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Euler equations in fluid dynamics: Good and bad news

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Abstract

We review several recent results concerning the properties of a barotropic Euler system. In particular, the following topics will be addressed:

- Density of the so called "wild data" in the Lebesgue topology.
- Strong continuity in time of weak solutions.
- Measurable semigroup selection.
- Euler system and its relevance to turbulence.
- Numerical solutions of barotropic Euler system and visualisation of oscillatory approximations.

Key words: Barotropic Euler system, weak solution, convex integration, semigroup selection, turbulence, finite volume approximation

1 Introduction

This is a survey of some recent results on qualitative properties of solutions of the *Euler system of equations* describing the motion of a barotropic compressible perfect fluid:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= 0, \end{aligned} \tag{1.1}$$

where $\varrho = \varrho(t, x)$ is the mass density, $\mathbf{u} = \mathbf{u}(t, x)$ the velocity, and p the pressure related to the density by an equation of state $p = p(\varrho)$.

The following properties of the Euler system are well known, see e.g. Benzoni-Gavage and Serre [3], Dafermos [11] , Smoller [25]:

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- **Local existence.**

  The problem admits a classical solution for smooth initial data defined on a short time interval.

- **Finite time blow–up.**

  The classical solutions develop a singularity (shock wave) in a finite time for a fairly general class of smooth initial data.

- **Ill posedness in the class of weak solutions.**

  The problem is ill posed in the class of weak solutions unless some extra admissibility conditions are prescribed.

The expected remedy for ill posedness in the class of weak solutions is imposing some admissibility conditions to be satisfied by the weak solutions. Formally, the Euler systems conserves energy, specifically

$$\partial_t E(\varrho, \mathbf{u}) + \mathrm{div}_x \left[\left(E(\varrho, \mathbf{u}) + p(\varrho) \right) \mathbf{u} \right] = 0,\ E(\varrho, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = 0. \tag{1.2}$$

We say that a weak solution of the Euler system is *admissible* if the energy inequality

$$\partial_t E(\varrho, \mathbf{u}) + \mathrm{div}_x \left[\left(E(\varrho, \mathbf{u}) + p(\varrho) \right) \mathbf{u} \right] \leq 0 \tag{1.3}$$

holds in the sense of distributions. If the problems is considered in a (bounded) domain $\Omega \subset R^d$ with impermeable boundary, the system of equations is accompanied by the no-flux boundary condition

$$\vec{u} \cdot \vec{n}|_{\partial\Omega} = 0. \tag{1.4}$$

In this case, the energy inequality (1.3) may be replaced by its weaker form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} E(\varrho, \mathbf{u})\ \mathrm{d}x \leq 0. \tag{1.5}$$

We call any weak solution satisfying (1.5) *weakly admissible.*

2 Wild data

There is a number of negative results concerning the well posedness of the Euler system in the class of weak solutions. Most of them are based on the recent development of the method of *convex integration* in the framework of fluid mechanics due to the series of works of Chiodaroli et al. [7], [8], De Lellis and Székelihydi [12], [13] among many others.

First, consider weak solutions of the Euler system emanating from the initial data

$$\varrho(0, \cdot) = \varrho_0,\ (\varrho\vec{u})(0, \cdot) = \varrho_0 \vec{u}_0. \tag{2.1}$$

We say that the initial data $\varrho_0, \vec{u}_0$ are *wild* if there exists a positive time $T_{\rm w} > 0$ such that the Euler system admits infinitely many *admissible* (weak) solutions $(\varrho, \vec{u})$ on any interval $[0,T]$, $0 < T < T_{\rm w}$ belonging to the class

$$\varrho \in L^\infty((0,T)\times\Omega),\ \ \varrho > 0,\ \ \vec{u} \in L^\infty((0,T)\times\Omega; R^d).$$

The following results was proved in [9, Theorem 1.3]

Theorem 2.1 (Density of wild data). *Let $d = 2,3$. Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \le a < b \le \infty$.*

Then for any

$$\varrho_0 \in W^{k,2}(\Omega),\ \ a < \inf_\Omega \varrho_0 \le \sup_\Omega \varrho_0 < b, \vec{u}_0 \in W^{k,2}(\Omega; R^d),\ \ k > \frac{d}{2} + 1,$$

