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**Indestructibility of some compactness  
principles over models of PFA**

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# INDESTRUCTIBILITY OF SOME COMPACTNESS PRINCIPLES OVER MODELS OF PFA

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ABSTRACT. We show that PFA (Proper Forcing Axiom) implies that adding any number of Cohen subsets of  $\omega$  will not add an  $\omega_2$ -Aronszajn tree or a weak  $\omega_1$ -Kurepa tree, and moreover no  $\sigma$ -centered forcing can add a weak  $\omega_1$ -Kurepa tree (a tree of height and size  $\omega_1$  with at least  $\omega_2$  cofinal branches). This partially answers an open problem whether ccc forcings can add  $\omega_2$ -Aronszajn or  $\omega_1$ -Kurepa trees.

We actually prove more: We show that a consequence of PFA, namely the *guessing model principle*, GMP, which is equivalent to the *ineffable slender tree property*, ISP, is preserved by adding any number of Cohen subsets of  $\omega$ . And moreover, GMP implies that no  $\sigma$ -centered forcing can add a weak  $\omega_1$ -Kurepa tree (see Section 2.1 for definitions).

For more generality, we study variations of the principle GMP at higher cardinals and the indestructibility consequences they entail, and as applications we answer a question of Mohammadpour about guessing models at weakly but not strongly inaccessible cardinals and show that there is a model in which there are no weak  $\aleph_{\omega+1}$ -Kurepa trees and no  $\aleph_{\omega+2}$ -Aronszajn trees.

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## 1. INTRODUCTION

It has been a question of some interest whether “small” forcings (either in terms of size or chain condition) can add “large” trees. Depending on the meaning of “small” and “large”, there are both negative and positive results. Here are some examples for the positive answer:

- By a result of Shelah, a single Cohen subset of  $\omega$  adds an  $\omega_1$ -Suslin tree.
- Rinot showed in [25] that a cofinality preserving forcing of size  $\omega_3$  can add a special Aronszajn tree at  $\aleph_{\omega_1+1}$  (this requires large cardinals).
- Jin and Shelah [16] showed that an  $\omega_1$ -distributive forcing of size  $\omega_1$  can add an  $\omega_1$ -Kurepa tree.

However, if we take “small” to be countable or just ccc, and “large” to mean an  $\omega_2$ -Aronszajn tree or a (weak)  $\omega_1$ -Kurepa tree, the question is open. Even the simplest question whether a single Cohen subset of  $\omega$  can add an  $\omega_2$ -Aronszajn tree or an  $\omega_1$ -Kurepa tree over some model of ZFC remains unanswered. Similar cases can be considered at larger cardinals as well (see for instance [11] for  $\aleph_{\omega+1}$ ).

For the negative answer, more results are known. The main reason is that many principles inherited from large cardinals (“compactness principles”) tend to prohibit the existence of certain trees, and by starting with a carefully chosen model where such principles hold, one can sometimes show that no trees of the given type are added by small forcings. But as an answer to the question above, this approach may appear unconvincing, inasmuch as it depends on the particular model in question (see a brief summary of these results in Section 1.1).

A more convincing approach is to find an assumption  $\varphi$  such that ZFC +  $\varphi$  prohibits the existence of certain trees, and  $\varphi$  itself is always preserved by small forcings, not just over some particular model under consideration.<sup>1</sup> For instance Chang’s Conjecture, CC, plays this role for  $\omega_1$ -Kurepa trees: by a well-known theorem, CC is preserved by all ccc forcings, and hence no ccc forcing can add an  $\omega_1$ -Kurepa tree over any model of CC. If we subscribe to ZFC + CC, then we conclude that no ccc forcing can add an  $\omega_1$ -Kurepa tree.

In this paper we show that the *guessing model principle*, GMP,<sup>2</sup> which is a consequence of PFA, plays a similar role for  $\omega_2$ -Aronszajn trees and weak  $\omega_1$ -Kurepa trees (see Section 1.1 for definitions, and Lemma 2.5, Corollary 2.16 and Theorem 3.5 for proofs):

**Theorem.** *ZFC + GMP proves that adding any number of Cohen subsets of  $\omega$  will not add an  $\omega_2$ -Aronszajn tree or a weak  $\omega_1$ -Kurepa tree. Moreover, it proves that no  $\sigma$ -centered forcing can add a weak  $\omega_1$ -Kurepa tree.*

GMP is a certain compactness principle introduced by Viale and Weiss in [31]. It follows from PFA, but it is strictly weaker: for instance, it does not put any bound on the value of  $2^\omega$ , apart from contradicting CH. This principle can either be formulated in terms of guessing models, or equivalently in

<sup>1</sup> $\varphi$  can be just the sentence that there are no trees of the given type, but in the known examples, a stronger principle is usually required.

<sup>2</sup>We generalize this principle to larger cardinals, so we will have  $\text{GMP} = \text{GMP}_{\omega_2}$  in what follows.

terms of slender lists (generalizations of trees), and is known to capture the “combinatorial core” of supercompactness; see [32] and [31] for more details. For our proof, we find it more convenient to work with guessing models.

The paper is structured as follows. In Section 1.1 we review the basic definitions and provide a brief survey of results related to preservation of the tree property (i.e. not adding Aronszajn trees) or the negation of the weak Kurepa Hypothesis (i.e. not adding weak Kurepa trees).

In Section 2, generalizing the principle  $\mathbf{GMP}$  from [31], we define for regular cardinals  $\mu < \kappa$  principles  $\mathbf{GMP}_{\mu^+, \kappa}$  (if  $\kappa = \mu^{++}$ , we simply write  $\mathbf{GMP}_{\kappa}$ , and note that  $\mathbf{GMP} = \mathbf{GMP}_{\omega_2}$ ) and show that, if  $\mu = \mu^{<\mu}$ , it is preserved by adding any number of Cohen subsets to  $\mu$  (Corollary 2.16). Since  $\mathbf{GMP}_{\mu^{++}}$  implies that there are no  $\mu^{++}$ -Aronszajn trees and no weak  $\mu^+$ -Kurepa trees (Lemma 2.5), we obtain the desired result. At the end of the section, we apply our results to answer a question of Mohammadpour from [24] by proving the consistency of a particular guessing model property at a cardinal that is weakly but not strongly inaccessible.

In Section 3, Theorem 3.5, we prove a slightly stronger result that over models of  $\mathbf{GMP}_{\mu^{++}}$ , no  $\mu^+$ -centered forcing (see Definition 3.1) can add a weak  $\mu^+$ -Kurepa tree.

In Section 4, we give an application of this result, showing that starting with some large cardinals there is a model in which there are no weak  $\aleph_{\omega+1}$ -Kurepa trees and no  $\aleph_{\omega+2}$ -Aronszajn trees.

In Section 5 we state some open questions.

## 1.1. PRELIMINARIES

Suppose  $\lambda$  is a regular cardinal. We say that the *tree property* at  $\lambda$ ,  $\mathbf{TP}(\lambda)$ , holds if every  $\lambda$ -tree has a cofinal branch; equivalently, there are no  $\lambda$ -Aronszajn trees. We say that the *Kurepa Hypothesis* at  $\lambda$ ,  $\mathbf{KH}(\lambda)$ , holds if there is a  $\lambda$ -tree with at least  $\lambda^+$ -many cofinal branches (we call such a tree a  $\lambda$ -Kurepa tree); we say that the *weak Kurepa Hypothesis* at  $\lambda$ ,  $\mathbf{wKH}(\lambda)$ , holds if there is a tree of height and size  $\lambda$  with at least  $\lambda^+$ -many cofinal branches; we call such a tree a weak  $\lambda$ -Kurepa tree.<sup>3</sup> A  $\lambda$ -Aronszajn tree is an incompact object because it has chains of every size  $< \lambda$ , but no chains of size  $\lambda$ ; similarly a (weak)  $\lambda$ -Kurepa tree is an incompact object because every level of the tree has size  $< \lambda$  (or  $\leq \lambda$ ), yet there are  $\lambda^+$ -many cofinal branches.

It is known that together with inaccessibility, these properties are related to large cardinals (see for instance Devlin [5]):

**Fact 1.1.** *Suppose  $\lambda$  is an inaccessible cardinal.*

- (i)  *$\lambda$  is weakly compact if and only if  $\mathbf{TP}(\lambda)$ .*
- (ii) *If  $\lambda$  is ineffable, then  $\neg\mathbf{KH}(\lambda)$ .<sup>4</sup> Moreover, if  $V = L$ , then the converse is true as well.*

<sup>3</sup> Let us emphasize that the levels of a weak  $\lambda$ -Kurepa tree are allowed to have size  $\lambda$ .

<sup>4</sup>If  $\lambda$  is inaccessible, we say that  $T$  is a  $\lambda$ -tree if for every  $\omega \leq \alpha < \lambda$ ,  $|T_\alpha| \leq |\alpha|$  (sometimes such trees are called *slim*  $\lambda$ -trees). This prevents the full binary tree  $2^{<\lambda}$  from being a witness for  $\mathbf{KH}(\lambda)$ . However  $\mathbf{wKH}(\lambda)$  is always true for an inaccessible  $\lambda$ , with  $2^{<\lambda}$  being a witness.

Notice that the characterization of weak compactness by  $\text{TP}$  is provable in  $\text{ZFC}$ , while the characterization of ineffability by  $\neg\text{KH}$  requires  $V = L$  (to our knowledge, it is open whether the assumption  $V = L$  can be removed).

Unlike the inaccessibility of  $\lambda$ , the principles  $\text{TP}(\lambda)$  and  $\neg\text{KH}(\lambda)$  are more robust, and may be viewed as the “combinatorial core” of the respective large cardinal notions. For instance adding  $\lambda$  many Cohen subsets of  $\omega$  to an ineffable cardinal  $\lambda$  will destroy the inaccessibility of  $\lambda$ , but will not add any  $\lambda$ -Aronszajn trees or  $\lambda$ -Kurepa trees.

