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ISOMETRIES OF COMBINATORIAL TSIRELSON SPACES

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Abstract. We extend existing results that characterize isometries on the Tsirelson-type spaces $T\big[\frac{1}{n}, \mathcal{S}_1\big]$ $(n \in \mathbb{N}, n \geqslant 2)$ to the class $T[\theta, \mathcal{S}_\alpha]$ $\big(\theta \in \big(0, \frac{1}{2}\big], 1 \leqslant \alpha < \omega_1\big)$, where $\mathcal{S}_\alpha$ denote the Schreier families of order α. We prove that every isometry on $T[\theta, \mathcal{S}_1]$ $\big(\theta \in \big(0, \frac{1}{2}\big]\big)$ is determined by a permutation of the first $\lceil \theta^{-1} \rceil$ elements of the canonical unit basis followed by a possible sign-change of the corresponding coordinates together with a sign-change of the remaining coordinates. Moreover, we show that for the spaces $T[\theta, \mathcal{S}_\alpha]$ $\big(\theta \in \big(0, \frac{1}{2}\big], 2 \leqslant \alpha < \omega_1\big)$ the isometries exhibit a more rigid character, namely, they are all implemented by a sign-change operation of the vector coordinates.

1. Introduction and the main result

The well-known Tsirelson space T (in the setting of Figiel and Johnson [4], i.e., the dual of the space constructed by Tsirelson [6], the first example of a space containing no isomorphic copies of c_0 or ℓ_p for $1 \leqslant p < \infty$) may be regarded as special instance of a space from a double-parameter family of Banach spaces $T[\theta, \mathcal{S}_\alpha]$ $\big(\theta \in \big(0, \frac{1}{2}\big], 1 \leqslant \alpha < \omega_1\big)$, where α is a countable ordinal and S_α is the Schreier family of order α. For brevity, we call members of this family *combinatorial Tsirelson spaces*, which appears to be in line with the terminology used, e.g., in [2]. (These are, of course, special cases of the so-called mixed Tsirelson spaces whose definition allows the parameter θ to vary, but by employing this name we want to emphasize the underlying family of sets rather than the numeric parameter.)

The aim of this paper is to delineate the structure of isometries on combinatorial Tsirelson spaces. We refer to the recent excellent survey [1] for further results and references concerning the problem of characerization of isometries on Banach (sequence) spaces.

In [1, Theorem 4.1] the authors provide a characterization of (linear) isometries of the spaces $T\big[\frac{1}{n}, \mathcal{S}_1\big]$ for $n \in \mathbb{N}$, $n \geqslant 2$, which we take as a departure point for our considerations

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and extend it to the whole scale of spaces $T\big[\theta, \mathcal{S}_1\big]$ $\big(\theta \in \big(0, \frac{1}{2}\big]\big)$. Let us then record the first main result. (In this paper, all considered Banach spaces are real; by an *isometry* we understand a linear isometry.)

Theorem A. Let $\theta \in \big(0, \frac{1}{2}\big]$. If $U\colon T\big[\theta, \mathcal{S}_1\big] \to T\big[\theta, \mathcal{S}_1\big]$ is an isometry, then

$$Ue_i = \begin{cases} \varepsilon_i e_{\pi(i)}, & 1 \leqslant i \leqslant \lceil \theta^{-1} \rceil \\ \varepsilon_i e_i, & i > \lceil \theta^{-1} \rceil \end{cases} \qquad (i \in \mathbb{N})$$

for some $\{-1, 1\}$-valued sequence $(\varepsilon_i)_{i=1}^{\infty}$ and a permutation π of $\big\{1, 2, \dots, \lceil \theta^{-1} \rceil\big\}$.

(Here $(e_i)_{i=1}^{\infty}$ is the standard unit vector basis of $T\big[\theta, \mathcal{S}_1\big]$ and $\lceil \theta^{-1} \rceil$ is the ceil of θ^{-1}, i.e., the least integer that θ^{-1} does not exceed.)

Then we answer the question from [1]. Indeed, we characterize the linear isometries of the spaces $T[\theta, \mathcal{S}_\alpha]$ $(\theta \in \big(0, \frac{1}{2}\big], 2 \leqslant \alpha < \omega_1)$ by proving the following second main result:

Theorem B. Let $\theta \in \big(0, \frac{1}{2}\big]$ and let $\alpha \geqslant 2$ be a countable ordinal. Then an operator $U\colon T[\theta, \mathcal{S}_\alpha] \to T[\theta, \mathcal{S}_\alpha]$ is an isometry if and only if $Ue_i = \varepsilon_i e_i$ for $i \in \mathbb{N}$ and some $\{-1, 1\}$-valued sequence $(\varepsilon_i)_{i=1}^{\infty}$.

Let us record the following observation that we draw directly from the proofs of Theorems A–B, which may be of independent interest.

