=\int_0^\tau\int_{\mathbb{T}^d}(\widetilde{p}-p_h-\partial_\varrho\widetilde{p}(\widetilde{\varrho}-\varrho_h)-\partial_\vartheta\widetilde{p}(\widetilde{\vartheta}-\vartheta_h))\mathrm{div}_x\widetilde{\boldsymbol{u}}\,\mathrm{d}x\mathrm{d}t,$$
$$R_5=-\int_0^\tau\int_{\mathbb{T}^d}\Big((\varrho_h-\widetilde{\varrho})(s_h-\widetilde{s})+\widetilde{\varrho}\big(s_h-\widetilde{s}-\partial_\varrho\widetilde{s}(\varrho_h-\widetilde{\varrho})-\partial_\vartheta\widetilde{s}(\vartheta_h-\widetilde{\vartheta})\big)\Big)(\partial_t\widetilde{\vartheta}+\widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\vartheta})\,\mathrm{d}x\mathrm{d}t,$$
$$R_{\mathbb{S}}=\int_0^\tau\int_{\mathbb{T}^d}\Big(\boldsymbol{u}_h\cdot\mathrm{div}_x\widetilde{\mathbb{S}}+\nabla_h\boldsymbol{u}_h:\widetilde{\mathbb{S}}\Big)\,\mathrm{d}x\mathrm{d}t,$$
$$R_\vartheta=\kappa\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\vartheta_h^2\vartheta_h^{\mathrm{out}}-\widetilde{\vartheta}^3}{\widetilde{\vartheta}^2\vartheta_h\vartheta_h^{\mathrm{out}}}|\nabla_x\widetilde{\vartheta}|^2+\vartheta_h\mathrm{div}_x\frac{\nabla_x\widetilde{\vartheta}}{\widetilde{\vartheta}}+\frac{2\widetilde{\vartheta}-\vartheta_h^{\mathrm{out}}}{\vartheta_h\vartheta_h^{\mathrm{out}}}\nabla_x\widetilde{\vartheta}\cdot\nabla_{\mathcal{E}}\vartheta_h\right)\,\mathrm{d}x\mathrm{d}t.\tag{D.2}$$

**Step 1.** Our first goal is to obtain a relative energy inequality directly from the consistency formulation (4.10) and the energy balance (4.1). We reformulate the relative energy as

$$R_E\Big((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\Big)=\sum_{i=1}^4T_i,\tag{D.3}$$

where

$$T_1=\int_{\mathbb{T}^d}\left(\frac{1}{2}\varrho_h|\boldsymbol{u}_h|^2+\mathcal{H}_{\widetilde{\vartheta}}(\varrho_h,\boldsymbol{u}_h)\right)\,\mathrm{d}x,\quad T_2=\int_{\mathbb{T}^d}\varrho_h\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2-\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho}\right)\,\mathrm{d}x,$$
$$T_3=-\int_{\mathbb{T}^d}\varrho_h\boldsymbol{u}_h\cdot\widetilde{\boldsymbol{u}}\,\mathrm{d}x,\quad T_4=\int_{\mathbb{T}^d}\left(\widetilde{\varrho}\frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho}-\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})\right)\,\mathrm{d}x=\int_{\mathbb{T}^d}\widetilde{p}\,\mathrm{d}x.$$

Hence, for the first term $T_1$, by integrating (4.1) from $t=0$ to $t=\tau$ and subtracting (4.10c) with $\phi=\widetilde{\vartheta}$, we obtain

$$\begin{aligned}[T_1]_{t=0}^{\tau} \leq & -\int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa\widetilde{\vartheta}}{\vartheta_h^{\text{out}}\vartheta_h} |\nabla_{\mathcal{E}}\vartheta_h|^2 \,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}_h : \nabla_h \boldsymbol{u}_h \frac{\widetilde{\vartheta}}{\vartheta_h} \,\mathrm{d}x\mathrm{d}t \\ & - \int_0^\tau \int_{\mathbb{T}^d} \varrho_h s_h (\partial_t\widetilde{\vartheta} + \boldsymbol{u}_h\cdot\nabla_x\widetilde{\vartheta}) \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \frac{\kappa}{\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_x\widetilde{\vartheta} \,\mathrm{d}x\mathrm{d}t - e_s\left(\widetilde{\vartheta}, \Delta t, h, \tau\right).\end{aligned} \tag{D.4}$$