Silver in [26] was the first one to show that the non-existence of Kurepa trees can also hold on a successor cardinal: using the Levy collapse of an inaccessible, he showed that  $\neg\text{KH}(\omega_1)$  is consistent (with  $\text{CH}$ ). Soon afterwards, Mitchell, in [23], devised a different collapse which simultaneously adds subsets of  $\omega$ , and showed that  $\text{TP}(\omega_2)$  and  $\neg\text{wKH}(\omega_1)$  are consistent; see Abraham’s [1] for the now-standard presentation of the forcing. We shall denote the forcing devised by Mitchell by  $\mathbb{M}(\mu, \kappa)$ , where it is understood that  $\mu$  is a regular cardinal satisfying  $\mu^{<\mu} = \mu$  and  $\kappa$  is (at least) an inaccessible cardinal.  $\mathbb{M}(\mu, \kappa)$  collapses cardinals in the open interval  $(\mu^+, \kappa)$  and turns  $\kappa$  into  $\mu^{++}$ . By [23] or [1] if  $\kappa$  is weakly compact, then  $\text{TP}(\mu^{++})$  is true in  $V[\mathbb{M}(\mu, \kappa)]$ . A similar argument shows that if  $\kappa$  is inaccessible, then  $\neg\text{KH}(\mu^+)$  and in fact  $\neg\text{wKH}(\mu^+)$  holds in  $V[\mathbb{M}(\mu, \kappa)]$ .<sup>5</sup>

With the discovery that many compactness principles can hold at successor cardinals, it was natural to inquire which forcing notions can destroy them and which will preserve them: the motivation being a general interest, and also an interest in developing a technical tool for forcing constructions. Todorćević showed the compatibility of  $\text{MA}_{\omega_1}$  with  $\neg\text{wKH}(\omega_1)$  in [28], and essentially proved that over the Mitchell model  $V[\mathbb{M}(\omega, \kappa)]$ ,  $\kappa$  inaccessible, a finite support iteration of length  $\omega_2$  of ccc forcings of size at most  $\omega_1$  which do not add cofinal branches to  $\omega_1$ -Suslin trees does not add weak  $\omega_1$ -Kurepa trees; by Remark 1.7 in the same paper, it follows that the same iteration does not add  $\omega_2$ -Aronszajn trees over  $V[\mathbb{M}(\omega, \kappa)]$ ,  $\kappa$  weakly compact. Unger studied the indestructibility over the Mitchell model explicitly in his [29], and further results appeared in [12] and [13]. Apart from Todorćević’s [28], the preservation of  $\neg\text{wKH}(\omega_1)$  over the Mitchell model has not been studied explicitly,<sup>6</sup> but it is likely that the analogues of the results for the tree property obtained in [12] also apply to the negation of the weak Kurepa Hypothesis. One can extend this line of inquiry to consider other variants of Mitchell forcing, and also other methods for obtaining  $\text{TP}(\omega_2)$  and  $\neg\text{wKH}(\omega_1)$ , for instance the Sacks forcing (see [17]).

In all these examples, however, the question of preservation of the compactness principles is asked over specific models, and the strategy of proofs follows the same pattern: Since we know the “history” of the cardinal  $\kappa$  in

<sup>5</sup>Notice the apparent disparity between the parameters in these two principles after the collapse: while the tree property is still considered at  $\kappa$ , the negation of the (weak) Kurepa hypothesis has  $\mu^+$  as the parameter (the large cardinal  $\kappa$  is used to control the number of cofinal branches of trees of size  $\mu^+$ ).

<sup>6</sup>There is a preservation result for  $\neg\text{KH}(\omega_1)$  over the Levy collapse, see [15]. This however does not fit with our topic here because in that model  $\text{CH}$  is true, so necessarily both  $\text{TP}(\omega_2)$  and  $\neg\text{wKH}(\omega_1)$  fail.

our model, we can return back to the ground model where  $\kappa$  is a large cardinal, and we can fix an appropriate elementary embedding and do a lifting argument. For instance, to argue that  $\mathbb{P}$  preserves  $\text{TP}(\omega_2)$  over the model  $V[\mathbb{M}(\omega, \kappa)]$ , the typical argument is to lift an elementary embedding with critical point  $\kappa$  to the forcing  $\mathbb{M}(\omega, \kappa) * \dot{\mathbb{P}}$ . But what if we do not have an elementary embedding, and all we know is that  $\text{TP}(\omega_2)$  or  $\neg\text{wKH}(\omega_1)$  are true: can we say something about their preservation?

This leads to the question of considering preservation of compactness principles over theories extending ZFC, not just models. Here are some known results in this direction:

- Chang’s Conjecture is preserved by all ccc forcings; hence, over models of Chang’s Conjecture, so is its consequence  $\neg\text{KH}(\omega_1)$ .
- Foreman showed in [6] that  $\mu^{++}$ -saturated ideals over  $\mu^+$ ,  $\mu$  regular, are preserved by  $\mu^+$ -centered forcing notions (in the sense that they generate saturated ideals in the extension).
- Gitik and Krueger showed in [10] that the negation of the approachability property at  $\mu^{++}$ ,  $\mu$  regular, is preserved by all  $\mu^+$ -centered forcings.<sup>7</sup>
- The first and the third author of the present paper showed in [13] that stationary reflection at  $\mu^+$ ,  $\mu$  regular, is preserved by all  $\mu$ -cc forcing notions. They further showed in [13] that if  $\mu^{<\mu} = \mu$ , then club stationary reflection at  $\mu^{++}$  is preserved by Cohen forcing at  $\mu$  and Prikry forcing at  $\mu$ .<sup>8</sup>

We extend this list in this paper by showing (see Definition 2.1 and Corollary 2.16):

- If  $\mu^{<\mu} = \mu$ , then  $\text{GMP}_{\mu^{++}}$  is preserved by adding any number of Cohen subsets of  $\mu$ , and hence, over models of  $\text{GMP}_{\mu^{++}}$ , so are its consequences  $\text{TP}(\mu^{++})$  and  $\neg\text{wKH}(\mu^+)$ .

## 2. PRESERVATION OF THE GUESSING MODEL PRINCIPLE BY COHEN FORCING

### 2.1. GUESSING MODELS

In [32], building on work of Jech [14] and Magidor [21] providing combinatorial characterizations of strongly compact and supercompact cardinals, Weiss introduced the notion of a *slender  $\mathcal{P}_\kappa(\lambda)$ -list* and used this to formulate a powerful compactness principle called the *ineffable slender tree property*,  $\text{ISP}_\kappa$ . For inaccessible  $\kappa$ ,  $\text{ISP}_\kappa$  characterizes supercompactness and can be seen as capturing the combinatorial core of supercompactness, but  $\text{ISP}_\kappa$  can also consistently hold at an accessible  $\kappa$ . Viale and Weiss [31] proved that

<sup>7</sup>  $\mathbb{P}$  is  $\mu^+$ -centered if it can be written as the union of a family  $\{\mathbb{P}_\alpha \subseteq \mathbb{P} \mid \alpha < \mu\}$ , where for every  $\alpha$  and  $p, q \in \mathbb{P}_\alpha$  there is  $r \in \mathbb{P}_\alpha$  with  $r \leq p, q$ . See Definition 3.1 for more details. Note that sometimes “ $\mu^+$ -centered” is called “ $\mu$ -centered” instead (for instance in [10]). We prefer to say  $\mathbb{P}$  is  $\mu^+$ -centered to agree with the notation for the chain condition (every  $\mu^+$ -centered  $\mathbb{P}$  is  $\mu^+$ -cc in our notation).

<sup>8</sup>This preservation result has been recently extended to all  $\mu^+$ -linked forcings in [9] (being  $\mu^+$ -linked is slightly weaker than  $\mu^+$ -centered).

PFA implies  $\text{ISP}_{\omega_2}$ , and moreover provided an equivalent characterization of  $\text{ISP}_{\omega_2}$  in terms of *guessing models*.

In this paper, we work with this equivalent characterization in terms of guessing models and its generalizations. We state the definition in the form relevant for us, though more general versions are clearly possible. Recall the notation  $\mathcal{P}_\kappa(x)$  which denotes the set of all subsets of  $x$  of size  $< \kappa$ .

**Definition 2.1.** Let  $\mu < \theta$  be regular cardinals, and let  $M \prec H(\theta)$ .

- (1) Given a set  $x \in M$ , and a subset  $d \subseteq x$ , we say that
  - (a)  $d$  is  $(\mu^+, M)$ -*approximated* if, for every  $z \in M \cap \mathcal{P}_{\mu^+}(M)$ , we have  $d \cap z \in M$ ;
  - (b)  $d$  is  $M$ -*guessed* if there is  $e \in M$  such that  $d \cap M = e \cap M$ .
- (2) For  $x \in M$ ,  $M$  is a  $\mu^+$ -*guessing model for  $x$*  if every  $(\mu^+, M)$ -approximated subset of  $x$  is  $M$ -guessed.
- (3)  $M$  is a  $\mu^+$ -*guessing model* if, for every  $x \in M$ , it is a  $\mu^+$ -guessing model for  $x$ .

Let  $\kappa$  be a regular cardinal such that  $\mu < \kappa \leq \theta$  and  $\kappa$  is  $(< \mu)$ -inaccessible, i.e.,  $\nu^{< \mu} < \kappa$  for all  $\nu < \kappa$ . We denote by  $\text{GMP}_{\mu^+, \kappa}(\theta)$  the assertion that the set of  $M \in \mathcal{P}_\kappa(H(\theta))$  such that  ${}^{< \mu} M \subseteq M$  and  $M$  is a  $\mu^+$ -guessing model is stationary in  $\mathcal{P}_\kappa(H(\theta))$ . We write  $\text{GMP}_{\mu^+, \kappa}$  if  $\text{GMP}_{\mu^+, \kappa}(\theta)$  holds for every regular  $\theta \geq \kappa$ . If  $\kappa = \mu^{++}$ , then we will simply write  $\text{GMP}_\kappa(\theta)$  or  $\text{GMP}_\kappa$  in place of  $\text{GMP}_{\mu^+, \kappa}(\theta)$  or  $\text{GMP}_{\mu^+, \kappa}$ , e.g.,  $\text{GMP}_{\omega_2}$  is the same as  $\text{GMP}_{\omega_1, \omega_2}$ .

Note that in the principles defined above we specifically require that the guessing models in question are closed under sequences of length  $< \mu$ . In other words, we require in  $\text{GMP}_{\mu^+, \kappa}(\theta)$  that the guessing models concentrate on the stationary set of all  $x \in \mathcal{P}_\kappa(H(\theta))$  such that  ${}^{< \mu} x \subseteq x$  (this set is in fact closed under increasing unions of cofinality at least  $\mu$ ). If  $\mu = \omega$ , this is automatic because it just means that the models are closed under finite sequences; however for  $\mu > \omega$  we need to require this property to show that the concept behaves as expected (see for instance Lemma 2.4).

Viale and Weiss essentially proved in [31] the following (see [31] for the definition of  $\text{ISP}_{\omega_2}$ ):

- Fact 2.2.** (i)  $\text{GMP}_{\omega_2}$  is equivalent to  $\text{ISP}_{\omega_2}$ .  
(ii) PFA implies  $\text{GMP}_{\omega_2}$ .

Models with  $\text{GMP}_{\mu^{++}}$  can be obtained starting with sufficiently large cardinals:

**Fact 2.3.** Suppose that  $\mu = \mu^{< \mu}$  is regular and  $\kappa > \mu$  is supercompact. Then in the Mitchell model  $V[\mathbb{M}(\mu, \kappa)]$ , which turns  $\kappa$  into  $\mu^{++}$ ,  $\text{GMP}_{\mu^{++}}$  holds.