$\vec{u}_0$ satisfying $\vec{u}_0 \cdot \vec{n}|_{\partial\Omega} = 0$ and the relevant higher order compatibility conditions, any $\varepsilon > 0$, and any $1 \le p < \infty$, there exist wild data $\varrho_{0,\varepsilon}$, $\vec{u}_{0,\varepsilon}$ such that

$$\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^p(\Omega)} < \varepsilon,\ \ \|\vec{u}_{0,\varepsilon} - \vec{u}_0\|_{L^p(\Omega;R^d)} < \varepsilon.$$

In other words, the set of wild data is dense in the L^p topology. Chen, Vasseur and You [6] established density of wild data for the isentropic Euler system ($p = a\varrho^\gamma$) in the class of weak solutions satisfying the total energy inequality

$$\int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\right](\tau,\cdot)\ {\rm d}x \le \int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho_0|\vec{u}_0|^2 + P(\varrho_0)\right]\ {\rm d}x \text{ for any } \tau > 0.$$

These solutions are global in time, however, the associated total energy profile

$$\int_\Omega \left[\frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\right](\tau,\cdot)\ {\rm d}x$$

may not be non–increasing in time.

3 Continuity in time of weak solutions

Next, let us focus on the continuity in time of the weak solutions necessary for a meaningful definition of the initial data. For definiteness, we may consider the space periodic boundary conditions,

$$x \in \Omega = \mathbb{T}^d,\ \ d = 2,3,$$

or the physically more realistic impermeability condition

$$\vec{u}\cdot\vec{n}|_{\partial\Omega} = 0,$$

where $\Omega \subset R^d$ is the physical domain occupied by the fluid.

It follows directly from the weak formulation that integrable weak solutions of the Euler system belong automatically to the class

$$[\varrho, (\varrho\vec{u})] \in C_{\rm weak}([0,T]; L^\gamma(\Omega)) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)).$$

We show this result is sharp in the class of weak solutions, meaning for any initial data

$$\varrho(0,\cdot) = \varrho_0 \text{ Riemann integrable in } \Omega,$$
$$\varrho\vec{u}(0,\cdot) = \varrho_0, \vec{u}_0, \ {\rm div}(\varrho_0\vec{u}_0) \text{ Riemann integrable in } \Omega, \ \vec{u}_0 \cdot \vec{n}|_{\partial\Omega} = 0,$$

and any countable set of times $\{\tau_i\}_{i=1}^\infty \subset (0,T)$, the Euler system admits infinitely many weak solutions that are not strongly continuous at any τ_i. In addition, these solutions satisfy the weaker for of the energy inequality,

$$\frac{\rm d}{\rm dt}\int_\Omega \left[\frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\right] \ {\rm d}x \le 0, \ P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma.$$

For a compact set $Q \subset R^M$, consider the class of Riemann integrable functions, specifically set

$$\mathcal{R}(Q) \equiv \left\{ v : Q \to \mathbb{R} \ \middle| \ {\rm meas}\left\{ y \in Q \ \middle| \ v \text{ is not continuous at } y \right\} = 0 \right\}$$

where the symbol "meas" stands for the Lebesgue measure.

The following result was proved in [1, Theorem 2.1].

Theorem 3.1. *Let $\Omega \subset R^d$, $d = 2,3$, be a bounded domain with C^2 boundary. Let the initial data ϱ_0, $\vec{u}_0$ be given,*

$$0 < \underline{\varrho} \le \varrho_0(x) \le \overline{\varrho} \text{ for all } x \in \overline{\Omega}, \ \varrho_0 \in \mathcal{R}(\overline{\Omega}),$$
$$\vec{u}_0 \in \mathcal{R}(\overline{\Omega}; R^d), \ {\rm div}_x(\varrho_0\vec{u}_0) \in \mathcal{R}(\overline{\Omega}), \ \varrho_0\vec{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $E(t)$ be an arbitrary function satisfying

$$0 \le E(t) \le \overline{E} \text{ for all } t \in [0,T], \ E \in \mathcal{R}[0,T].$$

Then there exists $E_0 \ge 0$ such that the Euler system admits infinitely many solutions $[\varrho, \mathbf{u}]$ in $(0,T)\times\Omega$ satisfying

$$\frac{1}{2}\underline{\varrho} \le \varrho(t,x) \le 2\overline{\varrho} \text{ for all } (t,x) \in (0,T)\times\Omega, \ \vec{u} \in L^\infty(0,T)\times\Omega; R^d),$$

$$\int_\Omega \left[\frac{1}{2}\varrho|\vec{u}|^2 + P(\varrho)\right](\tau,\cdot) \ {\rm d}x = E_0 + E(\tau) \text{ for a.a. } \tau \in (0,T).$$

In particular, we may choose the function E to be non–increasing and discontinuous at a given countable set of time $(\tau_i)_{i\in T} \subset (0,T)$ to deduce the following corollary, cf. [1, Theorem 2.3].