For the second term $T_2$, by taking $\phi = \frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - \frac{\partial \mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial \varrho}$ in (4.10a) we obtain

$$\begin{aligned}[T_2]_{t=0}^{\tau} = & \int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h \partial_t\widetilde{\boldsymbol{u}}\cdot\widetilde{\boldsymbol{u}} + \varrho_h\widetilde{\boldsymbol{u}}\otimes\boldsymbol{u}_h : \nabla_x\widetilde{\boldsymbol{u}}\right) \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \varrho_h\widetilde{s}(\partial_t\widetilde{\vartheta} + \boldsymbol{u}_h\cdot\nabla_x\widetilde{\vartheta})\,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^\tau \int_{\mathbb{T}^d} \frac{\varrho_h}{\widetilde{\varrho}}(\partial_t\widetilde{p} + \boldsymbol{u}_h\cdot\nabla_x\widetilde{p})\,\mathrm{d}x\,\mathrm{d}t + e_\varrho\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - \frac{\partial\mathcal{H}_{\widetilde{\vartheta}}(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho}, \Delta t, h, \tau\right).\end{aligned} \tag{D.5}$$

For the third term $T_3$, by setting $\boldsymbol{\phi} = \widetilde{\boldsymbol{u}}$ in (4.10b) we obtain

$$\begin{aligned}[T_3]_{t=0}^{\tau} = & -\int_0^\tau \int_{\mathbb{T}^d} \left(\varrho_h\boldsymbol{u}_h\cdot\partial_t\widetilde{\boldsymbol{u}} + \varrho_h\boldsymbol{u}_h\otimes\boldsymbol{u}_h : \nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t \\ & + \int_0^\tau \int_{\mathbb{T}^d} (\mathbb{S}_h - p_h\mathbb{I}) : \nabla_x\widetilde{\boldsymbol{u}}\,\mathrm{d}x\mathrm{d}t - e_{\boldsymbol{m}}\left(\widetilde{\boldsymbol{u}}, \Delta t, h, \tau\right).\end{aligned} \tag{D.6}$$

For the fourth the term $T_4$ we simply write

$$[T_4]_{t=0}^{\tau} = \int_0^\tau \int_{\mathbb{T}^d} \partial_t\widetilde{p}\,\mathrm{d}x\mathrm{d}t. \tag{D.7}$$

Now, summing up (D.4)–(D.7) we obtain from $\left[R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right)\right]_0^\tau = \sum_{i=1}^4 [T_i]_{t=0}^\tau$ that

$$\left[R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right)\right]_0^\tau + \Xi_l \leq R_C - \int_0^\tau\int_{\mathbb{T}^d} \varrho_h(s_h-\widetilde{s})(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})\cdot\nabla_x\widetilde{\vartheta}\,\mathrm{d}x\mathrm{d}t + \Xi_r \tag{D.8}$$