*Proof.* The argument is essentially the same as in [32] and [31]. Let us only mention why we can assume that the guessing models concentrate on sets closed under  $< \mu$ -sequences, as we require in our definition of  $\text{GMP}_{\mu^{++}}$ . This follows from the fact that the characterization of  $\theta$ -supercompactness of a supercompact cardinal  $\kappa$  by means of  $\theta$ -ineffability (see [21, p. 281]) ensures the stationarity of the required set by showing the stronger property, namely that the set in question is an element of a normal ultrafilter on  $\mathcal{P}_\kappa(\theta)$ . When reformulated for  $\mathcal{P}_\kappa(H(\theta))$ , with  $\theta = |H(\theta)|$ , it follows that we can start with



stationary sets which concentrate on the set of all submodels of  $H(\theta)$  which are closed under sequences of length  $< \nu$ , for any fixed  $\nu < \kappa$ . Since  $\mathbb{M}(\mu, \kappa)$  does not add new sequences of length  $< \mu$  (but adds many new subsets of  $\mu$ ), the argument proceeds with  $\nu = \mu$ .  $\square$

The following lemma generalizes the analogous lemma in [3] which was formulated for  $\mu = \omega$ .

**Lemma 2.4.** *Suppose that  $\mu < \mu^{++} \leq \theta$  are infinite regular cardinals, and let  $M \prec H(\theta)$  be a  $\mu^+$ -guessing model with  ${}^{<\mu}M \subseteq M$ . Let  $\nu \in M$  be a cardinal with cofinality  $\geq \mu^+$ .*

- (i) *Then  $\text{cf}(\sup(M \cap \nu)) \geq \mu^+$ , and in particular  $\mu^+ \subseteq M$  and so  $M \cap \mu^{++}$  is an ordinal.*
- (ii) *If  $\theta \geq \mu^{+3}$ , then moreover  $\text{cf}(M \cap \mu^{++}) \geq \mu^+$ .*

*Proof.* (i). This is like Lemma 2.3 from [3] using the fact that  ${}^{<\mu}M \subseteq M$ .

- (ii). If  $\theta \geq \mu^{+3}$ , then  $\mu^{++} \in M$ , and the claim follows by (i).  $\square$

Generalizing the results known for  $\text{GMP}_{\omega_2}$ , let us review the argument that  $\text{GMP}_{\mu^{++}}$  implies  $\text{TP}(\mu^{++})$  and  $\neg\text{wKH}(\mu^+)$ .

**Lemma 2.5.**  *$\text{GMP}_{\mu^{++}}$  implies  $\text{TP}(\mu^{++})$  and  $\neg\text{wKH}(\mu^+)$ .*

*Proof.* Assume  $\text{GMP}_{\mu^{++}}$  holds. Let us first show that  $\text{TP}(\mu^{++})$  holds. Suppose  $T$  is a  $\mu^{++}$ -tree; we wish to show that it contains a cofinal branch. Choose a  $\mu^+$ -guessing model  $M \prec H(\mu^{+3})$  of size  $\mu^+$  with  $T \in M$  (note that automatically  $\mu^+ \cup \{\mu^+\} \subseteq M$  by Lemma 2.4(i)). Let  $t$  be any node in  $T$  on level  $\delta = M \cap \mu^{++}$ . Denote by  $d = \{s \mid s <_T t\}$  the set of predecessors of  $t$  in  $T$ ; notice that  $d \subseteq M$ , and  $d \subseteq T \in M$ . The set  $d$  is  $(\mu^+, M)$ -approximated: if  $z \in M$  has size  $\mu$ , then since  $\delta$  has cofinality  $\mu^+$  (Lemma 2.4(ii)), there is  $t^* \in d$  such that  $z \cap d$  is definable in  $M$  as the set of all predecessors of  $t^*$  which are in  $z$ , and so  $z \cap d \in M$ . Since  $M$  is a  $\mu^+$ -guessing model, and we showed that  $d$  is  $(\mu^+, M)$ -approximated, there is  $e \in M$  with  $e \cap M = d \cap M$ . It follows  $M \models (e \text{ is a cofinal branch in } T)$ , and by elementarity this is true in  $H(\mu^{+3})$ , and hence in  $V$ .

In a similar way we can show that  $\neg\text{wKH}(\mu^+)$  holds. Suppose for contradiction that  $T$  is a tree of size and height  $\mu^+$  which has more than  $\mu^+$ -many branches. Choose a  $\mu^+$ -guessing model  $M \prec H(\mu^{++})$  of size  $\mu^+$  with  $T \in M$  (note that automatically  $\mu^+ \cup \{\mu^+\} \subseteq M$  by Lemma 2.4(i)). Every cofinal branch  $b \subseteq T$  is  $(\mu^+, M)$ -approximated, using the fact that  $T$  has height  $\mu^+$ . It follows that every cofinal branch  $b$  must be in  $M$ , but this contradicts the fact that  $M$  has size  $\mu^+$ .  $\square$

Note that we need just the principle  $\text{GMP}_{\mu^{++}}(\mu^{+3})$  for obtaining  $\text{TP}(\mu^{++})$ , and  $\text{GMP}_{\mu^{++}}(\mu^{++})$  for obtaining  $\neg\text{wKH}(\mu^+)$ .

## 2.2. A PRESERVATION THEOREM

In [2], Cox and Krueger prove that  $\text{GMP}_{\omega_2}$  is compatible with any consistent value of the continuum greater than  $\omega_1$  by producing a specific model of ZFC over which  $\text{GMP}_{\omega_2}$  is indestructible under adding any number of Cohen reals. In this section, we remove the dependence of their result on a particular choice of ground model by proving that, over *any* model of ZFC, if  $\mu^{<\mu} = \mu$ ,

then  $\text{GMP}_{\mu^{++}}$  is preserved by adding any number of Cohen subsets of  $\mu$ . In fact, we will prove a more general result; see Theorem 2.8 below for the precise statement.<sup>9</sup>

The following lemma is essentially due to Krueger [18], presented here in a slightly more general form.

**Lemma 2.6.** *Suppose that  $\mu < \theta$  are infinite regular cardinals and  $M \prec H(\theta)$  is a  $\mu^+$ -guessing model such that  ${}^{<\mu}M \subseteq M$ . Then  $M$  is  $\mu^+$ -internally unbounded, i.e., for every  $z \in M$  and every  $x \in [z \cap M]^\mu$ , or equivalently every  $x \in [z]^\mu$  with  $x \subseteq M$ , there is  $y \in [z]^\mu \cap M$  such that  $x \subseteq y$ .*

*Proof.* Assume for the sake of contradiction that there are  $z \in M$  and  $x \in [z \cap M]^\mu$  such that there is no  $y \in [z]^\mu \cap M$  for which  $x \subseteq y$ . Injectively enumerate  $x$  as  $\langle a_\alpha \mid \alpha < \mu \rangle$  and, for each  $\beta < \mu$ , let  $x_\beta := \{a_\alpha \mid \alpha < \beta\}$ . Since  ${}^{<\mu}M \subseteq M$ , we have  $x_\beta \in M$  for all  $\beta < \mu$ . Let  $x^* := \{x_\beta \mid \beta < \mu\}$ . Then  $x^* \subseteq [z]^{<\mu} \in M$ .

**Claim 2.7.**  *$x^*$  is  $(\mu^+, M)$ -approximated.*

*Proof.* Fix  $w \in M$  with  $|w| \leq \mu$ . Since  $x^* \subseteq [z]^{<\mu} \in M$ , we can assume that  $w \subseteq [z]^{<\mu}$ . If it were the case that  $|w \cap x^*| = \mu$ , then  $\bigcup w$  would be an element of  $[z]^\mu \cap M$  covering  $x$ , contradicting our assumption. Therefore,  $|w \cap x^*| < \mu$ , so, since  ${}^{<\mu}M \subseteq M$ , we have  $w \cap x^* \in M$ .  $\square$

Since  $M$  is a  $\mu^+$ -guessing model, we can find  $e \in M$  such that  $e \cap M = x^* \cap M = x^*$ . If it were the case that  $|e| > \mu$ , then there would be an injection  $f : \mu^+ \rightarrow e$  with  $f \in M$ , and since, by Lemma 2.4,  $\mu^+ \subseteq M$ , we would have  $|e \cap M| > \mu$ , contradicting the fact that  $e \cap M = x^*$  and  $|x^*| = \mu$ . Therefore,  $|e| = \mu$ . Since  $\mu \subseteq M$ , it follows that  $e \subseteq M$ , and therefore we in fact have  $e = x^*$ . It then follows that  $\bigcup e = x$  is an element of  $M$ , again contradicting our assumption and completing the proof.  $\square$

**Theorem 2.8.** *Let  $\nu \leq \mu < \theta$  be infinite regular cardinals such that  $\mu^{<\nu} = \mu$ , let  $\chi \geq \nu$  be a cardinal, and let  $\mathbb{P} := \text{Add}(\nu, \chi)$ . Suppose that  $M \prec H(\theta)$  is a  $\mu^+$ -guessing model such that  ${}^{<\mu}M \subseteq M$  and  $\mathbb{P} \in M$ . Then, in  $V[\mathbb{P}]$ ,  $M[\mathbb{P}]$  is a  $\mu^+$ -guessing model.*

*Proof.* Conditions in  $\mathbb{P}$  are partial functions  $p : \chi \rightarrow 2$ , with  $|p| < \nu$ . The fact that  $\nu \leq \mu$  and  ${}^{<\mu}M \subseteq M$  implies that the empty condition is  $(M, \mathbb{P})$ -generic. Therefore,  $1_{\mathbb{P}} \Vdash "M[\mathbb{P}] \cap \text{On} = M \cap \text{On}"$ . It thus suffices to show that  $M[\mathbb{P}]$  is forced to be a  $\mu^+$ -guessing model for  $\delta$  for every ordinal  $\delta \in M$ .

To this end, fix an ordinal  $\delta \in M$ , a  $\mathbb{P}$ -name  $\dot{d}$  for a subset of  $\delta$ , and a condition  $p \in \mathbb{P}$  such that  $p \Vdash "\dot{d} \text{ is } (\mu^+, M[\mathbb{P}])\text{-approximated}"$ . Note that, by Proposition 2.6,  $M$  is  $\mu^+$ -internally unbounded, so  $M \cap [\delta]^\mu$  is  $\subseteq$ -unbounded in  $[M \cap \delta]^\mu$ . Let  $f : [M \cap \delta]^\mu \rightarrow M \cap [\delta]^\mu$  be a function such that  $w \subseteq f(w)$  for all  $w \in [M \cap \delta]^\mu$  and  $f(w) = w$  for all  $w \in M \cap [\delta]^\mu$ .