Remark 1. Every isometry on $T[\theta, \mathcal{S}_\alpha]$ $\big(\theta \in \big(0, \frac{1}{2}\big], 1 \leqslant \alpha < \omega_1\big)$ is surjective.

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2. Preliminaries

2.1. **Combinatorial spaces.** We will denote by $(e_i)_{i=1}^{\infty}$ the standard unit vector basis of c_{00} and by $[\mathbb{N}]^{<\omega}$ the family of finite subsets of $\mathbb{N}$. For the sets $F_1, F_2 \in [\mathbb{N}]^{<\omega}$ we use the following notation: $F_1 < F_2$, whenever $\max F_1 < \min F_2$ and in such case we say that these sets are *consecutive*. Moreover, for $n \in \mathbb{N}$, we write $F_1 < n$ instead of $F_1 < \{n\}$.

Definition 2. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *regular*, whenever it is simultaneously

- *hereditary* $\big(F \in \mathcal{F}$ and $G \subset F \implies G \in \mathcal{F}\big)$;
- *spreading* $\big(\{l_1, l_2, \dots, l_n\} \in \mathcal{F}$ and $l_i \leqslant k_i \implies \{k_1, k_2, \dots, k_n\} \in \mathcal{F}\big)$;
- *compact* as a subset of the Cantor set $\{0, 1\}^{\mathbb{N}}$ via the natural identification of $F \in \mathcal{F}$ with
$$\chi_F = \sum_{i \in F} e_i \in \{0, 1\}^{\mathbb{N}}.$$

If $\mathcal{F}$ is a regular family, we say that $F \in \mathcal{F}$ is *maximal*, whenever there is no $n \in \mathbb{N}$ with $\max F < n$ so that $F \cup \{n\} \in \mathcal{F}$. The simplest examples of regular families include

$$\mathcal{A}_n := \big\{F \in [\mathbb{N}]^{<\omega} \colon |F| \leqslant n\big\} \quad (n \in \mathbb{N})$$

i.e., for a given $n \in \mathbb{N}$, the family comprising subsets of $\mathbb{N}$ of cardinality at most n. We employ these families to define the family of Schreier sets in the following way.

Definition 3. Given a countable ordinal α, we define inductively the Schreier family of order α as follows:

- $\mathcal{S}_0 := \mathcal{A}_1$;
- if α is a successor ordinal, i.e., $\alpha = \beta + 1$ for some $\beta < \omega_1$, then
$$\mathcal{S}_\alpha := \left\{ \bigcup_{i=1}^{d} S_\beta^i \colon d \leqslant S_\beta^1 < S_\beta^2 < \cdots < S_\beta^d,\ \big\{S_\beta^i\big\}_{i=1}^{d} \subset \mathcal{S}_\beta \text{ and } d \in \mathbb{N} \right\} \cup \big\{\emptyset\big\};$$
- if α is a non-zero limit ordinal and $(\alpha_n)_{n=1}^{\infty}$ is a fixed strictly increasing sequence of successor ordinals converging to α with $\mathcal{S}_{\alpha_n} \subset \mathcal{S}_{\alpha_{n+1}}$ for all $n \in \mathbb{N}$, we set
$$\mathcal{S}_\alpha := \big\{S_{\alpha_n} \in [\mathbb{N}]^{<\omega} \colon S_{\alpha_n} \in \mathcal{S}_{\alpha_n},\ n \leqslant \min S_{\alpha_n} \text{ for some n}\big\} \cup \big\{\emptyset\big\}.$$

We emphasize that in the case where α is a limit ordinal, we require the sequence $(\alpha_n)_{n=1}^{\infty}$ cofinal in α to comprise successor ordinals as needed in the proof of Theorem B. We may (and do) also assume that $S_{\alpha_n} \subset S_{\alpha_{n+1}}$ for all $n \in \mathbb{N}$ (see [3, Proposition 3.2.]), which we will also utilize in the proof of Theorem B. Elements of $\mathcal{S}_\alpha$ are called *$\mathcal{S}_\alpha$-sets*.

Note that the Schreier families $\{\mathcal{S}_\alpha\}_{\alpha<\omega_1}$ do depend on the choice of the sequences $\{\alpha_n\}_{n=1}^{\infty}$ converging to each limit ordinal α. It is a well-known fact $\big($[3][Proposition 3.2] or [5]$\big)$ that they are always regular families.

2.2. **Combinatorial Tsirelson spaces.** For a regular family $\mathcal{F}$ and $\theta \in \big(0, \frac{1}{2}\big]$, we define the Banach space $T[\theta, \mathcal{F}]$ that we shall later specialize to a combinatorial Tsirelson space $T[\theta, \mathcal{S}_\alpha]$ for some countable ordinal α.