with

$$\begin{aligned}\Xi_l = & \int_0^\tau\int_{\mathbb{T}^d}\left(\mathbb{S}_h : \nabla_h\boldsymbol{u}_h\frac{\widetilde{\vartheta}}{\vartheta_h} - \mathbb{S}_h : \nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\kappa\widetilde{\vartheta}}{\vartheta_h^{\text{out}}\vartheta_h}|\nabla_{\mathcal{E}}\vartheta_h|^2 - \frac{\kappa}{\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_x\widetilde{\vartheta}\right)\mathrm{d}x\mathrm{d}t, \\ \Xi_r = & \int_0^\tau\int_{\mathbb{T}^d}\left(\varrho_h(\widetilde{\boldsymbol{u}}-\boldsymbol{u}_h)\cdot\partial_t\widetilde{\boldsymbol{u}} + \varrho_h(\widetilde{\boldsymbol{u}}-\boldsymbol{u}_h)\otimes\boldsymbol{u}_h : \nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\mathbb{T}^d}\varrho_h(\widetilde{s}-s_h)(\partial_t\widetilde{\vartheta} + \widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\vartheta})\,\mathrm{d}x\mathrm{d}t \\ & + \int_0^\tau\int_{\mathbb{T}^d}\left(\partial_t\widetilde{p} - p_h\mathrm{div}_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t\,\mathrm{d}t - \int_0^\tau\int_{\mathbb{T}^d}\frac{\varrho_h}{\widetilde{\varrho}}(\partial_t\widetilde{p} + \boldsymbol{u}_h\cdot\nabla_x\widetilde{p})\,\mathrm{d}x.\end{aligned}$$

We point out that the derivative-terms of $\Xi_r$ are the first order derivative-terms of the exact solution $(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})$ so that all of them could be bounded by some constants.

**Step 2.** The second goal is to reformulate $\Xi_r$ into the difference between $(\varrho_h, \boldsymbol{u}_h, \vartheta_h)$ and $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$ so that it can be controlled by the relative energy with (5.2).

The key is to utilize that $(\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta})$ is the strong solution satisfying

$$\begin{aligned}
&\partial_t \widetilde{\varrho} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\varrho} = -\widetilde{\varrho} \mathrm{div}_x \widetilde{\boldsymbol{u}}, \quad \widetilde{\varrho}(\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}}) + \nabla_x \widetilde{p} = \mathrm{div}_x \widetilde{\mathbb{S}}, \\
&\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta} + \frac{\widetilde{\vartheta}}{c_v} \mathrm{div}_x \widetilde{\boldsymbol{u}} = \frac{1}{c_v \widetilde{\varrho}} \left( \widetilde{\mathbb{S}} : \nabla_x \widetilde{\boldsymbol{u}} + \frac{\kappa |\nabla_x \widetilde{\vartheta}|^2}{\widetilde{\vartheta}} + \vartheta \mathrm{div}_x \left( \frac{\kappa \nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} \right) \right).
\end{aligned}$$

Firstly, with

$$\begin{aligned}
&\int_{\mathbb{T}^d} (\varrho_h(\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot \partial_t \widetilde{\boldsymbol{u}} + \varrho_h(\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \otimes \boldsymbol{u}_h : \nabla_x \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \\
= &\int_{\mathbb{T}^d} \varrho_h(\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot (\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}}) \, \mathrm{d}x - \int_{\mathbb{T}^d} \varrho_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \otimes (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) : \nabla_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x \\
= &\int_{\mathbb{T}^d} \frac{\varrho_h}{\widetilde{\varrho}} (\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot (\mathrm{div}_x \mathbb{S}(\nabla_x \widetilde{\boldsymbol{u}}) - \nabla_x \widetilde{p}) \, \mathrm{d}x - \int_{\mathbb{T}^d} \varrho_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \otimes (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) : \nabla_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x
\end{aligned}$$

and $\int_{\mathbb{T}^d} (\widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{p} + \widetilde{p} \mathrm{div}_x \widetilde{\boldsymbol{u}}) \, \mathrm{d}x = 0$, we obtain

$$\Xi_r = -\int_0^\tau \int_{\mathbb{T}^d} \varrho_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) \otimes (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}) : \nabla_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \frac{\varrho_h}{\widetilde{\varrho}} (\widetilde{\boldsymbol{u}} - \boldsymbol{u}_h) \cdot \mathrm{div}_x \widetilde{\mathbb{S}} \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \Xi_{r,1} \, \mathrm{d}x\mathrm{d}t,$$

$$\Xi_{r,1} = \varrho_h(\widetilde{s} - s_h)(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) + \frac{\widetilde{\varrho} - \varrho_h}{\widetilde{\varrho}} (\partial_t \widetilde{p} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{p}) + (\widetilde{p} - p_h) \mathrm{div}_x \widetilde{\boldsymbol{u}}.$$