For each  $\alpha < \delta$ , let  $A_\alpha$  be a maximal antichain of  $\mathbb{P}$  below  $p$  consisting of conditions deciding the statement  $\alpha \in \dot{d}$ , and let  $u_\alpha = \bigcup \{\text{dom}(p) \mid p \in A_\alpha\}$ . Since  $\mu^{<\nu} = \mu$ ,  $\mathbb{P}$  has the  $\mu^+$ -cc, so we know that  $|u_\alpha| \leq \mu$ . Moreover, for

<sup>9</sup>We would like to thank to Menachem Magidor who shared with us his unpublished proof that  $\text{GMP}_{\omega_2}$  is preserved by adding a single Cohen subset of  $\omega$ , which helped lead us to our proof of Theorem 2.8.

every  $q \leq p$ , if  $q$  decides the statement  $\alpha \in \dot{d}$ , then  $q \upharpoonright u_\alpha$  already decides the statement in the same way.

For any  $\mathbb{P}$ -name  $\dot{x}$  for a subset of  $\delta$ , let  $\hat{x}$  denote the function from  $\mathbb{P} \times \delta$  to 3 such that, for all  $(q, \alpha) \in \mathbb{P} \times \delta$ , we have

- $\hat{x}(q, \alpha) = 0$  if and only if  $q \Vdash \alpha \in \dot{x}$ ;
- $\hat{x}(q, \alpha) = 1$  if and only if  $q \Vdash \alpha \notin \dot{x}$ ;
- $\hat{x}(q, \alpha) = 2$  if and only if  $q$  does not decide the statement  $\alpha \in \dot{x}$ .

For each  $z \in M \cap [\delta]^\mu$ , using the fact that  $p$  forces  $\dot{d}$  to be  $(\mu^+, M[\mathbb{P}])$ -approximated, find  $p_z \leq p$  and a  $\mathbb{P}$ -name  $\dot{d}_z \in M$  such that  $p_z \Vdash \dot{d} \cap z_\eta = \dot{d}_z$ .

Let  $\Xi$  be a sufficiently large regular cardinal, and let  $\mathcal{N}$  denote the set of  $N \prec H(\Xi)$  such that

- $|N| = \mu \subseteq N$ ;
- $f, \dot{d}, M, \langle (\dot{d}_z, p_z) \mid z \in M \cap [\delta]^\mu \rangle, \langle u_\alpha \mid \alpha < \delta \rangle \in N$ ;
- ${}^{<\nu}N \subseteq N$ .

Since  $\mu^{<\nu} = \mu$ ,  $\mathcal{N}$  is stationary in  $\mathcal{P}_{\mu^+}(H(\Xi))$ . For each  $N \in \mathcal{N}$ , let  $z_N = f(N \cap M \cap \delta)$ ,  $\dot{d}_N = \dot{d}_{z_N}$ , and  $p_N = p_{z_N}$ . Since each element of  $\mathcal{N}$  is closed under  $(< \nu)$ -sequences, we have  $p_N \cap N \in N$  for all  $N \in \mathcal{N}$ . Therefore, by Fodor's Lemma, we can find a stationary  $\mathcal{N}' \subseteq \mathcal{N}$  and a condition  $p_\emptyset \in \mathbb{P}$  such that  $p_N \cap N = p_\emptyset$  for all  $N \in \mathcal{N}'$ . Let  $r = \text{dom}(p_\emptyset)$  and, for all  $N \in \mathcal{N}'$ , let  $s_N = \text{dom}(p_N) \setminus r$ . We will prove that  $p_\emptyset$  forces  $\dot{d}$  to be  $M[\mathbb{P}]$ -guessed. Since  $p_\emptyset \leq p$  and  $p$  was chosen arbitrarily, this suffices to prove the theorem.

**Claim 2.9.** *Suppose that  $N \in \mathcal{N}'$ ,  $\alpha \in M \cap \delta$ , and  $q \in \mathbb{P} \cap M$  are such that  $\alpha \in N$ ,  $q \leq p_\emptyset \cap M$ , and  $q \parallel p_N$ . Then  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = \hat{d}(q \cup p_N, \alpha)$ .*

*Proof.* We will prove that  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = 0$  if and only if  $\hat{d}(q \cup p_N, \alpha) = 0$ . The proof of the rest of the claim is the same, *mutatis mutandis*. For the forward direction, suppose that  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = 0$ . Then  $q \cup (p_N \cap M) \Vdash \alpha \in \dot{d}_N$  and  $p_N \Vdash \dot{d} \cap z_N = \dot{d}_N$ , so it follows that  $q \cup p_N \Vdash \alpha \in \dot{d}$ , and hence  $\hat{d}(q \cup p_N, \alpha) = 0$ .

For the backward direction, suppose that  $\hat{d}_N(q \cup (p_N \cap M), \alpha) \neq 0$ . Then, by elementarity, we can find  $q' \leq q \cup (p_N \cap M)$  in  $M$  such that  $q' \Vdash \alpha \notin \dot{d}_N$ . Since  $q' \in M$  and  $q' \leq p_N \cap M$ , it follows that  $q' \parallel p_N$ . Then  $q' \cup p_N \Vdash \dot{d} \cap z_N = \dot{d}_N$ , and hence, since  $\alpha \in N$ , we have  $q' \cup p_N \Vdash \alpha \notin \dot{d}$ . Since  $q' \cup p_N \leq q \cup p_N$ , it follows that  $q \cup p_N \not\Vdash \alpha \in \dot{d}$ , i.e.,  $\hat{d}(q \cup p_N, \alpha) \neq 0$ .  $\square$

Let  $D := \{(q, \alpha) \in (\mathbb{P} \times \delta) \cap M \mid q \leq p_\emptyset \cap M\}$ . For each  $(q, \alpha) \in D$ , let  $B_{q,\alpha}$  be the set of all  $N \in \mathcal{N}'$  such that  $q, \alpha \in N$ .

**Claim 2.10.** *For all  $(q, \alpha) \in D$  and  $N \in B_{q,\alpha}$ , we have  $s_N \cap (u_\alpha \cup \text{dom}(q)) = \emptyset$  and  $q \parallel p_N$ .*

*Proof.* Fix  $(q, \alpha) \in D$  and  $N \in B_{q,\alpha}$ . Since  $q, \alpha \in N$  and  $\mu \subseteq N$ , we have  $u_\alpha \cup \text{dom}(q) \subseteq N$ . By construction,  $r = \text{dom}(p_N) \cap N$ , so  $s_N \cap N = \emptyset$ . Therefore,  $s_N \cap (u_\alpha \cup \text{dom}(q)) = \emptyset$ . Since, additionally, we have  $q \in M$  and  $q \leq p_\emptyset \cap M$ , it follows that  $q \parallel p_N$ .  $\square$

**Claim 2.11.** *For all  $(q, \alpha) \in D$  and  $N \in B_{q, \alpha}$ , we have*

$$\hat{d}_N(q \cup (p_N \cap M), \alpha) = \hat{d}(q \cup p_\emptyset, \alpha).$$

*Proof.* Fix  $(q, \alpha) \in D$  and  $N \in B_{q, \alpha}$ . We will prove that  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = 0$  if and only if  $\hat{d}(q \cup p_\emptyset, \alpha) = 0$ . The proof of the rest of the claim is the same. For the forward direction, suppose that  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = 0$ . Then, by Claim 2.9,  $\hat{d}(q \cup p_N, \alpha) = 0$ , i.e.,  $q \cup p_N \Vdash \alpha \in \dot{d}$ . It follows that  $(q \cup p_N) \upharpoonright u_\alpha \Vdash \alpha \in \dot{d}$ . But, by Claim 2.10, we have  $s_N \cap u_\alpha = \emptyset$ , so  $q \cup p_\emptyset \leq (q \cup p_N) \upharpoonright u_\alpha$ , and hence  $\hat{d}(q \cup p_\emptyset, \alpha) = 0$ .

For the backward direction, suppose that  $\hat{d}(q \cup p_\emptyset, \alpha) = 0$ . Then, *a fortiori*,  $\hat{d}(q \cup p_N, \alpha) = 0$ , so, by Claim 2.9, we have  $\hat{d}_N(q \cup (p_N \cap M), \alpha) = 0$ , as desired.  $\square$

Now define a function  $\tau : D \rightarrow \mathfrak{3}$  by letting  $\tau(q, \alpha) := \hat{d}(q \cup p_\emptyset, \alpha)$  for all  $(q, \alpha) \in D$ .

**Claim 2.12.**  *$\tau$  is  $(\mu^+, M)$ -approximated.*

*Proof.* It suffices to show that, for every  $y \in M$  with  $|y| = \mu$ , we have  $\tau \upharpoonright y \in M$ . Fix such a  $y$ . We can assume that  $y \subseteq \mathbb{P} \times \delta$  and, for all  $(q, \alpha) \in y$ , we have  $q \leq p_\emptyset \cap M$ . Since  $|y| = \mu$ , we can find  $N \in \mathcal{N}'$  such that  $y \subseteq N$ , and hence  $N \in \bigcap \{B_{q, \alpha} \mid (q, \alpha) \in y\}$ .

By Claim 2.11, it follows that, for all  $(q, \alpha) \in y$ , we have  $\tau(q, \alpha) = \hat{d}_N(q \cup (p_N \cap M), \alpha)$ . Since  $\hat{d}_N$ ,  $p_N \cap M$ , and  $y$  are in  $M$ , it follows that  $\tau \upharpoonright y$  is definable in  $M$  and is therefore an element of  $M$ .  $\square$

Note that  $\tau$  is a subset of  $(\mathbb{P} \times \delta) \times \mathfrak{3}$ , which is an element of  $M$ . Since  $M$  is a  $\mu^+$ -guessing model, we can find  $\sigma \in M$  such that  $\sigma \cap M = \tau \cap M = \tau$ . By elementarity,  $\sigma$  is a function from  $\{q \in \mathbb{P} \mid q \leq p_\emptyset \cap M\} \times \delta$  to  $\mathfrak{3}$ .

Let  $G \subseteq \mathbb{P}$  be a  $V$ -generic filter with  $p_\emptyset \in G$ . Let  $E := \{N \in \mathcal{N}' \mid p_N \in G\}$ .