For a vector $x = (a_1, a_2, \ldots, a_n) \in c_{00}$ and a finite set $E \subset \mathbb{N}$, we use the same symbol Ex to denote the projection of x onto the space $[e_i \colon i \in E]$, given by

$$E\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i \in E} a_i e_i. \tag{2.1}$$

If the cardinality of set E is equal k, then we say that E is a *k-element projection*.

We denote by $\|\cdot\|_0$ the supremum norm on c_{00}. Suppose that for some $n \in \mathbb{N}$ the norm $\|\cdot\|_n$ has been defined. Let

$$\|x\|_{n+1} = \max\big\{\|x\|_n, \|x\|_{T_n}\big\} \quad (n \in \mathbb{N}),$$

where

$$\|x\|_{T_n} = \sup\left\{\theta \sum_{i=1}^{d} \big\|E_i x\big\|_n \colon E_1 < \cdots < E_d,\ d \in \mathbb{N},\ \{E_i\}_{i=1}^d \subset [\mathbb{N}]^{<\omega},\ \{\min E_i\}_{i=1}^d \in \mathcal{F}\right\}.$$

We define the norm $\|x\|_{\theta,\mathcal{F}} := \sup_{n\in\mathbb{N}} \|x\|_n$ and denote by $T[\theta,\mathcal{F}]$ the completion of c_{00} with respect to it.

A proof by induction shows that this norm is majorized by the ℓ_1-norm and that satisfies the following implicit formula for $x \in T[\theta,\mathcal{F}]$:

$$\|x\|_{\theta,\mathcal{F}} = \max\big\{\|x\|_\infty, \|x\|_T\big\}, \tag{2.2}$$

where

$$\|x\|_{T} = \sup\left\{\theta \sum_{i=1}^{d} \big\|E_i x\big\|_{\theta,\mathcal{F}} : E_1 < \cdots < E_d,\ d \in \mathbb{N},\ \{E_i\}_{i=1}^d \subset [\mathbb{N}]^{<\omega},\ \{\min E_i\}_{i=1}^d \in \mathcal{F}\right\}.$$

It follows easily from the definition that the unit vectors $(e_i)_{i=1}^\infty$ form an 1-unconditional basis of the space $T[\theta,\mathcal{S}_\alpha]$ for a countable ordinal α.

For $x_1, x_2 \in c_{00}$, we write $x_1 < x_2$ whenever $\operatorname{supp} x_1 < \operatorname{supp} x_2$ and for $n \in \mathbb{N}$ we simplify the notation of $\operatorname{supp} x_1 < n$ to $x_1 < n$.

In this paper, we will use the following convention: we say that the norm of an element $x \in T[\theta,\mathcal{F}]$ is given by sets $E_1 < E_2 < \cdots < E_d$ for some $d \in \mathbb{N}$ (with $\{\min E_i\}_{i=1}^d \in \mathcal{F}$) precisely when

$$\|x\|_{\theta,\mathcal{F}} = \theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\|_{\theta,\mathcal{F}}.$$

It follows easily from the definition that if $(x_i)_{i=1}^d$ is a block sequence in $T[\theta,\mathcal{F}]$ (i.e., $x_1 < x_2 < \ldots < x_d$) with $\{\min \operatorname{supp} x_i\}_{i=1}^d \in \mathcal{F}$ we have

$$\Big\|\sum_{i=1}^{n} x_i\Big\|_{\theta,\mathcal{F}} \geqslant \theta \sum_{i=1}^{n} \|x_i\|_{\theta,\mathcal{F}}. \tag{2.3}$$

It follows that in the case of the space $T[\theta,\mathcal{S}_1]$, (2.3) yields that if $d \leqslant x_1 < \cdots < x_d$, then

$$\Big\|\sum_{i=1}^{d} x_i\Big\|_{\theta,\mathcal{S}_1} \geqslant \theta \sum_{i=1}^{d} \|x_i\|_{\theta,\mathcal{S}_1} \tag{2.4}$$

For brevity, we write $\|\cdot\|$ instead of $\|\cdot\|_{\theta,\mathcal{S}_\alpha}$, where $\theta \in \big(0,\frac{1}{2}\big]$, $1 \leqslant \alpha < \omega_1$. Let us record the following lemma that we shall later use extensively.