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^2 boundary. Let the initial data ϱ_0, $\vec{u}_0$ be given,*

$$0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho} \text{ for all } x \in \overline{\Omega},\ \varrho_0 \in \mathcal{R}(\overline{\Omega}),$$

$$\vec{u}_0 \in \mathcal{R}(\overline{\Omega}; R^d),\ \mathrm{div}_x(\varrho_0, \mathbf{u}_0) \in \mathcal{R}(\overline{\Omega}),\ \varrho_0 \vec{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\mathcal{S} = \{\tau_n\}_{n=1}^{\infty} \subset (0, T)$ be an arbitrary (countable) set of times.

Then the Euler system admits infinitely many weakly admissible (weak) solutions that are not strongly continuous at any τ_n, $n = 1, 2, \ldots$

Recall that *weakly admissible* means the total energy is equal to a non–increasing function for a.a. time as specified in (1.5). In particular, the solutions need not be strongly continuous at $t = 0$.

The proof of the above results is based on a generalized version of the "oscillatory lemma" of De Lellis and Székelyhidi [12, Proposition 3] extended to Riemann integrable parameters, see [1, Lemma 3.3].

Lemma 3.3 (**Oscillatory Lemma – Riemann version**). *Let*

$$Q = (t_1, t_2) \times \Pi_{i=1}^d (a_i, b_i),\ t_1 < t_2,\ a_i < b_i,\ i = 1, \ldots, d,$$

be a block. Suppose that

$$\vec{v} \in \mathcal{R}(\overline{Q}; \mathbb{R}^d),\ \mathbb{U} \in \mathcal{R}(\overline{Q}; \mathbb{R}^{d\times d}_{0,\mathrm{sym}}),\ e \in \mathcal{R}(\overline{Q}),\ r \in \mathcal{R}(\overline{Q})$$

be given such that

$$0 < \underline{r} \leq r(t,x) \leq \overline{r},\ e(t,x) \leq \overline{e} \text{ for all } (t,x) \in \overline{Q},$$
$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{\vec{v} \otimes \vec{v}}{r} - \mathbb{U} \right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences of vector functions $(\vec{w}_n)_{n=1}^{\infty}$, $(\mathbb{V}_n)_{n=1}^{\infty}$,

$$\vec{w}_n \in C_c^{\infty}(Q; \mathbb{R}^d),\ \mathbb{V}_n \in C_c^{\infty}(Q; \mathbb{R}^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \vec{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \vec{w}_n = 0 \text{ in } Q,$$
$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[\frac{(\vec{v} + \vec{w}_n) \otimes (\vec{v} + \vec{w}_n)}{r} - (\mathbb{U} + \mathbb{V}_n) \right] < \inf_{\overline{Q}} e \text{ for all } n = 1, 2, \ldots,$$
$$\vec{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([t_1, t_2]; L^2(\Pi_{i=1}^d (a_i, b_i); \mathbb{R}^d)) \text{ as } n \to \infty,$$
$$\liminf_{n\to\infty} \int_Q \frac{|\vec{w}_n|^2}{r} \,\mathrm{d}x\,\mathrm{d}t \geq c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} \frac{|\vec{v}|^2}{r} \right)^2 \mathrm{d}x\,\mathrm{d}t.$$

Here, the symbol $\lambda_{\max}[\mathbb{A}]$ denotes the maximal eigenvalue of a symmetric matrix $\mathbb{A}$. As demonstrated in [14], Lemma 3.3 can be applied to other problems involving the compressible Euler system.

3.1 Measurable semigroup selection

Despite the number of ill–posedness results – the "bad news" – reported above, there is a semigroup selection of suitable *generalized* solutions to the Euler system. The semigroup mapping is Borel measurable but not continuous in general.