**Claim 2.13.**  *$E$  is  $\subseteq$ -cofinal in  $(\mathcal{P}_{\mu^+}(H(\Xi)))^V$ .*

*Proof.* By a standard density argument, it suffices to show that, for every  $w \in (\mathcal{P}_{\mu^+}(H(\Xi)))^V$  and every  $q \leq p_\emptyset$ , there is  $N \in \mathcal{N}'$  such that  $w \subseteq N$  and  $q \parallel p_N$ . Fix such a  $w$  and  $q$ . Find  $N \in \mathcal{N}'$  such that  $w \subseteq N$  and  $q \in N$ . By Claim 2.10, we have  $q \parallel p_N$ , as desired.  $\square$

Let  $d$  be the interpretation of  $\dot{d}$  in  $V[G]$ , and, for all  $N \in \mathcal{N}'$ , let  $d_N$  be the interpretation of  $\dot{d}_N$ . Let

$$e = \{\alpha < \delta \mid \exists q \in G [\sigma(q, \alpha) = 0]\}.$$

Everything needed to define  $e$  is in  $M[G]$ , so  $e \in M[G]$ . We will be done if we show that  $e \cap M[G] = d \cap M[G]$ . We first prove two preliminary claims.

**Claim 2.14.** *Let  $\alpha \in \delta \cap M$ .*

- (1)  $\alpha \in d$  if and only if there is  $q \in G \cap M$  such that  $\tau(q, \alpha) = 0$ .
- (2)  $\alpha \notin d$  if and only if there is  $q \in G \cap M$  such that  $\tau(q, \alpha) = 1$ .

*Proof.* We prove (1). The proof of (2) follows by a symmetric argument. For the forward direction, suppose that  $\alpha \in d$ . Find  $N \in E$  such that  $\alpha \in N$ , and hence  $s_N \cap u_\alpha = \emptyset$ . Then  $d \cap z_N = d_N$ , so we can find  $q \in G \cap M$  such that  $q \Vdash \alpha \in \dot{d}_N$ . It follows that  $q \cup p_N \Vdash \alpha \in \dot{d}$ , and therefore that  $(q \cup p_N) \upharpoonright u_\alpha \Vdash \alpha \in \dot{d}$ . Since  $s_N \cap u_\alpha = \emptyset$ , we have  $q \cup p_\emptyset \leq (q \cup p_N) \upharpoonright u_\alpha$ . Therefore,  $q \cup p_\emptyset \Vdash \alpha \in \dot{d}$ , and hence  $\tau(q, \alpha) = 0$ .

For the backward direction, suppose that  $q \in G \cap M$  and  $\tau(q, \alpha) = 0$ . Find  $N \in E \cap B_{q, \alpha}$ . By Claim 2.11, we have  $\dot{d}_N(q \cup (p_N \cap M), \alpha) = 0$ . Since  $q \cup p_N \in G$ , we therefore have  $\alpha \in d_N$ . Again since  $p_N \in G$ , we have  $d \cap z_N = d_N$ , and hence  $\alpha \in d$ .  $\square$

**Claim 2.15.** *For all  $\alpha \in \delta$  and  $q_0, q_1 \in \mathbb{P}$  with  $q_0, q_1 \leq p_\emptyset \cap M$ , if  $\sigma(q_0, \alpha) = 0$  and  $\sigma(q_1, \alpha) = 1$ , then  $q_0$  and  $q_1$  are incompatible in  $\mathbb{P}$ .*

*Proof.* This is immediate from the elementarity of  $M$  and the fact that  $\sigma \cap M = \tau \cap M$ .  $\square$

We are now ready to show that  $e \cap M[G] = d \cap M[G]$ . Fix  $\alpha \in \delta \cap M$ , and suppose first that  $\alpha \in d$ . By Claim 2.14, there is  $q \in G \cap M$  such that  $\tau(q, \alpha) = 0$ . But then we also have  $\sigma(q, \alpha) = 0$ , so  $\alpha \in e$ . For the other direction, suppose that  $\alpha \notin d$ . Again by Claim 2.14, there is  $q \in G \cap M$  such that  $\tau(q, \alpha) = 1$ , and hence  $\sigma(q, \alpha) = 1$ . By Claim 2.15, there cannot be  $q' \in G$  such that  $\sigma(q', \alpha) = 0$ , and therefore  $\alpha \notin e$ .  $\square$

We immediately obtain the following corollary.

**Corollary 2.16.** *Suppose that  $\mu < \kappa$  are regular cardinals such that  $\mu^{<\mu} = \mu$  and  $\text{GMP}_{\mu^+, \kappa}$  holds. Then  $\text{GMP}_{\mu^+, \kappa}$  is preserved by adding any number of Cohen subsets to  $\mu$ .*

*Proof.* Fix a cardinal  $\chi \geq \mu$ , let  $\mathbb{P} := \text{Add}(\mu, \chi)$ , and let  $\theta > \chi$  be a sufficiently large regular cardinal. Let  $\dot{C}$  be a  $\mathbb{P}$ -name for a club in  $(\mathcal{P}_\kappa(H(\theta)))^{V[\mathbb{P}]}$ . It suffices to prove that  $\mathbb{P}$  forces that there is a  $\mu^+$ -guessing model  $\dot{N} \in \dot{C}$  such that  ${}^{<\mu}\dot{N} \subseteq \dot{N}$ . To this end, find a  $\mu^+$ -guessing model  $M \prec H(\theta)$  such that  $\mathbb{P}, \dot{C} \in M$ ,  $|M| < \kappa$ , and  ${}^{<\mu}M \subseteq M$ . By Theorem 2.8,  $M[\mathbb{P}]$  is forced to be a guessing model in  $V[\mathbb{P}]$ . Moreover,  $\dot{C} \cap M[\mathbb{P}]$  is forced to be a directed subset of  $\dot{C}$  of size less than  $\kappa$  whose union is all of  $M[\mathbb{P}]$ . Since  $\dot{C}$  is forced to be a club in  $(\mathcal{P}_\kappa(H(\theta)))^{V[\mathbb{P}]}$ , it follows that  $M[\mathbb{P}]$  is forced to be in  $\dot{C}$ . Finally, since  ${}^{<\mu}M \subseteq M$  and  $\mathbb{P}$  is  $\mu$ -closed, it is forced to be the case that  ${}^{<\mu}M[\mathbb{P}] \subseteq M[\mathbb{P}]$ . Therefore,  $\text{GMP}_{\mu^+, \kappa}$  continues to hold in  $V[\mathbb{P}]$ .  $\square$

Note that in the previous proof, we showed that in  $V[\mathbb{P}]$ ,  $\text{GMP}_{\mu^+, \kappa}(\theta)$  holds for all sufficiently large regular  $\theta \geq \kappa$ , whereas  $\text{GMP}_{\mu^+, \kappa}$  asserts this for *all* regular  $\theta \geq \kappa$ . These are obviously the same, though, since, if  $\kappa \leq \theta \leq \theta'$  are regular cardinals and  $M \prec H(\theta')$  is a  $\mu^+$ -guessing model with  $\theta \in M$ , then  $M \cap H(\theta)$  is also a  $\mu^+$ -guessing model.

**Corollary 2.17.** *Assume PFA holds. Then both  $\text{TP}(\omega_2)$  and  $\neg \text{wKH}(\omega_1)$  are preserved by adding any number of Cohen subsets of  $\omega$ .*

*Proof.* This follows from Fact 2.2, Lemma 2.5, and Corollary 2.16.  $\square$

**Remark 2.18.** Note that even a single Cohen subset of  $\omega$  destroys PFA because it adds an  $\omega_1$ -Suslin tree. So it is essential to isolate a principle weaker than PFA for our preservation result.

In [24, Problem 4.33], Mohammadpour asks whether it is consistent for there to exist a weakly inaccessible but not strongly inaccessible cardinal  $\kappa$  such that, for all regular  $\theta \geq \kappa$ , the set of  $\kappa$ -guessing models is stationary in  $\mathcal{P}_\kappa H(\theta)$ . As another application of our preservation theorem, we give a positive answer to this, in fact proving something stronger.

**Corollary 2.19.** *Suppose that  $\kappa$  is supercompact and  $\mathbb{P}$  is the forcing to add any number of Cohen reals. Then  $\text{GMP}_{\omega_1, \kappa}$  holds in  $V[\mathbb{P}]$ . In particular, there is a forcing extension in which  $\kappa$  is weakly inaccessible but not strongly inaccessible and  $\text{GMP}_{\omega_1, \kappa}$  holds.*

*Proof.* By a result of Magidor [21], the supercompactness of  $\kappa$  implies that  $\text{GMP}_{\omega_1, \kappa}$  holds (in fact, it implies more; cf. [30, Theorem 3.3]). Then Corollary 2.16 implies that  $\text{GMP}_{\omega_1, \kappa}$  continues to hold in  $V[\mathbb{P}]$ . In particular, if  $\mathbb{P} = \text{Add}(\omega, \kappa)$ , then in  $V[\mathbb{P}]$   $\kappa$  is weakly inaccessible but not strongly inaccessible and  $\text{GMP}_{\omega_1, \kappa}$  holds.  $\square$

### 3. PRESERVATION OF THE NEGATION OF THE WEAK KUREPA HYPOTHESIS BY CENTERED FORCINGS

#### 3.1. CENTERED FORCINGS

**Definition 3.1.** Let  $\mathbb{P}$  be a forcing and suppose  $\mu$  is a cardinal. We say that  $\mathbb{P}$  is  $\mu^+$ -centered if  $\mathbb{P}$  can be written as the union of a family  $\{\mathbb{P}_\alpha \subseteq \mathbb{P} \mid \alpha < \mu\}$  such that for every  $\alpha < \mu$ :

$$(3.1) \quad \text{for every } p, q \in \mathbb{P}_\alpha \text{ there exists } r \in \mathbb{P}_\alpha \text{ with } r \leq p, q.$$

If  $\mu = \omega$ , we say that  $\mathbb{P}$  is  $\sigma$ -centered.

It follows that  $\mathbb{P}$  can be written as a union of  $\mu$ -many filters if we close each  $\mathbb{P}_\alpha$  upwards. We require (3.1) to ensure nice properties of the system  $S(\dot{T})$  defined in Section 3.2 (in particular the transitivity of  $<_i$ ).

Let us add two remarks with regard to Definition 3.1. Firstly, as we already mentioned above, sometimes “ $\mu^+$ -centered” is called “ $\mu$ -centered” instead. We prefer to say  $\mathbb{P}$  is  $\mu^+$ -centered to agree with the notation for the chain condition (every  $\mu^+$ -centered  $\mathbb{P}$  is  $\mu^+$ -cc in our notation).

Secondly, some definitions of  $\mu^+$ -centeredness require just the compatibility of the conditions, with a witness not necessarily in  $\mathbb{P}_\alpha$ . The condition (3.1) in this case reads:

$$(3.2) \quad \text{for every } n < \omega \text{ and every sequence } p_0, p_1, \dots, p_{n-1} \\ \text{of conditions in } \mathbb{P}_\alpha \text{ there exists } r \in \mathbb{P} \text{ with } r \leq p_i \text{ for every } 0 \leq i < n.$$

The conditions (3.1) and (3.2) are not in general equivalent (see Kunen [19], before Exercise III.3.27), but the distinction is not so important for us because common forcings such as Cohen forcing and Prikry forcings are all centered in the stronger sense of (3.1). Also note that the conditions are equivalent for Boolean algebras: the definition (3.2) means that each  $\mathbb{P}_\alpha$  is

a system with the FIP (finite intersection property), and as such can be extended to a filter.