Lemma 4. *Let $\theta \in \big(0,\frac{1}{2}\big]$ and $1 \leqslant \alpha < \omega_1$. Suppose that $x \in T[\theta,\mathcal{S}_\alpha]$ is a vector whose coordinates are either 0 or 1. Fix $k \geqslant 2$. Let E be a k-element set. Then the norm given by the set E is not greater than the norm given by the $k-1$ many singleton projections. (For* *simplicity we assume we only project to non-zero coordinates and all of these projections are admissible.)*

Proof. Fix $k \geqslant 2$. The norm given by $k-1$ many 1-element projections is

$$\max\{1,\, \theta(k-1)\}.$$

Let E be a k-element set and let $E_1 < E_2 < \cdots < E_d$. Then

$$\begin{aligned}\theta \cdot \|Ex\| &= \theta \cdot \max\left\{1,\, \theta \cdot \sum_{i=1}^{d} \|E_i x\|\right\}\\ &\leqslant \theta \cdot \max\left\{1,\, \theta \cdot \sum_{i=1}^{d} \|E_i x\|_{\ell_1}\right\}\\ &= \theta \cdot \max\left\{1,\, \theta \cdot \sum_{i=1}^{d} |E_i|\right\}\\ &\leqslant \theta \cdot \max\{1,\, \theta k\}.\end{aligned}$$

Since $\theta \in \left(0, \frac{1}{2}\right]$ and $k \geqslant 2$, so $\theta(k-1) \geqslant \theta^2 k$. $\square$

Lemma 5. *Let $\theta \in \left(0, \frac{1}{2}\right]$ and $1 \leqslant \alpha < \omega_1$. Suppose that $x \in T[\theta, \mathcal{S}_\alpha]$ is given by the formula*

$$x = e_i + \sum_{j \in A} e_j,$$

where $A = \left\{j_1, j_2, \ldots, j_{|A|}\right\}$ is an $\mathcal{S}_\alpha$-set with $A > i$ and $|A| \geqslant \lceil \theta^{-1} \rceil$. If $A \cup \{i\}$ is not an $\mathcal{S}_\alpha$-set, then $\|x\| = \theta \cdot |A|$.

Proof. Since A is an $\mathcal{S}_\alpha$-set, we have

$$\|x\| \geqslant \theta \cdot |A| \geqslant 1 = \|x\|_\infty.$$

Assume that there are projections $E_1 < E_2 < \cdots < E_d$ for which we obtain greater norm when applied to x. If $\min E_1 > i$, then by Lemma 4 we arrive at a contradiction. If $\min E_1 \leqslant i$, then we have at least one k-element projection ($k \geqslant 2$), because by the hypothesis, $A \cup \{i\} \notin S_\alpha$. This contradicts Lemma 4 likewise.

$\square$

3. Isometries on $T[\theta, \mathcal{S}_1]$ spaces for $\theta \in \left(0, \frac{1}{2}\right]$

For the space $T[\theta, \mathcal{S}_1]$ $\big(\theta \in (0,1)\big)$ the norm given by the formula (2.2) takes the following form:

$$\|x\| = \max\big\{\|x\|_\infty, \|x\|_T\big\}, \tag{3.1}$$

where

$$\|x\|_T = \sup\left\{\theta\sum_{i=1}^{d}\big\|E_i x\big\| : d \leqslant E_1 < E_2 < \cdots < E_d,\, d\in\mathbb{N},\, \{E_i\}_{i=1}^d \subset [\mathbb{N}]^{<\omega}\right\}. \tag{3.2}$$

We are now ready to prove Theorem A; the proof emulates the one of [1, Theorem 4.1].

Proof. Let $Ue_n := \sum_{i=1}^\infty a_i^n e_i$ $(n\in\mathbb{N})$.

Claim 1. For any $n \geqslant \lceil\theta^{-1}\rceil$ we have $U([e_1, e_2, \dots, e_n]) \subset [e_1, e_2, \dots, e_n]$.

Let $n \geqslant \lceil\theta^{-1}\rceil$ and $j \in \big\{1, 2, \dots, n\big\}$. Define $x := \sum_{i=n+1}^\infty a_i^j e_i$ and fix $\varepsilon > 0$.

As $(Ue_i)_{i=1}^\infty$ is weakly null we may find indices $n < j_1 < j_2 < \cdots < j_n$ and vectors

$$n+1 \leqslant x' < y_1 < y_2 < \cdots < y_n,$$

so that $\|x'-x\| < \varepsilon$ and $\big\|Ue_{j_i} - y_i\big\| < \varepsilon$ for $1 \leqslant i \leqslant n$.

By Lemma 5 and since U is an isometry we have

$$\theta n = \left\|e_j + \sum_{i=1}^n e_{j_i}\right\| = \left\|\sum_{i=1}^n a_i^j e_i + x + \sum_{i=1}^n Ue_{j_i}\right\|$$

Hence by triangle inequality we obtain

$$\left\|\sum_{i=1}^n a_i^j e_i + x' + \sum_{i=1}^n y_i\right\| \leqslant \theta n + (n+1)\varepsilon \tag{3.3}$$

On the other hand

$$\left\|\sum_{i=1}^n a_i^j e_i + x' + \sum_{i=1}^n y_i\right\| \geqslant \theta\left(\|x'\| + \sum_{i=1}^n \|y_i\|\right) > \theta\Big(\|x'\| + n(1-\varepsilon)\Big). \tag{3.4}$$