We start by introducing the *dissipative* solutions to the compressible Euler system satisfying the following system of equations in the sense of distributions:

$$\partial_t \varrho + \mathrm{div}(\vec{m}) = 0,$$
$$\partial_t \vec{m} + \mathrm{div}\left(\frac{\vec{m} \otimes \vec{m}}{\varrho}\right) + \nabla p(\varrho) = -\mathrm{div}\ \mathcal{R},$$

together with the energy inequality

$$\int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\mathrm{d}x + \mathfrak{E}(\tau,\cdot) \leq \int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right](0,\cdot)\mathrm{d}x,$$

where $\mathcal{R} \in L^\infty(0,T;\mathcal{M}^+_{\mathrm{sym}}(\Omega;R^{d\times d}))$, and $\mathcal{R} \geq 0$,

$$\int_\Omega \mathrm{trace}[\mathcal{R}]\ \mathrm{d}x \lesssim \mathfrak{E}, \tag{3.1}$$

see [4]. In this context, it more convenient to consider the *conservative variables* $(\varrho, \vec{m} = \varrho\vec{u})$. The symbols $\mathcal{M}^+_{\mathrm{sym}}$ denotes the space of non–negative tensor valued measures on Ω. Roughly speaking, dissipative solutions satisfy the Euler system modulo the tensor $\mathcal{R}$ termed *Reynolds stress*. The crucial property of dissipative solutions is the compatibility relation (3.1) between the trace of the Reynolds stress and the total energy defect. Unlike the conventional weak solutions, the dissipative solutions are identified with the *triple* $(\varrho, \vec{m}, E)$, where E denotes the "augmented" energy

$$E(\tau) = \int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot)\mathrm{d}x + \mathfrak{E}(\tau,\cdot).$$

As shown in [4], the Euler system admits a measurable semiflow selection in the class of *dissipative* solutions. Specifically, we consider three state variables $(\varrho, \vec{m})$ together with the augmented energy

$$\mathcal{E} = \int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right]\mathrm{d}x + \mathfrak{E}$$

as functions of the time $t \in [0,\infty)$. Then there is a Borel measurable mapping

$$\mathcal{S} : t \geq 0,\ (\varrho_0, \vec{m}_0, \mathcal{E}_0) \mapsto (\varrho(t,\cdot), \vec{m}(t,\cdot), \mathcal{E}(t))(\varrho_0, \vec{m}_0, \mathcal{E}_0),$$

such that

$$\mathcal{S}[0;\varrho_0,\vec{m}_0,\mathcal{E}_0] = (\varrho_0,\vec{m}_0,\mathcal{E}_0),\ \mathcal{S}[t+s;\varrho_0,\vec{m}_0,\mathcal{E}_0] = \mathcal{S}\Big[s;\mathcal{S}[t;\varrho_0,\vec{m}_0,\mathcal{E}_0]\Big],\ s \geq 0.$$

Specifically, the selection enjoys the following properties, see [4]:

- **Compatibility.** The mapping

$$t \in [0, \infty) \mapsto \mathcal{S}[t, \varrho_0, \vec{m}_0, E_0]$$

is a dissipative solution of the Euler system emanating form the initial data $(\varrho_0, \vec{m}_0, E_0)$.

- **Stability of strong solutions.** Let the Euler system admit a strong C^1 solution $\widehat{\varrho}$, $\widehat{\mathbf{m}}$, with the associated energy

$$E_0 = \int_\Omega \left[\frac{|\vec{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \mathrm{d}x,$$

defined on a maximal time interval $[0, T_{\max})$.

Then we have

$$\mathcal{S}[t, \varrho_0, \vec{m}_0, E_0] = (\widehat{\varrho}, \widehat{\mathbf{m}}, E_0)(t) \text{ for all } t \in [0, T_{\max}).$$

This reflects the fact that dissipative solutions satisfy the weak–strong uniqueness principle.

- **Maximal dissipation.** Let the Euler system admit a dissipative solution $\widehat{\varrho}$, $\widehat{\mathbf{m}}$, with the associated energy $\widehat{E}$ such that

$$\widehat{E}(t) \leq E(t) \text{ for all } t \geq 0,$$

where E is the energy of the solution semiflow $\mathcal{S}[t, \varrho_0, \mathbf{m}_0, E_0]$.