### 3.2. SYSTEMS AND DERIVED SYSTEMS

Suppose  $\mathbb{P}$  is a forcing notion. In order to show that certain objects cannot exist in a generic extension  $V[\mathbb{P}]$  (such as a weak Kurepa tree), we will work in the ground model and work with a system derived from a  $\mathbb{P}$ -name for the object in question. We give the general definition of a *system* here and discuss systems derived from names in Definition 3.4 below. We formulate the definition of the system to fit our purpose, which gives a slightly less general concept than the one introduced in [22].

**Definition 3.2.** Let  $\kappa \leq \lambda$  be cardinals and let  $D \subseteq \lambda$  be unbounded in  $\lambda$ . For each  $\alpha \in D$ , let  $S_\alpha \subseteq \{\alpha\} \times \kappa$  and let  $S = \bigcup_{\alpha \in D} S_\alpha$ .<sup>10</sup> Moreover, let  $I$  be an index set of cardinality  $\leq \kappa$  and  $\mathcal{R} = \{<_i \mid i \in I\}$  a collection of binary relations on  $S$ . We say that  $\langle S, \mathcal{R} \rangle$  is a  $(\kappa, \lambda)$ -*system* if the following hold:

- (i) For each  $i \in I$ ,  $\alpha, \beta \in D$  and  $\gamma, \delta < \kappa$ ; if  $(\alpha, \gamma) <_i (\beta, \delta)$  then  $\alpha < \beta$ .
- (ii) For each  $i \in I$ ,  $<_i$  is irreflexive and transitive.
- (iii) For each  $i \in I$ ,  $\alpha < \beta < \gamma$  in  $D$ ,  $x \in S_\alpha$ ,  $y \in S_\beta$  and  $z \in S_\gamma$ , if  $x <_i z$  and  $y <_i z$ , then  $x <_i y$ .
- (iv) For all  $\alpha < \beta$  in  $D$  there are  $y \in S_\beta$  and  $x \in S_\alpha$  and  $i \in I$  such that  $x <_i y$ .

We call a  $(\kappa, \lambda)$ -system  $\langle S, \mathcal{R} \rangle$  a *strong*  $(\kappa, \lambda)$ -*system* if the following strengthening of item (iv) holds:

- (iv') For all  $\alpha < \beta$  in  $D$  and for every  $y \in S_\beta$  there are  $x \in S_\alpha$  and  $i \in I$  such that  $x <_i y$ .

If  $\langle S, \mathcal{R} \rangle$  is a  $(\kappa, \lambda)$ -system, we say that the system has height  $\lambda$  and width  $\kappa$ . We call  $S_\alpha$  the  $\alpha$ -th level of  $S$ .

For the purposes of this paper we introduce the following definition:

**Definition 3.3.** Suppose  $\kappa \leq \lambda$  are cardinals and let  $\langle S, \mathcal{R} \rangle$  be a  $(\kappa, \lambda)$ -system. We call  $\langle S, \mathcal{R} \rangle$  *well-behaved* if  $|\mathcal{R}| < \kappa$ , i.e. the number of relations is strictly smaller than the width of the system.

A *branch* of the system is a subset  $B$  of  $S$  such that for some  $i \in I$ , and for all  $a \neq b \in B$ ,  $a <_i b$  or  $b <_i a$ . A branch  $B$  is *cofinal* if for each  $\alpha < \lambda$  there are  $\beta \geq \alpha$  and  $b \in B$  on level  $\beta$ .

Systems appear naturally when we wish to analyse in the ground model a  $\mathbb{P}$ -name  $\dot{T}$  for a tree which is added by a forcing notion  $\mathbb{P}$ . We give the definition for the context in which we will use it (more general definitions are possible).

**Definition 3.4.** Assume  $\mu$  is a regular cardinal. Assume  $\mathbb{P}$  is a  $\mu^+$ -centered forcing notion; let  $\mathbb{P} = \bigcup_{\alpha < \mu} \mathbb{P}_\alpha$  where each  $\mathbb{P}_\alpha$  is a filter. Assume further that  $\mathbb{P}$  forces that  $\dot{T}$  is a tree of height and size  $\lambda$ , where  $\lambda \geq \mu^+$  is regular. We assume that the domain of  $T$  is  $\lambda \times \lambda$ , where the  $\beta$ -th level of  $\dot{T}$  consists

<sup>10</sup>The elements of  $S$  are therefore ordered pairs of ordinals; if the ordinals are not important, we denote the pairs of ordinals by letters  $x, y, \dots$ , etc.

of pairs in  $\{\beta\} \times \lambda$ . We say that  $S(\dot{T}) = \langle \lambda \times \lambda, \mathcal{R} \rangle$  is a *derived system* (with respect to  $\mathbb{P}$  and  $\dot{T}$ ) if it is a system with domain  $\lambda \times \lambda$  which is equipped with binary relations  $\mathcal{R} = \{<_\alpha \mid \alpha < \mu\}$ , where

$$x <_\alpha y \leftrightarrow (\exists p \in \mathbb{P}_\alpha) p \Vdash x <_{\dot{T}} y.$$

Following the terminology of Definitions 3.2 and 3.3,  $S(\dot{T})$  is a strong well-behaved  $(\lambda, \lambda)$ -system.

### 3.3. A PRESERVATION THEOREM

We can prove a slightly more general result for the negation of the weak Kurepa hypothesis at  $\mu^+$ : we show that it is preserved over any model of  $\mathbf{GMP}_{\mu^{++}}$  by all  $\mu^+$ -centered forcings. Note that, if  $\mu^{<\mu} = \mu$ , then  $\text{Add}(\mu, \mu^+)$  is  $\mu^+$ -centered, and since a weak Kurepa tree at  $\mu^+$  has size only  $\mu^+$ , Theorem 3.5 implies that  $\neg\mathbf{wKH}(\mu^+)$  is preserved by adding any number of Cohen subsets of  $\mu$  to a model of  $\mathbf{GMP}_{\mu^{++}}$  in which  $\mu^{<\mu} = \mu$  (see Corollary 3.6). Also recall that  $\mathbf{MA}_{\omega_1}$  implies that every ccc forcing of size  $\omega_1$  is  $\sigma$ -centered, so our result implies that over models of  $\mathbf{GMP}_{\omega_2} + \mathbf{MA}_{\omega_1}$ , and hence also of  $\mathbf{PFA}$ ,  $\neg\mathbf{wKH}(\omega_1)$  is preserved by all ccc forcings of size  $\omega_1$  (see Corollary 3.7).

**Theorem 3.5.**  *$\mathbf{GMP}_{\mu^{++}}$  implies that  $\neg\mathbf{wKH}(\mu^+)$  is preserved by any  $\mu^+$ -centered forcing.*

*Proof.* Suppose  $\mathbb{P} = \bigcup_{i < \mu} \mathbb{P}_i$  is a  $\mu^+$ -centered forcing. Assume for a contradiction that  $\dot{T}$  is forced by the weakest condition in  $\mathbb{P}$  to be a weak  $\mu^+$ -Kurepa tree, and let  $S(\dot{T})$  be the derived system with respect to  $\dot{T}$ , as in Definition 3.4. Let  $M$  be a  $\mu^+$ -guessing model  $M \prec H(\mu^{++})$  of size  $\mu^+$  with  $S(\dot{T}) \in M$ . Since  $S(\dot{T})$  has domain  $\mu^+ \times \mu^+$ , the system with the relations is a subset of  $M$ .

Let us fix a sequence  $\langle \dot{b}_\alpha \mid \alpha < \mu^{++} \rangle$  of  $\mathbb{P}$ -names such that

$$(3.3) \quad 1_{\mathbb{P}} \Vdash \text{“}\langle \dot{b}_\alpha \mid \alpha < \mu^{++} \rangle \text{ are pairwise distinct cofinal branches in } \dot{T}\text{.”}$$

Working in  $V$ , there must be some  $i < \mu$ , such that for some  $I \subseteq \mu^{++}$  of size  $\mu^{++}$ , and all  $\alpha \in I$ , there are cofinally many  $x$  for which there are  $p_x \in \mathbb{P}_i$  with

$$p_x \Vdash x \in \dot{b}_\alpha.$$

Let us fix such an  $i < \mu$  and  $I \subseteq \mu^{++}$ .

For each  $\alpha \in I$ , let us define

$$B_\alpha = \{x \in S(\dot{T}) \mid (\exists p \in \mathbb{P}_i) p \Vdash x \in \dot{b}_\alpha\}.$$

Note that  $B_\alpha$  is a cofinal branch in  $S(\dot{T})$ . We finish the proof by showing:

- (i) For each  $\alpha \in I$ ,  $B_\alpha$  is an element of  $M$ .
- (ii) For all  $\alpha \neq \beta \in I$ ,  $B_\alpha \neq B_\beta$ .

Items (i) and (ii) imply that  $M$  has size at least  $\mu^{++}$ , which is a contradiction.

With regard to (i), we will show that each  $B_\alpha$  is  $(\mu^+, M)$ -approximated, and therefore is an element of  $M$ .<sup>11</sup> Let us fix  $\alpha \in I$  and  $a \in M$  of size  $\mu$ .

<sup>11</sup>In general, if some set  $d$  is  $(\mu^+, M)$ -approximated, and  $M$  is  $\mu^+$ -guessing, there is some  $e \in M$  such that  $d \cap M = e \cap M$ , where  $d \neq e$  is possible. However, in the present case, since  $S(\dot{T}) \subseteq M$ , if  $B_\alpha$  is  $(\mu^+, M)$ -approximated, and  $M$  is  $\mu^+$ -guessing, then  $B_\alpha \in M$ .



We need to show that  $B_\alpha \cap a$  is in  $M$ . Since the system  $S(\dot{T})$  has height  $\mu^+$  and is a subset of  $M$ , there is some  $y \in B_\alpha \cap M$  which is above every element of  $B_\alpha \cap a$  in  $<_i$ . It follows that

$$B_\alpha \cap a = \{x \in a \cap S(\dot{T}) \mid (\exists p \in \mathbb{P}_i) p \Vdash x \in \dot{b}_\alpha\} = \{x \in a \cap S(\dot{T}) \mid x <_i y\}.$$

For the identity between the second and third set, fix  $p' \in \mathbb{P}_i$  such that  $p' \Vdash y \in \dot{b}_\alpha$ , and note that if  $p \in \mathbb{P}_i$  and  $p \Vdash x \in \dot{b}_\alpha$ , then the existence of a lower bound of  $p, p'$  in  $\mathbb{P}_i$  implies  $x <_i y$ ; and conversely, if  $p \Vdash x <_{\dot{T}} y$  for some  $p \in \mathbb{P}_i$ , then the existence of a lower bound in  $\mathbb{P}_i$  implies that for some  $r \in \mathbb{P}_i$ ,  $r \Vdash x \in \dot{b}_\alpha$ . Since the third expression determines a set in  $M$  (because all parameters are in  $M$ ),  $B_\alpha \cap a$  is in  $M$ .