Secondly, applying the product rule for $\widetilde{p}$ and $\widetilde{s}$ as functions of $(\widetilde{\varrho}, \widetilde{\vartheta})$

$$\begin{aligned}
&(\partial_t \widetilde{p} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{p}) = (\partial_t \widetilde{\varrho} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\varrho}) \partial_\varrho \widetilde{p} + (\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) \partial_\vartheta \widetilde{p} = -\widetilde{\varrho} \mathrm{div}_x \widetilde{\boldsymbol{u}} \partial_\varrho \widetilde{p} + \widetilde{\varrho}(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) \\
&\varrho_h(s_h - \widetilde{s}) = (\varrho_h - \widetilde{\varrho})(s_h - \widetilde{s}) + \widetilde{\varrho}(s_h - \widetilde{s} - \partial_\varrho \widetilde{s}(\varrho_h - \widetilde{\varrho}) - \partial_\vartheta \widetilde{s}(\vartheta_h - \widetilde{\vartheta})) + \widetilde{\varrho}(\partial_\varrho \widetilde{s}(\varrho_h - \widetilde{\varrho}) + \partial_\vartheta \widetilde{s}(\vartheta_h - \widetilde{\vartheta}))
\end{aligned}$$

we have

$$\begin{aligned}
\Xi_{r,1} = &\Big( \widetilde{p} - p_h - \partial_\varrho \widetilde{p}(\widetilde{\varrho} - \varrho_h) \Big) \mathrm{div}_x \widetilde{\boldsymbol{u}} + \Big( \varrho_h(\widetilde{s} - s_h) + (\widetilde{\varrho} - \varrho_h) \Big) (\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) \\
= &\Big( \widetilde{p} - p_h - \partial_\varrho \widetilde{p}(\widetilde{\varrho} - \varrho_h) - \partial_\vartheta \widetilde{p}(\widetilde{\vartheta} - \vartheta_h) \Big) \mathrm{div}_x \widetilde{\boldsymbol{u}} \\
&- \Big( (\varrho_h - \widetilde{\varrho})(s_h - \widetilde{s}) + \widetilde{\varrho}(s_h - \widetilde{s} - \partial_\varrho \widetilde{s}(\varrho_h - \widetilde{\varrho}) - \partial_\vartheta \widetilde{s}(\vartheta_h - \widetilde{\vartheta})) \Big) (\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) \\
&- (\vartheta_h - \widetilde{\vartheta}) \Big( \widetilde{\varrho} \partial_\vartheta \widetilde{s}(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) + \partial_\vartheta \widetilde{p} \mathrm{div}_x \widetilde{\boldsymbol{u}} \Big).
\end{aligned}$$

With $\partial_\vartheta \widetilde{s} = \frac{c_v}{\widetilde{\vartheta}}, \partial_\vartheta \widetilde{p} = \widetilde{\varrho}$, the last term above can be rewritten as

$$-(\vartheta_h - \widetilde{\vartheta}) \Big( \widetilde{\varrho} \partial_\vartheta \widetilde{s}(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta}) + \partial_\vartheta \widetilde{p} \mathrm{div}_x \widetilde{\boldsymbol{u}} \Big) = -\frac{\vartheta_h - \widetilde{\vartheta}}{\widetilde{\vartheta}} \left( \widetilde{\mathbb{S}} : \nabla_x \widetilde{\boldsymbol{u}} + \frac{\kappa |\nabla_x \widetilde{\vartheta}|^2}{\widetilde{\vartheta}} + \widetilde{\vartheta} \mathrm{div}_x \left( \frac{\kappa \nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} \right) \right).$$