Then we have

$$E(t) = \widehat{E}(t) \text{ for all } t \geq 0.$$

The dissipative solutions respects the maximal dissipation criterion proposed by Dafermos [10] for general systems of hyperbolic conservation laws.

- **Stability of stationary states.** Let $\overline{\varrho} > 0$, $\vec{m} \equiv 0$ be a stationary solution of the Euler system. Suppose that

$$\varrho(T, \cdot) = \overline{\varrho},\ \vec{m}(T, \cdot) = 0 \text{ for some } T \geq 0,$$

where ϱ, $\vec{m}$ are the density and the momentum components of a solution semiflow $\mathcal{S}[t, \varrho_0, \vec{m}_0, E_0]$.

Then we have

$$\varrho(t, \cdot) = \overline{\varrho},\ \vec{m}(t, \cdot) = 0 \text{ for all } t \geq T.$$

If the system reaches a stationary state where the density is constant and the momentum vanishes, it remains in this state for all future times.

The semiflow selection is based on the general method proposed by Krylov [24] and its adaptation by Cardona and Kapitanski [5]. The fact that the selection complies with the maximal dissipation principle has an interesting consequence (see [20]), namely:

$$E(\tau) - \int_{\Omega} \left[\frac{1}{2} \frac{|\vec{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \to 0 \text{ as } \tau \to \infty.$$

for any dissipative solution maximizing the energy dissipation. In particular the Reynolds stress $\mathcal{R}$ vanishes in the asymptotic limit for large times. This fact may be seen as another piece of evidence supporting the physical relevance of dissipative solutions.

3.2 Euler system as vanishing viscosity limit, relevance to models of turbulence

A largely used approach in the mathematical studies of hydrodynamic turbulence is to add certain stochastic perturbation to the model. For instance, Yakhot and Orszak [26] suggested that a stochastically perturbed Navier–Stokes system shall possess the same *statistical properties* as the deterministic Navier–Stokes system with general inhomogeneous boundary conditions. Another approach how to observe turbulent behaviour is increasing the Reynolds number, meaning following the so-called vanishing viscosity regime.

Consider the Euler system

$$\begin{aligned} \mathrm{d}\varrho + \mathrm{div}(\varrho\vec{u})\mathrm{d}t &= 0, \\ \mathrm{d}(\varrho\vec{u}) + \mathrm{div}(\varrho\vec{u} \otimes \vec{u})\mathrm{d}t + \nabla p(\varrho)\mathrm{d}t &= \vec{F}(\varrho, \varrho\vec{u})\mathrm{d}W, \end{aligned} \tag{3.2}$$

driven by a stochastic forcing represented by a multiplicative cylindrical white noise W. We address the question, to which extent solutions of the Euler system can be "statistically equivalent" to the inviscid limit of the Navier–Stokes system

$$\begin{aligned} \partial_t\varrho + \mathrm{div}(\varrho\vec{u}) &= 0, \\ \partial_t(\varrho\vec{u}) + \mathrm{div}(\varrho\vec{u} \otimes \vec{u}) + \nabla p(\varrho) &= \mu\Delta\vec{u} + \lambda\nabla\mathrm{div}\vec{u}, \ \mu, \lambda \searrow 0 \end{aligned} \tag{3.3}$$

past a convex obstacle in R^3.

Following [17], we consider a domain

$$\Omega = R^d \setminus K, \ d = 2, 3$$

exterior to compact convex obstacle K. Let $(\varrho_n, \vec{u}_n)_{n=1}^{\infty}$ be a sequence of (weak) solutions of the Navier–Stokes system (3.3) in $(0, T) \times \Omega$ satisfying the boundary conditions

$$\vec{u}_n|_{\partial K} = 0, \ \vec{u}_n \to \vec{u}_\infty, \ \varrho_n \to \varrho_\infty \text{ as } |x| \to \infty$$

for constant field $\vec{u}_\infty$, $\varrho_\infty \geq 0$, and vanishing viscosity coefficients

$$\mu_n \searrow 0, \ \lambda_n \searrow 0.$$

In addition, consider the sequence of probability measures – empirical means –

$$\mathcal{V}_N = \frac{1}{N}\sum_{n=1}^{\infty} \delta_{(\varrho_n, \varrho_n \mathbf{u}_n)},$$

where δ is the Dirac distribution "sitting" on a suitable (infinite dimensional) trajectory space $\mathcal{T}$, see [17] for details.