With regard to (ii): suppose for a contradiction  $B_\alpha = B_\beta$  for some  $\alpha \neq \beta \in I$ . Fix for every  $x \in B_\alpha = B_\beta$  some conditions  $p_x^\alpha$  and  $p_x^\beta$  in  $\mathbb{P}_i$  such that

$$p_x^\alpha \Vdash x \in \dot{b}_\alpha \text{ and } p_x^\beta \Vdash x \in \dot{b}_\beta.$$

Let  $p_x \in \mathbb{P}_i$  be some lower bound of  $p_x^\alpha, p_x^\beta$ .

Suppose first that  $\{p_x \mid x \in B_\alpha = B_\beta\}$  has size at most  $\mu$ . Then there exists some  $p$  such that  $p = p_x$  for  $\mu^+$ -many  $x$ . This  $p$  forces  $\dot{b}_\alpha = \dot{b}_\beta$ , which contradicts (3.3).

Suppose now that  $\{p_x \mid x \in B_\alpha = B_\beta\}$  has size  $\mu^+$ . The  $\mu^+$ -cc of  $\mathbb{P}$  implies that there is a condition  $p$  which forces that  $\dot{G}$  has an intersection with  $\{p_x \mid x \in B_\alpha = B_\beta\}$  of size  $\mu^+$ . In particular,  $p$  forces  $\dot{b}_\alpha = \dot{b}_\beta$ , which contradicts (3.3).  $\square$

Theorem 3.5 gives an alternative proof that  $\neg\text{wKH}(\mu^+)$  is preserved by adding any number of Cohen subsets of  $\mu$  over models of  $\text{GMP}_{\mu^{++}}$ .

**Corollary 3.6.** *Over models of  $\text{GMP}_{\mu^{++}}$  with  $\mu^{<\mu} = \mu$ ,  $\neg\text{wKH}(\mu^+)$  is preserved by adding any number of Cohen subsets of  $\mu$ .*

*Proof.* It is known that if  $\mu^{<\mu} = \mu$ , then for every  $\alpha \leq 2^\mu$ , Cohen forcing  $\text{Add}(\mu, \alpha)$  is  $\mu^+$ -centered. We only need that  $\text{Add}(\mu, \mu^+)$  is  $\mu^+$ -centered for our argument, but note that since  $\text{GMP}_{\mu^{++}}$  implies  $2^\mu \geq \mu^{++}$ , longer Cohen forcings are  $\mu^+$ -centered.

Suppose for a contradiction that  $\text{Add}(\mu, \gamma)$  adds a weak Kurepa tree  $T$  at  $\mu^+$  for some  $\gamma$ . Using the fact that  $T$  has size  $\mu^+$ ,  $T$  is added already by  $\text{Add}(\mu, \mu^+)$ . By Theorem 3.5,  $T$  has at most  $\mu^+$  cofinal branches in  $V[\text{Add}(\mu, \mu^+)]$ . Since Cohen forcing at  $\mu$  is  $\mu^+$ -Knaster, and in particular its product with itself is  $\mu^+$ -cc, it follows that Cohen forcing at  $\mu$  cannot add new cofinal branches to  $T$  over  $V[\text{Add}(\mu, \mu^+)]$ . This is a contradiction.  $\square$

If  $\mu = \omega$  and we additionally assume  $\text{MA}_{\omega_1}$ , then ccc forcings are more well-behaved (for instance they are all Knaster). Moreover, if they have size at most  $\omega_1$ , they are even  $\sigma$ -centered (see [33, Theorem 4.5]), so we obtain the following Corollary.

**Corollary 3.7.**  *$\neg\text{wKH}(\omega_1)$  is preserved over models of  $\text{GMP}_{\omega_2} + \text{MA}_{\omega_1}$  by any ccc forcing of size  $\omega_1$ . In particular over models of  $\text{PFA}$ ,  $\neg\text{wKH}(\omega_1)$  is preserved by all ccc forcings of size  $\omega_1$ .*

## 4. AN APPLICATION

In this section, we sketch an application of the indestructibility result for the negation of the weak Kurepa Hypothesis to provide a proof of the consistency of  $\neg\text{wKH}(\aleph_{\omega+1})$ . Familiarity with lifting arguments, as they appear for instance in [1], [12] or [4], is assumed.

Suppose  $\mu < \kappa$  are regular cardinals with  $\mu^{<\mu} = \mu$  and  $\kappa$  inaccessible. The Mitchell forcing  $\mathbb{M}(\mu, \kappa)$  can be written as  $\text{Add}(\mu, \kappa) * \dot{R}$  for some quotient forcing  $\dot{R}$  which is forced to be  $\mu^+$ -distributive (see [1] for more details). If  $G_{\mathbb{M}}$  is  $\mathbb{M}(\mu, \kappa)$ -generic, we write  $G_{\mathbb{M}} = G_0 * G_1$  to denote the corresponding  $\text{Add}(\mu, \kappa) * \dot{R}$ -generic (this will be relevant for Theorem 4.3).

**Theorem 4.1.** *Suppose  $\mu < \kappa$  are supercompact cardinals and  $\mu$  is Laver-indestructibly supercompact. Let  $G = G_{\mathbb{M}} \times G_A$  be  $\mathbb{M}(\mu, \kappa) \times \text{Add}(\mu, \kappa^+)$ -generic, where  $\mathbb{M}(\mu, \kappa)$  is the Mitchell forcing and  $\text{Add}(\mu, \kappa^+)$  is the Cohen forcing for adding  $\kappa^+$ -many subsets of  $\mu$ . In  $V[G]$ ,  $\mu$  is supercompact,  $2^\mu = \mu^{+3}$ , and  $\text{GMP}_{\mu^{++}}$  holds. Let  $\mathbb{Q}$  be the Prikry forcing with interleaved collapses which turns  $\mu$  into  $\aleph_\omega$ . Suppose  $F$  is  $\mathbb{Q}$ -generic over  $V[G]$ . Then in  $V[G][F]$ , we have  $2^{\aleph_\omega} = \aleph_{\omega+3}$  and  $\neg\text{wKH}(\aleph_{\omega+1})$ .*

*Proof.* Note that the preparatory forcing  $\text{Add}(\mu, \kappa^+)$  is included to obtain a “guiding generic” required for the definition of  $\mathbb{Q}$ . More details can be found in [27] and [4].<sup>12</sup> By Fact 2.3,  $\text{GMP}_{\mu^{++}}$  holds in  $V[G_{\mathbb{M}}]$ . By the preservation theorem we proved earlier, Corollary 2.16,  $\text{GMP}_{\mu^{++}}$  continues to hold in  $V[G_{\mathbb{M}} \times G_A] = V[G]$ . The compatibility of conditions in the Prikry forcing with collapses  $\mathbb{Q}$  (defined with respect to some guiding generic which exists in  $V[G]$  due to the preparatory forcing  $\text{Add}(\mu, \kappa^+)$ ) is determined only by the stems, and so  $\mathbb{Q}$  is  $\mu^+$ -centered. Then the present theorem follows by Theorem 3.5 applied to  $\mathbb{Q}$ .  $\square$

**Remark 4.2.** By an argument using a quotient analysis as in [27] and [4], the tree property holds at  $\aleph_{\omega+2}$  in the model  $V[G][F]$  as well.

If  $\mu$  is not turned into  $\aleph_\omega$ , but only singularised (to an arbitrary cofinality), we can apply an indestructibility result also for the tree property, following [12].

**Theorem 4.3.** *Suppose  $\mu < \kappa$  are supercompact cardinals and  $\mu$  is Laver-indestructibly supercompact. Let  $G_{\mathbb{M}}$  be  $\mathbb{M}(\mu, \kappa)$ -generic and let us write  $G_{\mathbb{M}}$  as  $G_0 * G_1$  (see our convention before Theorem 4.1). In  $V[G_{\mathbb{M}}] = V[G_0][G_1]$ ,  $\mu$  is supercompact,  $2^\mu = \mu^{++}$ , and  $\text{GMP}_{\mu^{++}}$  holds. Let  $\mathbb{Q}$  be the Prikry or Magidor forcing which turns  $\mu$  into a singular cardinal without collapsing any cardinals; choose  $\mathbb{Q}$  to be an element of  $V[G_0]$ .<sup>13</sup> Suppose  $F$  is  $\mathbb{Q}$ -generic over  $V[G_{\mathbb{M}}]$ . Then in  $V[G_{\mathbb{M}}][F]$  we have  $2^\mu = \mu^{++}$  and  $\neg\text{wKH}(\mu^+)$  and  $\text{TP}(\mu^{++})$  both hold.*

*Proof.* The part regarding the negation of the weak Kurepa Hypothesis is as in Theorem 4.1 (but it is easier since the preparatory forcing is not necessary because no guiding generic needs to be constructed this time), and the tree

<sup>12</sup>To avoid possible confusion, let us mention that this preparatory forcing has been omitted from an analogous argument in [4], but it should have been included there as well.

<sup>13</sup>This is possible; see the construction in [13] for more details.

property holds because the forcing  $\mathbb{Q}$  lives in  $V[G_0]$ , so the indestructibility result for  $\mu^+$ -cc forcings from [12] applies.  $\square$

Let us summarize the arguments in this section succinctly as follows:

**Corollary 4.4.** *Assuming the consistency of the existence of two supercompact cardinals, it is consistent that  $\neg\text{wKH}(\aleph_{\omega+1})$  and  $\text{TP}(\aleph_{\omega+2})$  both hold.*

**Remark 4.5.** Let us make some comments on the preparatory forcing in Theorem 4.1. It increases the value of  $2^\kappa$  to  $\kappa^{+3}$ , resulting in  $2^{\aleph_\omega} = \aleph_{\omega+3}$  in the final model, and its purpose is to ensure one can construct a guiding generic for the definition of the Prikry forcing with collapses. A natural question is whether Theorem 4.1 can hold with  $2^{\aleph_\omega} = \aleph_{\omega+2}$ . If we want to have just  $\neg\text{wKH}(\aleph_{\omega+1})$ , then the answer is yes because it suffices to collapse  $\aleph_{\omega+3}$  to  $\aleph_{\omega+2}$  with a  $\aleph_{\omega+2}$ -closed forcing which cannot add new trees of height and size  $\aleph_{\omega+1}$ , and thus preserves  $\neg\text{wKH}(\aleph_{\omega+1})$ . But this collapse may introduce new  $\aleph_{\omega+2}$ -Aronszajn trees, thus violating  $\text{TP}(\aleph_{\omega+2})$ . It seems unclear whether a modification of the method in Theorem 4.1 can ensure  $\neg\text{wKH}(\aleph_{\omega+1})$  and  $\text{TP}(\aleph_{\omega+2})$  together with  $2^{\aleph_\omega} = \aleph_{\omega+2}$ .