Thanks to

$$\int_{\mathbb{T}^d}\left(\widetilde{\boldsymbol{u}}\cdot\mathrm{div}_x\widetilde{\mathbb{S}}+\widetilde{\mathbb{S}}:\nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x=0,\quad \int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa|\nabla_x\widetilde{\vartheta}|^2}{\widetilde{\vartheta}}+\widetilde{\vartheta}\mathrm{div}_x\left(\frac{\kappa\nabla_x\widetilde{\vartheta}}{\widetilde{\vartheta}}\right)\mathrm{d}x\mathrm{d}t=0,$$

finally we reformulate $\Xi_r$ as

$$\begin{aligned}\Xi_r=&-\int_0^\tau\int_{\mathbb{T}^d}\varrho_h(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})\otimes(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}):\nabla_x\widetilde{\boldsymbol{u}}\,\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^d}\frac{\varrho_h-\widetilde{\varrho}}{\widetilde{\varrho}}(\widetilde{\boldsymbol{u}}-\boldsymbol{u}_h)\cdot\mathrm{div}_x\widetilde{\mathbb{S}}\,\mathrm{d}x\mathrm{d}t\\&+\int_0^\tau\int_{\mathbb{T}^d}\Big(\widetilde{p}-p_h-\partial_\varrho\widetilde{p}(\widetilde{\varrho}-\varrho_h)-\partial_\vartheta\widetilde{p}(\widetilde{\vartheta}-\vartheta_h)\Big)\mathrm{div}_x\widetilde{\boldsymbol{u}}\,\mathrm{d}x\mathrm{d}t\\&-\int_0^\tau\int_{\mathbb{T}^d}\Big((\varrho_h-\widetilde{\varrho})(s_h-\widetilde{s})+\widetilde{\varrho}(s_h-\widetilde{s}-\partial_\varrho\widetilde{s}(\varrho_h-\widetilde{\varrho})-\partial_\vartheta\widetilde{s}(\vartheta_h-\widetilde{\vartheta}))\Big)(\partial_t\widetilde{\vartheta}+\widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\vartheta})\,\mathrm{d}x\mathrm{d}t-\Xi_{r,1}\\\Xi_{r,1}=&\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\vartheta_h}{\widetilde{\vartheta}}\widetilde{\mathbb{S}}:\nabla_x\widetilde{\boldsymbol{u}}+\boldsymbol{u}_h\cdot\mathrm{div}_x\widetilde{\mathbb{S}}\right)\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^d}\frac{\vartheta_h}{\widetilde{\vartheta}}\left(\frac{\kappa|\nabla_x\widetilde{\vartheta}|^2}{\widetilde{\vartheta}}+\widetilde{\vartheta}\mathrm{div}_x\left(\frac{\kappa\nabla_x\widetilde{\vartheta}}{\widetilde{\vartheta}}\right)\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

**Step 3.** Back to the relative energy inequality (D.8), the rest is to reformulate $\Xi_l+\Xi_{r,1}$. We split $\Xi_l+\Xi_{r,1}$ into the velocity-gradient-terms and temperature-gradient-terms:

$$\begin{aligned}&\Xi_l+\Xi_{r,1}=T_{\mathbb{S}}+T_\vartheta,\\&T_{\mathbb{S}}=\int_0^\tau\int_{\mathbb{T}^d}\left(\mathbb{S}_h:\nabla_h\boldsymbol{u}_h\frac{\widetilde{\vartheta}}{\vartheta_h}-\mathbb{S}_h:\nabla_x\widetilde{\boldsymbol{u}}+\frac{\vartheta_h}{\widetilde{\vartheta}}\widetilde{\mathbb{S}}:\nabla_x\widetilde{\boldsymbol{u}}+\boldsymbol{u}_h\cdot\mathrm{div}_x\widetilde{\mathbb{S}}\right)\mathrm{d}x\mathrm{d}t,\\&T_\vartheta=\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa\widetilde{\vartheta}}{\vartheta_h^{\mathrm{out}}\vartheta_h}|\nabla_{\mathcal{E}}\vartheta_h|^2-\frac{\kappa}{\vartheta_h}\nabla_{\mathcal{E}}\vartheta_h\cdot\nabla_x\widetilde{\vartheta}+\frac{\vartheta_h}{\widetilde{\vartheta}}\left(\frac{\kappa|\nabla_x\widetilde{\vartheta}|^2}{\widetilde{\vartheta}}+\widetilde{\vartheta}\mathrm{div}_x\left(\frac{\kappa\nabla_x\widetilde{\vartheta}}{\widetilde{\vartheta}}\right)\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

**Term $T_{\mathbb{S}}$:** Denoting $R_{\mathbb{S}}=\int_0^\tau\int_{\mathbb{T}^d}\left(\boldsymbol{u}_h\cdot\mathrm{div}_x\widetilde{\mathbb{S}}+\mathbb{S}_h:\nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t$, we have

$$\begin{aligned}T_{\mathbb{S}}&=\int_0^\tau\int_{\mathbb{T}^d}\left(\frac{\widetilde{\vartheta}}{\vartheta_h}\mathbb{S}_h:\nabla_h\boldsymbol{u}_h-2\mathbb{S}_h:\nabla_x\widetilde{\boldsymbol{u}}+\frac{\vartheta_h}{\widetilde{\vartheta}}\widetilde{\mathbb{S}}:\nabla_x\widetilde{\boldsymbol{u}}\right)\mathrm{d}x\mathrm{d}t+R_{\mathbb{S}}\\&=\int_0^\tau\int_{\mathbb{T}^d}\frac{\widetilde{\vartheta}}{\vartheta_h}\left(2\mu\left|\mathbb{D}_h(\boldsymbol{u}_h)-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathbb{D}(\widetilde{\boldsymbol{u}})\right|^2+\lambda\left|\mathrm{div}_h\boldsymbol{u}_h-\frac{\vartheta_h}{\widetilde{\vartheta}}\mathrm{div}_x\widetilde{\boldsymbol{u}}\right|^2\right)\mathrm{d}x\mathrm{d}t+R_{\mathbb{S}}.\end{aligned}$$

**Term $T_\vartheta$:** We straightforward reformulate $T_\vartheta$ as

$$R_\vartheta=T_\vartheta-\int_0^\tau\int_{\mathbb{T}^d}\frac{\kappa\widetilde{\vartheta}}{\vartheta_h\vartheta_h^{\mathrm{out}}}|\nabla_{\mathcal{E}}\vartheta_h-\nabla_x\widetilde{\vartheta}|^2\,\mathrm{d}x\mathrm{d}t.$$

Finally, collecting the above and going back to (D.8) we obtain the relative energy inequality (D.1).

# E Estimates on $R_{\mathbb{S}}$ and $R_\vartheta$

In this part we prove (5.8), namely the following estimate.

$$|R_{\mathbb{S}}|\overset{<}{\sim}h\left\|\nabla_{\mathcal{E}}\boldsymbol{u}_h\right\|_{L^2L^2},|R_\vartheta|\overset{<}{\sim}h+\int_0^\tau R_E\left((\varrho_h,\boldsymbol{u}_h,\vartheta_h)\mid(\widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{\vartheta})\right)(t)\,\mathrm{d}t+\delta\left\|\nabla_{\mathcal{E}}\vartheta_h-\nabla_x\widetilde{\vartheta}\right\|_{L^2L^2}^2.$$