The arguments presented below are explained in detail in [17]. Using stochastic compactness method, we can show that the sequence $(\mathcal{V}_N)_{N=1}^{\infty}$ is tight and converges to a limit

$$\mathcal{V}_N \to \mathcal{V}_{NS} \text{ narrowly in } \mathfrak{P}[\mathcal{T}].$$

Next, let us introduce the concept of *statistical equivalence*:

Let $\mathcal{P}_i$, $i = 1, 2$, be Borel probability measures on $\mathcal{T}$. We say that the two measures are *statistically equivalent* if the following holds:

- **Equality of expectations of density and momentum.**

$$\mathbb{E}_{\mathcal{P}_1}\left[\int_0^T \int_\Omega r\varphi \ \mathrm{d}x\,\mathrm{d}t\right] = \mathbb{E}_{\mathcal{P}_2}\left[\int_0^T \int_\Omega r\varphi \ \mathrm{d}x\,\mathrm{d}t\right],$$
$$\mathbb{E}_{\mathcal{P}_1}\left[\int_0^T \int_\Omega \vec{w}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t\right] = \mathbb{E}_{\mathcal{P}_2}\left[\int_0^T \int_\Omega \vec{w}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t\right],$$

  for any $\varphi \in C_c^\infty((0,T)\times\Omega)$, $\boldsymbol{\varphi} \in C_c^\infty((0,T)\times\Omega; R^d)$.

- **Equality of expectations of kinetic, internal and angular energy.**

$$\mathbb{E}_{\mathcal{P}_1}\left[\int_0^T \int_\Omega 1_{r>0}\frac{|\vec{w}|^2}{r}\varphi \ \mathrm{d}x\,\mathrm{d}t\right] = \mathbb{E}_{\mathcal{P}_2}\left[\int_0^T \int_\Omega 1_{r>0}\frac{|\vec{w}|^2}{r}\varphi \ \mathrm{d}x\,\mathrm{d}t\right],$$
$$\mathbb{E}_{\mathcal{P}_1}\left[\int_0^T \int_\Omega P(r)\varphi \ \mathrm{d}x\,\mathrm{d}t\right] = \mathbb{E}_{\mathcal{P}_2}\left[\int_0^T \int_{\mathbb{T}^d} P(r)\varphi \ \mathrm{d}x\,\mathrm{d}t\right], \tag{3.4}$$
$$\mathbb{E}_{\mathcal{P}_1}\left[\int_0^T \int_\Omega 1_{r>0}\frac{1}{r}(\mathbb{J}_{x_0}\cdot\vec{w})\cdot\vec{w}\ \varphi \ \mathrm{d}x\,\mathrm{d}t\right] = \mathbb{E}_{\mathcal{P}_2}\left[\int_0^T \int_\Omega 1_{r>0}\frac{1}{r}(\mathbb{J}_{x_0}\cdot\vec{w})\cdot\vec{w}\ \varphi \ \mathrm{d}x\,\mathrm{d}t\right],$$

  for any $x_0 \in R^d$, and any $\varphi \in C_c^\infty((0,T)\times\Omega)$, where

$$\mathbb{J}_{x_0}(x) = |x - x_0|^2\mathbb{I} - (x - x_0)\otimes(x - x_0).$$

Finally, we anticipate the hypothesis that the stochastically driven Euler system (3.2) generating a solution $\mathcal{V}_E$ is statistically equivalent to the vanishing viscosity limit $\mathcal{V}_{NS}$. Then necessarily:

- $\mathcal{V}_{NS}$ is a weak statistical solution of the *deterministic* Euler system. In particular, the genuinely stochastic model becomes irrelevant.

- If, in addition, the barycenter

$$(\overline{\varrho}, \overline{\mathbf{m}}) = \int_{\mathcal{T}} (r, \vec{w}) \mathrm{d}\mathcal{V}_{NS} \in \mathcal{T}$$

is a weak solution to the Euler system, then the low viscosity limit is deterministic

$$\mathcal{V}_{NS} = \delta_{(\overline{\varrho}, \overline{\mathbf{m}})},$$

and the sequence $(\varrho_n, \vec{m}_n)$ statistically converges to $(\overline{\varrho}, \overline{\mathbf{m}})$, specifically,

$$\frac{1}{N} \# \left\{ n \leq N \middle| \|\varrho_n - \overline{\varrho}\|_{L^\gamma(M)} + \|\vec{m}_n - \overline{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(M;R^d)} > \varepsilon \right\} \to 0 \text{ as } N \to \infty \tag{3.5}$$

for any $\varepsilon > 0$, and any compact $M \subset [0, T] \times \Omega$.