The first inequality follows from (2.4) and second by the fact that $\|Ue_i\| = 1$ for any $i\in\mathbb{N}$. Therefore by (3.3) and (3.4) we obtain

$$\|x\| \leqslant \|x'-x\| + \|x'\| < \varepsilon + \Big(\big(n+1\big)\theta^{-1} + n\Big)\varepsilon.$$

Since ε was arbitrary, we get $\|x\| = 0$. Consequently,

$$U\big[e_1, e_2, \dots, e_n\big] \subset \big[e_1, e_2, \dots, e_n\big].$$

Claim 2. There exists a permutation π of $\big\{1, 2, \dots, \lceil\theta^{-1}\rceil\big\}$ such that $Ue_n = \pm e_{\pi(n)}$ for $n \in \big\{1, 2, \dots, \lceil\theta^{-1}\rceil\big\}$.

First we will show that the norm on $\big[e_1, e_2, \ldots, e_{\lceil\theta^{-1}\rceil}\big]$ is the supremum norm. Indeed, suppose that the norm of some

$$x = \sum_{i=1}^{\lceil\theta^{-1}\rceil} a_i e_i$$

is given by certain sets $d \leqslant E_1 < E_2 < \cdots < E_d$ for some $d \in \mathbb{N}$ in the sense that

$$\|x\| = \theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\|. \tag{3.5}$$

Suppose that $\min E_1 < \lceil\theta^{-1}\rceil$. Then

$$d \leqslant \min E_1 \leqslant \lceil\theta^{-1}\rceil - 1 < \theta^{-1},$$

so

$$\theta \cdot \sum_{i=1}^{d} \big\|E_i x\big\| \leqslant \theta \cdot d \cdot \|x\| < \|x\|.$$

Hence (3.5) cannot hold; a contradiction. Suppose that $\min E_1 \geqslant \lceil\theta^{-1}\rceil$. Then the only non-zero coordinate of $(E_1 \cup E_2 \cup \ldots \cup E_d)x$ is $a_{\lceil\theta^{-1}\rceil}$, so

$$\|x\| \leqslant \theta \cdot |a_{\lceil\theta^{-1}\rceil}| \leqslant \theta \cdot \|x\|_\infty < \|x\|_\infty.$$

This contradiction ends the proof that $\|x\| = \|x\|_\infty$. This means that for each $n \in \big\{1, 2, \ldots, \lceil\theta^{-1}\rceil\big\}$ there is at least one index $\pi(n)$ so that $\big|a^n_{\pi(n)}\big| = 1$. By the very definition of the norm and since U is an isometry we have

$$1 = \max\{\, 1,\, 2 \cdot \theta\,\} = \|e_n \pm e_i\| = \big\|U(e_n \pm e_i)\big\| \geqslant \big\|U(e_n \pm e_i)\big\|_\infty \geqslant \big|a^n_{\pi(n)} \pm a^i_{\pi(n)}\big|$$

for any $i \neq n$ in $\big\{1, \ldots, \lceil\theta^{-1}\rceil\big\}$. Therefore $\big|a^i_{\pi(n)}\big| = 0$ for any $i \neq n$, so π is the desired permutation.

Claim 3. $Ue_n = \pm e_n$ for $n > \lceil\theta^{-1}\rceil$.

Let $n = \lceil\theta^{-1}\rceil + 1$. Then, by Claim 1, $Ue_n = a^n_1 e_1 + a^n_2 e_2 + \cdots + a^n_n e_n$ and, by Claim 2, for $1 \leqslant j < n$ we have

$$1 = \max\{\, 1,\, 2 \cdot \theta\,\} = \|e_j \pm e_n\| = \big\|e_{\pi(j)} \pm Ue_n\big\| \geqslant \big|1 \pm a^n_{\pi(j)}\big|.$$

Consequently, $a^n_{\pi(j)} = 0$ for all such j, so $Ue_n = \pm e_n$.

We now proceed to the inductive step. Fix $n \in \mathbb{N}$, $n > \lceil\theta^{-1}\rceil$ and assume that for $k \in \mathbb{N}$ with $\lceil\theta^{-1}\rceil < k < n$ one has $Ue_k = \pm e_k$. Then by Claim 1 we have $Ue_n = a^n_1 e_1 + a^n_2 e_2 + \cdots + a^n_n e_n$. By Claim 2, for $1 \leqslant j \leqslant \lceil\theta^{-1}\rceil$ we have

$$1 = \max\{\, 1,\, 2 \cdot \theta\,\} = \|e_j \pm e_n\| = \big\|e_{\pi(j)} \pm Ue_n\big\| \geqslant \big|1 \pm a^n_{\pi(j)}\big|.$$

Therefore $a^n_{\pi(j)} = 0$ for all such j. Similarly, by the inductive hypothesis, we have

$$1 = \max\{\,1,\, 2\cdot\theta\,\} = \|e_j \pm e_n\| = \|e_j \pm Ue_n\| \geqslant \left|1 \pm a^n_j\right|$$

for $\lceil\theta^{-1}\rceil < j < n$, so $a^n_j = 0$ for all such j. Hence $Ue_n = \pm e_n$. This finishes the proof that the isometry has the desired form.