**Estimate of $R_{\mathbb{S}}$** Noticing the fact that $\nabla_h \boldsymbol{u}_h$ is piecewise constant, we have

$$
\begin{aligned}
R_{\mathbb{S}} &= \int_0^\tau \int_{\mathbb{T}^d} \left( \boldsymbol{u}_h \cdot \mathrm{div}_x \widetilde{\mathbb{S}} + \nabla_h \boldsymbol{u}_h : \Pi_Q \widetilde{\mathbb{S}} \right) \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\mathbb{T}^d} \boldsymbol{u}_h \cdot (\mathrm{div}_x \widetilde{\mathbb{S}} - \mathrm{div}_h \Pi_Q \widetilde{\mathbb{S}}) \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \sum_K \boldsymbol{u}_h|_K \cdot \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \left( \widetilde{\mathbb{S}} - \left\{\!\!\left\{ \Pi_Q \widetilde{\mathbb{S}} \right\}\!\!\right\} \right) \cdot \boldsymbol{n} \, \mathrm{d}S_x \, \mathrm{d}t \\
&= - \int_0^\tau \int_{\mathcal{E}} [\![ \boldsymbol{u}_h ]\!] \cdot \left( \widetilde{\mathbb{S}} - \left\{\!\!\left\{ \Pi_Q \widetilde{\mathbb{S}} \right\}\!\!\right\} \right) \cdot \boldsymbol{n} \, \mathrm{d}S_x \, \mathrm{d}t \lesssim h \, \|\widetilde{\boldsymbol{u}}\|_{L^\infty W^{2,\infty}} \int_0^\tau \int_{\mathcal{E}} |\, [\![ \boldsymbol{u}_h ]\!] \,| \, \mathrm{d}S_x \, \mathrm{d}t \\
&\lesssim h \left( \int_0^\tau \int_{\mathcal{E}} |\, [\![ \boldsymbol{u}_h ]\!] \,|^2 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} \left( \int_0^\tau \int_{\mathcal{E}} 1 \, \mathrm{d}S_x \, \mathrm{d}t \right)^{1/2} = h \, \|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|_{L^2 L^2} .
\end{aligned}
$$

**Estimate of $R_\vartheta$** We start with the reformulation $R_\vartheta = \sum_{i=1}^5 R_{\vartheta,i}$, where

$$
\begin{aligned}
R_{\vartheta,1} =& \kappa \int_0^\tau \int_{\mathbb{T}^d} \left( \vartheta_h \mathrm{div}_x \left( \frac{\nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} \right) + \nabla_{\mathcal{E}} \vartheta_h \cdot \frac{\nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} \right) \mathrm{d}x\mathrm{d}t, \\
R_{\vartheta,2} =& \kappa \int_0^\tau \int_{\mathbb{T}^d} \left( \frac{\vartheta_h}{\widetilde{\vartheta}} - \frac{\widetilde{\vartheta}}{\vartheta_h} \right) \left( \frac{1}{\widetilde{\vartheta}} - \frac{1}{\vartheta_h^{\mathrm{out}}} \right) |\nabla_x \widetilde{\vartheta}|^2 \, \mathrm{d}x\mathrm{d}t \, \mathrm{d}t, \\
R_{\vartheta,3} =& \kappa \int_0^\tau \int_{\mathbb{T}^d} \frac{1}{\vartheta_h^{\mathrm{out}}} \left( \frac{\vartheta_h}{\widetilde{\vartheta}} - \frac{\widetilde{\vartheta}}{\vartheta_h} \right) \left( \nabla_x \widetilde{\vartheta} - \nabla_{\mathcal{E}} \vartheta_h \right) \cdot \nabla_x \widetilde{\vartheta} \, \mathrm{d}x\mathrm{d}t, \\
R_{\vartheta,4} =& \kappa \int_0^\tau \int_{\mathbb{T}^d} \frac{\widetilde{\vartheta}}{\vartheta_h} \left( \frac{1}{\widetilde{\vartheta}} - \frac{1}{\vartheta_h^{\mathrm{out}}} \right) (\nabla_x \widetilde{\vartheta} - \nabla_{\mathcal{E}} \vartheta_h) \cdot \nabla_x \widetilde{\vartheta} \, \mathrm{d}x\mathrm{d}t, \\
R_{\vartheta,5} =& \kappa \int_0^\tau \int_{\mathbb{T}^d} \frac{1}{\widetilde{\vartheta}} \left( \frac{\vartheta_h}{\vartheta_h^{\mathrm{out}}} - 1 \right) \nabla_{\mathcal{E}} \vartheta_h \cdot \nabla_x \widetilde{\vartheta} \, \mathrm{d}x\mathrm{d}t = -h\kappa \int_0^\tau \int_{\mathbb{T}^d} \frac{1}{\widetilde{\vartheta} \vartheta_h^{\mathrm{out}}} |\nabla_{\mathcal{E}} \vartheta_h|^2 \nabla_x \widetilde{\vartheta} \, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Hence, with (3.3b), the assumption (4.5), the projection error (3.1) and the a priori bound on temperature (4.7a) we have