Note in this respect that having just  $\text{TP}(\aleph_{\omega+2})$  with  $2^{\aleph_\omega} = \aleph_{\omega+2}$  is possible by starting from a hypermeasurable cardinal  $\kappa$  (see [8]). It is an interesting fact that the construction from a hypermeasurable cardinal also needs a preparatory forcing, this time it is “roughly” equal to  $\text{Add}(\mu^+, \kappa)$ , which in this case is used not to construct a guiding generic but to lift an elementary embedding (it forces an extra generic on which a “surgery” is then performed). However, while  $\text{Add}(\mu^+, \kappa)$  is innocuous with respect to the tree property, it appears to force  $\text{wKH}(\aleph_{\omega+1})$  in the resulting model, by considering the cardinal arithmetic consequences identified in [20].

Finally note that there is another method for obtaining  $\text{TP}(\aleph_{\omega+2})$  which uses the iteration of the  $\kappa$ -Sacks forcing, and which does not require any preparatory forcing of this type (but can only yield  $2^{\aleph_\omega} = \aleph_{\omega+2}$ ); see [7]. It might be interesting to see whether  $\neg\text{wKH}(\aleph_{\omega+1})$  holds in this model.

## 5. OPEN QUESTIONS

- (1) Can we extend the preservation result for  $\text{TP}(\mu^{++})$  to all  $\mu^+$ -centered forcings? It seems that our method in Theorem 3.5 does not generalize to the tree property. The syntactical difference between the two principles may be important here: while  $\text{wKH}(\omega_1)$  is a  $\Sigma_1$  sentence,  $\neg\text{TP}(\omega_2)$  is a  $\Sigma_2$  sentence. This is also relevant for question (3) mentioned below.

Or more ultimately, can the preservation result under  $\text{GMP}_{\mu^{++}}$  be extended to all  $\mu^+$ -cc forcings? To our knowledge there is no known counterexample.

- (2) Is the assumption of  $\text{GMP}_{\mu^{++}}$  necessary for the preservation results? To our knowledge it is still open whether, for instance, there may be a model which satisfies  $\text{TP}(\omega_2)$  and/or  $\neg\text{wKH}(\omega_1)$ , and over which a single Cohen forcing at  $\omega$  adds an  $\omega_2$ -Aronszajn tree and/or a weak Kurepa tree at  $\omega_1$ .

- (3) Zapletal observed that  $\mathbf{MA}_{\omega_2}$  implies that  $\neg\mathbf{wKH}(\omega_1)$  is preserved by all ccc forcings: if  $\mathbb{P}$  is a ccc forcing notion and  $\dot{T}$  is a  $\mathbb{P}$ -name for a weak  $\omega_1$ -Kurepa tree, then with  $\mathbf{MA}_{\omega_2}$  one can use  $\dot{T}$  to define a weak Kurepa tree back in  $V$ . However, it is not known whether  $\mathbf{MA}_{\omega_2}$  is consistent with  $\neg\mathbf{wKH}(\omega_1)$ .

It is known that  $\mathbf{MA}_{\omega_2} + \neg\mathbf{KH}(\omega_1)$  is consistent due to a result of Jensen and Schlichta [15] who showed that starting with  $\mathbf{CH}$ , in the Levy collapse by countable conditions of a Mahlo cardinal  $\kappa$  to  $\omega_2$ ,  $\neg\mathbf{KH}(\omega_1)$  is preserved by all ccc forcings. In particular, it is preserved by any ccc finite-support iteration which forces  $\mathbf{MA}_{\omega_2}$  over this model. (We do not know an easier proof of the consistency of  $\mathbf{MA}_{\omega_2} + \neg\mathbf{KH}(\omega_1)$ .)

We can ask whether there is an analogous construction for  $\neg\mathbf{wKH}(\omega_1)$ : can one collapse a large cardinal  $\kappa$  to become  $\omega_2$  with  $2^\omega \geq \omega_2$  and obtain a model over which  $\neg\mathbf{wKH}(\omega_1)$  is preserved by all ccc forcings? Note that this line of argument cannot be used with  $\mathbf{PFA}$  because  $\mathbf{MA}_{\omega_2}$  implies  $2^\omega > \omega_2$ . More generally, if  $\mathbf{MA}_\theta$  holds for a regular  $\theta$ , then we can only argue that if  $\dot{T}$  is a  $\mathbb{P}$ -name for a weak Kurepa tree at  $\omega_1$  with at least  $\theta$  cofinal branches, there is one in  $V$  with at least  $\theta$  cofinal branches.

- (4) Is it possible to strengthen Theorem 3.5 along the lines of Corollary 2.16 and show that not only  $\neg\mathbf{wKH}(\mu^+)$ , but the principle  $\mathbf{GMP}_{\mu^{++}}$  itself is preserved by all  $\mu^+$ -centered forcings? Or perhaps by all  $\mu^+$ -cc forcings?

## REFERENCES

1. Uri Abraham, *Aronszajn trees on  $\aleph_2$  and  $\aleph_3$* , *Annals of Pure and Applied Logic* **24** (1983), no. 3, 213–230.
2. Sean Cox and John Krueger, *Quotients of strongly proper forcings and guessing models*, *The Journal of Symbolic Logic* **81** (2016), no. 1, 264–283.
3. ———, *Indestructible guessing models and the continuum*, *Fundamenta Mathematicae* **239** (2017), 221–258.
4. James Cummings, Sy-David Friedman, Menachem Magidor, Assaf Rinot, and Dima Sinapova, *The eightfold way*, *The Journal of Symbolic Logic* **83** (2018), no. 1, 349–371.
5. Keith J. Devlin, *Constructibility*, Springer, 1984.
6. Matthew Foreman, *More saturated ideals*, *Cabal seminar 79-81*, volume 1019 of *Lecture Notes in Math.* (A. S. Kechris, D. A. Martin, and Y. N. Moschovakis, eds.), Springer, Berlin, 1983, pp. 1–27.
7. Sy-David Friedman and Ajdin Halilović, *The tree property at  $\aleph_{\omega+2}$* , *The Journal of Symbolic Logic* **76** (2011), no. 2, 477–490.
8. Sy-David Friedman, Radek Honzik, and Šárka Stejskalová, *The tree property at  $\aleph_{\omega+2}$  with a finite gap*, *Fundamenta Mathematicae* **251** (2020), no. 3, 219–244.
9. Thomas Gilton and Šárka Stejskalová, *Compactness principles at  $\aleph_{\omega+2}$* , (2022), Submitted.
10. Moti Gitik and John Krueger, *Approachability at the second successor of a singular cardinal*, *The Journal of Symbolic Logic* **74** (2009), no. 4, 1211–1224.
11. Yair Hayut and Menachem Magidor, *Destructibility of the tree property at  $\aleph_{\omega+1}$* , *The Journal of Symbolic Logic* **84** (2019), no. 2, 621–631.
12. Radek Honzik and Šárka Stejskalová, *Indestructibility of the tree property*, *The Journal of Symbolic Logic* **85** (2020), no. 1, 467–485.
13. ———, *Small  $\mathfrak{u}(\kappa)$  at singular  $\kappa$  with compactness at  $\kappa^{++}$* , *Archive for Mathematical Logic* **61** (2022), 33–54.

14. Tomáš Jech, *Some combinatorial problems concerning uncountable cardinals*, Annals of Mathematical Logic **5** (1973), 165–198.
15. Ronald Jensen and Karl Schlechta, *Results on the generic Kurepa hypothesis*, Archive for Mathematical Logic **30** (1990), 13–27.
16. Renling Jin and Saharon Shelah, *Can a small forcing create Kurepa trees*, Annals of Pure and Applied Logic **85** (1997), 47–68.
17. Akihiro Kanamori, *Perfect-set forcing for uncountable cardinals*, Annals of Mathematical Logic **19** (1980), 97–114.
18. John Krueger, *Guessing models imply the singular cardinal hypothesis*, Proc. Amer. Math. Soc. **147** (2019), no. 12, 5427–5434. MR 4021101
19. Kenneth Kunen, *Set theory (studies in logic: Mathematical logic and foundations)*, College Publications, 2011.
20. Chris Lambie-Hanson and Šárka Stejskalová, *Guessing models, trees, and cardinal arithmetic*, Submitted 2023, <https://arxiv.org/abs/2303.01366>.
21. Menachem Magidor, *Combinatorial characterization of supercompact cardinals*, Proceedings of the American Mathematical Society **42** (1974), no. 1, 279–285.
22. Menachem Magidor and Saharon Shelah, *The tree property at successors of singular cardinals*, Annals of Mathematical Logic **35** (1996), no. 5-6, 385–404.
23. William J. Mitchell, *Aronszajn trees and the independence of the transfer property*, Annals of Mathematical Logic **5** (1972/1973), no. 1, 21–46.
24. Rahman Mohammadpour, *A road to compactness through guessing models*, (2022), Preprint.
25. Assaf Rinot, *A cofinality-preserving small forcing may introduce a special Aronszajn tree*, Archive for Mathematical Logic **48** (2009), no. 8, 934–946.
26. J. H. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Axiomatic Set Theory (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1971, pp. 383–390.
27. Dima Sinapova and Spencer Unger, *The tree property at  $\aleph_{\omega+1}$  and  $\aleph_{\omega+2}$* , The Journal of Symbolic Logic **82** (2018), 669–682.
28. Stevo Todorčević, *Some consequences of  $\text{MA} + \neg\text{wKH}$* , Topology and its Applications **12** (1981), no. 2, 187–202.
29. Spencer Unger, *Fragility and indestructibility of the tree property*, Archive for Mathematical Logic **51** (2012), no. 5–6, 635–645.
30. Matteo Viale, *Guessing models and generalized Laver diamond*, Annals of Pure and Applied Logic **163** (2012), no. 11, 1660–1678.
31. Matteo Viale and Christoph Weiss, *On the consistency strength of the proper forcing axiom*, Advances in Mathematics **228** (2011), 2672–2687.
32. Christoph Weiss, *The combinatorial essence of supercompactness*, Annals of Pure and Applied Logic **163** (2012), no. 11, 1710–1717.
33. William Weiss, *Versions of Martin's Axiom*, Handbook of set theoretic topology (K. Kunen and J. E. Vaughan, eds.), North-Holland Publishing Co., 1984, pp. 827–886.

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