□

Remark 6. The reverse implication in Theorem A need not hold. Indeed, let $\theta^{-1} = 2.1$. Then, of course, $\lceil\theta^{-1}\rceil = 3$. Let us consider the vectors:

(1) $x = (1, 0, 0, 1, 1, 0, \ldots)$
(2) $y = (0, 0, 1, 1, 1, 0, \ldots)$

The only $\mathcal{S}_1$-set with a minimum equal to 1 is $\{1\}$. Since $\{4, 5\} \in \mathcal{S}_1$, so from (3.2) we have

$$\|x\|_T \geqslant \frac{10}{21}\cdot(1+1) = \frac{20}{21}.$$

Now that vector x has only 3 ones, so by Lemma 4 this inequality is in fact equality. Since $\{3, 4, 5\} \in \mathcal{S}_1$, so again by (3.2) and Lemma 4 we obtain

$$\|y\|_T = \frac{10}{21}\cdot(1+1+1) = \frac{30}{21}.$$

Consequently, by (3.1), we have

(1) $\|x\| = \max\{\|x\|_\infty, \|x\|_T\} = \max\{1, \frac{20}{21}\} = 1$,
(2) $\|y\| = \max\{\|y\|_\infty, \|y\|_T\} = \max\left\{1, \frac{30}{21}\right\} = \frac{30}{21}$.

Let us define an operator U such that $Ux = y$. More formally,

$$Ue_1 = e_3,\ Ue_2 = e_2,\ Ue_3 = e_1,\ Ue_i = e_i \text{ for } i \geqslant 4.$$

It is clear that U is not an isometry.

4. Isometries on $T[\theta, \mathcal{S}_\alpha]$ for $\theta \in \left(0, \frac{1}{2}\right]$ and $2 \leqslant \alpha < \omega_1$

We are now ready to prove Theorem B.

Proof. Let $Ue_n := \sum_{i=1}^{\infty} a^n_i e_i$ $(n \in \mathbb{N})$.
Claim 1. For any ordinal $2 \leqslant \alpha < \omega_1$ and $n \in \mathbb{N}$ we have $U([e_1, e_2, \ldots, e_n]) \subset [e_1, e_2, \ldots, e_n]$.

Case 1. Fix a successor ordinal $2 \leqslant \alpha < \omega_1$ and $j \in \left\{1, 2, \ldots, n\right\}$, where $n \in \mathbb{N}$.

Define $x := \sum_{i=n+1}^{\infty} a^j_i e_i$, $\alpha := \beta + 1$ for some $\beta < \omega_1$ and fix $\varepsilon > 0$. Note that every $\mathcal{S}_\alpha$-set whose minimum is n is the union of at most n many $\mathcal{S}_\beta$-sets, so the idea of the proof of this case is to choose the indices $\lceil\theta^{-1}\rceil < j_1 < j_2 < \cdots < j_m$, for some $m \in \mathbb{N}$, so that they creates n many maximal and consecutive $\mathcal{S}_\beta$-sets. At the same time, we must ensure that the set

$$\{\min\operatorname{supp} x', \min\operatorname{supp} y_1, \min\operatorname{supp} y_2, \ldots, \min\operatorname{supp} y_m\}$$

associated with these indices was also $\mathcal{S}_\alpha$-set, i.e., it was the union of at most (not necessarily maximal) $n+1$ many $\mathcal{S}_\beta$-sets. We will choose these sets inductively.

Step 1. Set $\alpha = 2$.

Fix any $t \geqslant \max\{n, \lceil \theta^{-1} \rceil\}$. As $(Ue_i)_{i=1}^\infty$ is weakly null we find index $j_1 > t$ and vectors $n+1 \leqslant x' < y_1$, so that $\|x' - x\| < \varepsilon$ and $\left\|Ue_{j_1} - y_1\right\| < \varepsilon$. Since we have determined the index j_1, we know the cardinality of the first maximal Schreier set.

Similarly, we may choose an index $j_2 > j_1$ and vector $y_2 > \max\{j_1, \max\operatorname{supp} y_1\}$ so that $\left\|Ue_{j_2} - y_2\right\| < \varepsilon$. We continue the above procedure until we reach a maximal Schreier set built out of the indices $t < j_1 < j_2 < \cdots < j_{p_1}$, where $p_1 := 2j_1 - 1$. At the same time, we get vectors $n+1 \leqslant x' < y_1 < y_2 < \cdots < y_{p_1}$ so that $\|x' - x\| < \varepsilon$ and $\left\|Ue_k - y_k\right\| < \varepsilon$ for $k = 1, \ldots, p_1$.