As a corollary, we recover the dichotomy proved [16]: If

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^\infty(0, T; L^\gamma_{\rm loc}(\Omega)),\ \mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(\Omega)),$$

then either

$$\varrho_n \to \varrho \text{ in } L^\gamma_{\rm loc}([0, T] \times \Omega),\ \mathbf{m}_n \to \mathbf{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}([0, T] \times \Omega; R^d),$$

or $(\varrho, \mathbf{m})$ *is not* a solution of the (deterministic) Euler system. Note in passing that the arguments used in [17] rely essentially on the fact that the stochastic integral in the Euler system is understood in Itô's sense, in particular it is a martingale.

4 Computing oscillatory solutions to the Euler system – a short excursion in numerical analysis

In accordance with the theoretical results mentioned above and in the line with the research of the group Fjordholm, Käppeli, Mishra, and Tadmor [21], [22], the "oscillatory solutions" of the Euler system described in terms of parametrized measures (Young measures) are objects of practical interest worth of visualising by means of numerical methods.

It turns out that the "standard" approach to Young measures via computing the limits of non–linear superpositions $b(\varrho_n, \vec{m}_n)$ of a sequence of consistent numerical approximations is not suitable as the convergence is necessarily weak. Instead we propose an alternative approach inspired by Komlós theorem later adapted by Balder [2]. The celebrated Komlós theorem (see [23]) can be viewed as a generalization of the Banach–Saks theorem:

$$\{U_n\}_{n=1}^\infty \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty.$$

This motivated the introduction of the concept of (S)–convergence in [15] and elaborated in the monograph [18]. We say that a sequence of approximate solutions (S)–converges to a parametrized measure $(\mathcal{V}_{t,x})_{t\in(0,T),x\in\Omega}$

$$(\varrho_n, \vec{m}_n) \overset{(S)}{\to} \mathcal{V}$$

if

$$\frac{1}{N}\sum_{n=1}^{N} b(\varrho_n, \vec{m}_n)(t,x) \to \int_{R^{d+1}} b(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \ \mathrm{d}\mathcal{V}_{t,x} \text{ for any } b \in BC(R^{d+1}) \text{ and a.a. } (t,x).$$

Note that $\mathcal{V}$ may exist even if the sequence does not generate a Young measure and it coincides with the Young measure as soon as the latter exists. Moreover, by virtue of Komlós theorem, the measure $\mathcal{V}$ always exists for a suitable subsequence.

We say that a sequence $(\varrho_n, \vec{m}_n)_{n=1}^{\infty}$ is a *consistent approximation* of the Euler system if the following holds:

- **Approximate equation of continuity.**

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \vec{m}_n \cdot \nabla_x \varphi] \ \mathrm{d}x \, \mathrm{d}t = -\int_\Omega \varrho_0 \varphi \ \ \mathrm{d}x + e_{1,n}[\varphi]$$

  for any $\varphi \in C_c^\infty[0,T) \times \Omega)$.

- **Approximate momentum equation**

$$\int_0^T \int_\Omega \left[\vec{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\vec{m}_n \otimes \vec{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t$$

$$= -\int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi} \ \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

  for any $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \Omega; R^d)$.

- **Stability - approximate energy inequality.**

$$\int_\Omega \left[\frac{1}{2} \frac{|\vec{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] (\tau, \cdot) \, \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \frac{|\vec{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x + e_{3,n}$$

  for aa.a. $0 < \tau < T$.

- **Consistency.** The error terms vanish in the asymptotic limit:

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0, \ \ e_{3,n} \to 0 \text{ as } n \to \infty$$

An unconditionally consistent finite volume approximation of the Euler system was proposed in [18, Chapter 9]. Thus introducing a sequence of approximate empiric probability measures

$$\nu^n_{t,x} = \delta_{(\varrho_n, \vec{m}_n)(t,x)}$$

we conclude, up to a suitable subsequence,

$$\left\| \mathrm{dist}_M \left[\frac{1}{N} \sum_{k=1}^{M} \nu^{n_k}_{t,x}; \mathcal{V}_{t,x} \right] \right\|_{L^q((0,T)\times\Omega)} \to 0$$

for some $q > 1$, where dist_M denotes the Monge–Kantorowich distance. Similarly, we obtain convergence of the first variation,

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu^{n_k}_{t,x}; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x}; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

in $L^1(Q)$, where $\widetilde{\mathbf{U}}$ denotes the "dummy variables" for $(\varrho, \vec{m})$, see [19].