$$
\begin{aligned}
&|R_{\vartheta,1}| = \kappa \left| \int_0^\tau \int_{\mathbb{T}^d} \nabla_{\mathcal{E}} \vartheta_h \cdot \left( \frac{\nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} - \Pi_W \left( \frac{\nabla_x \widetilde{\vartheta}}{\widetilde{\vartheta}} \right) \right) \mathrm{d}x\mathrm{d}t \right| \lesssim h \, \|\nabla_{\mathcal{E}} \vartheta_h\|_{L^2 L^2} \left\| \widetilde{\vartheta} \right\|_{L^2 W^{2,\infty}}, \\
&|R_{\vartheta,2}| \lesssim \left\| \vartheta_h - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + \left\| \vartheta_h^{\mathrm{out}} - \widetilde{\vartheta} \right\|_{L^2 L^2}^2, \quad |R_{\vartheta,3}| \lesssim \frac{1}{\delta} \left\| \vartheta_h - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + \delta \left\| \nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta} \right\|_{L^2 L^2}^2, \\
&|R_{\vartheta,4}| \lesssim \frac{1}{\delta} \left\| \vartheta_h^{\mathrm{out}} - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + \delta \left\| \nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta} \right\|_{L^2 L^2}^2, \quad |R_{\vartheta,5}| \lesssim h \, \|\nabla_{\mathcal{E}} \vartheta_h\|_{L^2 L^2}^2 \lesssim h.
\end{aligned}
$$

Moreover, $\left\| \vartheta_h^{\mathrm{out}} - \widetilde{\vartheta} \right\|_{L^2 L^2}$ can be controlled as

$$
\left\| \vartheta_h^{\mathrm{out}} - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 \le \left\| \vartheta_h - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + \left\| \vartheta_h^{\mathrm{out}} - \vartheta_h^{\mathrm{in}} \right\|_{L^2 L^2}^2 = \left\| \vartheta_h - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + h^2 \, \|\nabla_{\mathcal{E}} \vartheta_h\|_{L^2 L^2}^2 \lesssim \left\| \vartheta_h - \widetilde{\vartheta} \right\|_{L^2 L^2}^2 + h^2.
$$

Consequently, with (5.2) we have

$$
|R_\vartheta| \lesssim h + \int_0^\tau R_E \left( (\varrho_h, \boldsymbol{u}_h, \vartheta_h) \mid (\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{\vartheta}) \right) (t) \, \mathrm{d}t + \delta \left\| \nabla_{\mathcal{E}} \vartheta_h - \nabla_x \widetilde{\vartheta} \right\|_{L^2 L^2}^2 .
$$

Collecting the above estimates, we complete the proof.