Proceeding analogously, we fix an index $j_{p_1+1} > j_{p_1}$, which will be the minimum of the second maximal Schreier set, and find vector $y_{p_1+1} > \max\{j_{p_1}, \max\operatorname{supp} y_{p_1}\}$ in such a way that $\left\|Ue_{p_1+1} - y_{p_1+1}\right\| < \varepsilon$. Then, the set

$$\{\min\operatorname{supp} y_2, \min\operatorname{supp} y_3, \ldots, \min\operatorname{supp} y_{p_1+1}\}$$

is Schreier because the Schreier family is spreading (see Definition 2). Define $p_2 := 2j_{p_1+1} - 1$.

Continuing the above procedure, we find indices

$$t < j_1 < j_2 < \cdots < j_{p_2}$$

and block vectors

$$\begin{aligned} n+1 \leqslant x' < y_1 \leqslant \max\{\, j_1, \max\operatorname{supp} y_1 \,\} < y_2 \leqslant \cdots \\ \cdots \leqslant \max\{\, j_{p_2-1}, \max\operatorname{supp} y_{p_2-1} \,\} < y_{p_2}, \end{aligned}$$

so that $\|x' - x\| < \varepsilon$ and $\left\|Ue_k - y_k\right\| < \varepsilon$ for $k = 1, \ldots, p_2$.

Consequently, we already have two Schreier sets. In the same way we may finally find indices

$$t < j_1 < j_2 < \cdots < j_p,$$

for some $p \in \mathbb{N}$, that form a union of n maximal Schreier sets with minima

$$\big\{j_1, j_{p_1+1}, \ldots, j_{p_{n-1}+1}\big\}.$$

Hence the conlusion follows because the indices $t < j_1 < j_2 < \cdots < j_p$ form a set that is the union of n maximal and consecutive Schreier sets with minimum greater than t and we may choose sets $n+1 \leqslant S_1^1 < S_1^2 < \cdots < S_1^{n+1}$ in a way that

– $S_1^1 = \{\min \operatorname{supp} x', \min \operatorname{supp} y_1\}$,
– $S_1^2 = \{\min \operatorname{supp} y_2, \min \operatorname{supp} y_3, \ldots, \min \operatorname{supp} y_{m_1+1}\}$,
– $\vdots$
– $S_1^{n+1} = \{\min \operatorname{supp} y_{m_{n-1}+2}, \min \operatorname{supp} y_{m_{n-1}+3}, \ldots, \min \operatorname{supp} y_m\}$.

Step 2. Set $\alpha = \beta + 1$ for some $\beta < \omega_1$ and suppose that for any $\gamma < \alpha$ and any $t \geqslant \max\{n, \lceil \theta^{-1} \rceil\}$ we may construct maximal $\mathcal{S}_\gamma$-set with minimum greater than t and built out of the desired indices.

By the inductive hypothesis we may choose a maximal $\mathcal{S}_\beta$-set created from the indices $t < j_1 < j_2 < \cdots < j_{m_1}$, for some $m_1 \in \mathbb{N}$. At the same time, we get vectors $n+1 \leqslant x' < y_1 < y_2 < \cdots < y_{m_1}$ so that $\|x' - x\| < \varepsilon$ and $\big\|Ue_k - y_k\big\| < \varepsilon$ for $k = 1, \ldots, m_1$. Next, we apply again the inductive hypothesis to find the second maximal $\mathcal{S}_\beta$-set with minimum j_{m_1+1} greater than j_{m_1} and block vectors

$$n+1 \leqslant x' < y_1 \leqslant \max\{\, j_1, \max \operatorname{supp} y_1 \,\} < y_2 \leqslant \cdots$$
$$\cdots \leqslant \max\{\, j_{m_2-1}, \max \operatorname{supp} y_{m_2-1} \,\} < y_{m_2},$$

for some $m_2 \in \mathbb{N}$, so that $\|x' - x\| < \varepsilon$ and $\big\|Ue_k - y_k\big\| < \varepsilon$ for $k = 1, \ldots, m_2$. Then, the set

$$\{\min \operatorname{supp} y_2, \min \operatorname{supp} y_3, \ldots, \min \operatorname{supp} y_{m_1+1}\}$$

is $\mathcal{S}_\beta$-set because the Schreier family (of order β) is spreading (see Definition 2).