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References

[1] A. Abbatiello and E. Feireisl. On strong continuity of weak solutions to the compressible Euler system. *J. Nonlinear Sci.*, **31**(2):Paper No. 33, 16, 2021.

[2] E.J. Balder. On weak convergence implying strong convergence in l^1 spaces. *Bull. Austral. Math. Soc.*, **33**:363–368, 1986.

[3] S. Benzoni-Gavage and D. Serre. *Multidimensional hyperbolic partial differential equations, First order systems and applications.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007.

[4] D. Breit, E. Feireisl, and M. Hofmanová. Solution semiflow to the isentropic Euler system. *Arch. Ration. Mech. Anal.*, **235**(1):167–194, 2020.

[5] J. E. Cardona and L. Kapitanski. Semiflow selection and Markov selection theorems. *Topol. Methods Nonlinear Anal.*, **56**(1):197–227, 2020.

[6] R. M. Chen, A. F. Vasseur, and Ch. Yu. Global ill-posedness for a dense set of initial data to the isentropic system of gas dynamics. *Adv. Math.*, 393:Paper No. 108057, 46, 2021.

[7] E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.*, **11**(3):493–519, 2014.

[8] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.*, **68**(7):1157–1190, 2015.

[9] E. Chiodaroli and E. Feireisl. On the density of "wild" initial data for the barotropic Euler system. **arxiv preprint No. 2208.04810**, 2022.

[10] C. M. Dafermos. The entropy rate admissibility criterion for solutions of hyperbolic conservation laws. *J. Differential Equations*, **14**:202–212, 1973.

[11] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.

[12] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**(1):225–260, 2010.

[13] C. De Lellis and L. Székelyhidi, Jr. The h-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, **49**(3):347–375, 2012.

[14] E. Feireisl. Weak solutions to problems involving inviscid fluids. In *Mathematical Fluid Dynamics, Present and Future*, volume 183 of *Springer Proceedings in Mathematics and Statistics*, pages 377–399. Springer, New York, 2016.

[15] E. Feireisl. (S)-convergence and approximation of oscillatory solutions in fluid dynamics. *Nonlinearity*, **34**(4):2327–2349, 2021.

[16] E. Feireisl and M. Hofmanová. On convergence of approximate solutions to the compressible Euler system. *Ann. PDE*, **6**(2):11, 2020.

[17] E. Feireisl and M. Hofmanová. Randomness in compressible fluid flows past an obstacle. *Journal of Statistical Physics*, **186**:32–, 2022.

[18] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová, and B. She. *Numerical analysis of compressible fluid flows*. Springer-Verlag, Cham, 2022.

[19] E. Feireisl, M. Lukáčová-Medvid'ová, H. Mizerová, B. She, and Y. Wang. Computing oscillatory solutions of the Euler system via $\mathcal{K}$-convergence. *Math. Models Methods Appl. Sci.*, 31(**3**):537–576, 2021.

[20] Eduard Feireisl. A note on the long-time behavior of dissipative solutions to the Euler system. *J. Evol. Equ.*, **21**(3):2807–2814, 2021.

[21] U. K. Fjordholm, R. Käppeli, S. Mishra, and E. Tadmor. Construction of approximate entropy measure valued solutions for hyperbolic systems of conservation laws. *Foundations Comp. Math.*, pages 1–65, 2015.

[22] U. S. Fjordholm, S. Mishra, and E. Tadmor. On the computation of measure-valued solutions. *Acta Numer.*, **25**:567–679, 2016.

[23] J. Komlós. A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.*, **18**:217–229, 1967.

[24] N. V. Krylov. The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. *Izv. Akad. Nauk SSSR Ser. Mat.*, **37**:691–708, 1973.

[25] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, 1967.

[26] V. Yakhot and S. A. Orszag. Renormalization group analysis of turbulence. I. Basic theory. *J. Sci. Comput.*, **1**(1):3–51, 1986.