Proceeding analogously, we finally arrive at indices

$$t < j_1 < j_2 < \cdots < j_m,$$

for some $m \in \mathbb{N}$, that form a union of n maximal $\mathcal{S}_\beta$-sets with minima

$$\big\{j_1, j_{m_1+1}, \ldots, j_{m_{n-1}+1}\big\},$$

so we got the conclusion, because we may choose sets

$$n+1 \leqslant S_\beta^1 < S_\beta^2 < \cdots < S_\beta^{n+1}, \tag{4.1}$$

where

– $S_\beta^1 = \{\min \operatorname{supp} x', \min \operatorname{supp} y_1\}$,
– $S_\beta^2 = \{\min \operatorname{supp} y_2, \min \operatorname{supp} y_3, \ldots, \min \operatorname{supp} y_{m_1+1}\}$,
– $\vdots$

– $S_\beta^{n+1} = \{\min\operatorname{supp}\, y_{m_{n-1}+2}, \min\operatorname{supp}\, y_{m_{n-1}+3}, \ldots, \min\operatorname{supp}\, y_m\}$.

Since the indices $\max\{n, \lceil\theta^{-1}\rceil\} < j_1 < j_2 < \cdots < j_m$, chosen as in Step 2 above, form a set that is the union of n maximal and consecutive $\mathcal{S}_\beta$-sets with minimum greater than n, by Lemma 5, we have

$$\left\| e_j + \sum_{i=1}^m e_{j_i} \right\| = \theta \cdot m. \tag{4.2}$$

Since U is an isometry, we obtain

$$\left\| e_j + \sum_{i=1}^m e_{j_i} \right\| = \left\| \sum_{i=1}^n a_i^j e_i + x + \sum_{i=1}^m U e_{j_i} \right\|. \tag{4.3}$$

By (4.2), (4.3), and the triangle inequality

$$\left\| \sum_{i=1}^n a_i^j e_i + x' + \sum_{i=1}^m y_i \right\| \leqslant \theta \cdot m + (m+1)\varepsilon. \tag{4.4}$$

On the other hand,

$$\left\| \sum_{i=1}^n a_i^j e_i + x' + \sum_{i=1}^m y_i \right\| \geqslant \theta \left(\|x'\| + \sum_{i=1}^m \|y_i\| \right) > \theta\big(\|x'\| + m - m\varepsilon\big); \tag{4.5}$$

the former inequality follows from (2.3) as we may choose sets as in (4.1), whereas the latter one holds because $\big\|Ue_i\big\| = 1$ $(i \in \mathbb{N})$. Thus, by (4.4) and (4.5), we have

$$\|x\| \leqslant \|x' - x\| + \|x'\| < \varepsilon + \big(\theta^{-1}(m+1) + m\big)\varepsilon,$$

so $\|x\| = 0$. Consequently, $Ue_j = a_1^j e_1 + a_2^j e_2 + \cdots + a_n^j e_n$.

Case 2: Fix a limit ordinal $2 \leqslant \alpha < \omega_1$ and $j \in \big\{1, 2, \ldots, n\big\}$, where $n \in \mathbb{N}$.

We proceed as in Case 1 for $\alpha = \alpha_n := \beta_n + 1$, where $(\alpha_i)_{i=1}^\infty$ is a fixed strictly increasing sequence of successor ordinals converging to α with $\mathcal{S}_{\alpha_j} \subset \mathcal{S}_{\alpha_n}$ for $j \leqslant n$. Indeed, an $\mathcal{S}_\alpha$-set whose minimum is n must be an $\mathcal{S}_{\alpha_n}$-set and sets $n+1 \leqslant S_{\beta_n}^1 < S_{\beta_n}^2 < \cdots < S_{\beta_n}^{n+1}$ give rise to an $\mathcal{S}_\alpha$-set (even an $\mathcal{S}_{\alpha_n}$-set).

Claim 2. $Ue_n = \pm e_n$ for $n \in \mathbb{N}$.

Set $n = 1$. Then by Case 1 of Claim 1 we have $Ue_1 = a_1^1 e_1$. Since $\big\|Ue_1\big\| = 1$, so $Ue_1 = \pm e_1$.

We now proceed to the inductive step. Fix $n \in \mathbb{N}$ and assume that for $k \in \mathbb{N}$ with $k < n$ one has $Ue_k = \pm e_k$. Then by Claim 1 we have $Ue_n = a_1^n e_1 + a_2^n e_2 + \cdots + a_n^n e_n$. By the very definition of the norm and the inductive hypothesis we have

$$1 = \max\{1,\, 2 \cdot \theta\} = \|e_k \pm e_n\| = \big\|e_k \pm Ue_n\big\| \geqslant \big\|e_k \pm Ue_n\big\|_\infty \geqslant \big|1 \pm a_k^n\big|$$

for $k \in \{1, 2, \dots, n-1\}$. Hence $a_k^n = 0$ for all such k. Since $\|Ue_i\| = 1$ ($i \in \mathbb{N}$), so $Ue_n = \pm e_n$.

□

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